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Topics in Stochastic Geometry

Lecture 1
Point processes and random measures

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1. Definition of random measures

Setting

- $(S, \mathcal{S})$ = measurable (state) space.
- $(\Omega, \mathcal{F}, \mathbb{P})$ = probability space.

Remark

Sometimes it is useful to assume that $\mathbb{P}$ is only σ-finite.

Definition

- $\mathbf{M} \equiv \mathbf{M}(S)$ space of all σ-finite measures on S.
- $\mathcal{M}$ = smallest σ-field of subsets of $\mathbf{M}$ making the mappings $\mu \mapsto \mu(B)$ for all $B \in \mathcal{S}$ measurable.
- $\mathbf{N}$ = space of all σ-finite counting measures on S.

Definition

- A random measure on S is a random element η in $\mathbf{M}$ that is uniformly σ-finite. This means that there are measurable sets $B_n \in \mathcal{S}$, $n \in \mathbb{N}$, such that $\eta(B_n) < \infty$ a.s.
- A point process is a random measure η such that $\mathbb{P}(\eta \notin \mathbf{N}) = 0$.

2. Intensity measure and distribution

Definition

The intensity measure of a random measure η is the measure Λ defined by

$$\Lambda(B) := \mathbb{E}\eta(B), \quad B \in \mathcal{S}.$$

Theorem (Campbell's theorem)

Let η be a random measure with intensity measure Λ. Then

$$\mathbb{E} \int f(s)\eta(ds) = \int f(s)\Lambda(ds)$$

for all measurable $f : S \to [0, \infty)$.

Example

Let $X_1, X_2, \dots$ be independent and identically distributed random elements in S.

- The point process

$$\eta := \sum_{j=1}^{n} \delta_{X_j}$$

  is a Binomial process of size n based on the distribution of X_1.
- If n above is replaced with a $\mathbb{N}_0$-valued random variable Y independent of (X_n), then η is a mixed Binomial process with random size Y.

Theorem

Let ξ and η be random measures on S. The following statements are equivalent:

1. $\xi \stackrel{d}{=} \eta$
2. $(\xi(B_1), \dots, \xi(B_k)) \stackrel{d}{=} (\eta(B_1), \dots, \xi(B_k))$ *for all* $B_1, \dots, B_k \in \mathcal{S}$ *and* $k \in \mathbb{N}$.
3. $\int f d\xi \stackrel{d}{=} \int f d\eta$ *for all measurable* $f : S \to [0, \infty)$.
4. *For all* $f : S \to [0, \infty)$

$$\mathbb{E} \exp \Big[- \int f(s) \xi(ds) \Big] = \mathbb{E} \exp \Big[- \int f(s) \eta(ds) \Big].$$

3. The Poisson process

Definition

A Poisson process on S is a point process η on S having the following two properties:

- The random variables $\eta(B_1), \dots, \eta(B_m)$ are stochastically independent whenever $B_1, \dots, B_m$ are measurable and pairwise disjoint.
- There is a measure Λ on S such that

$$\mathbb{P}(\eta(B) = k) = \frac{\Lambda(B)^k}{k!} \exp[-\Lambda(B)], \quad k \in \mathbb{N}_0, B \in \mathcal{S},$$

where $\infty^k e^{-\infty} := 0$ for all $k \in \mathbb{N}_0$.

Theorem

Let Λ be a σ-finite measure on S. Then a point process η on S is a Poisson process with intensity measure Λ if and only if

$$\mathbb{E} \exp\left[-\int f(s)\eta(ds)\right] = \exp\left[-\int \left(1 - e^{-f(s)}\right)\Lambda(ds)\right]$$

for all measurable $f : S \to [0, \infty)$.

Corollary

Assume that η is a mixed sample process with sample distribution F and a sample size distribution that is Poisson with intensity $\gamma \geq 0$. Then η is a Poisson process with intensity measure γF.

Theorem (Mecke 1967)

Let Λ be a σ-finite measure on S. Then a point process η on S is a Poisson process with intensity measure Λ if and only if

$$\mathbb{E} \int f(\eta, s)\eta(ds) = \mathbb{E} \int f(\eta + \delta_s, s)\Lambda(ds),$$

or, equivalently,

$$\mathbb{E} \int f(\eta - \delta_s, s)\eta(ds) = \mathbb{E} \int f(\eta, s)\Lambda(ds),$$

for all measurable $f : \mathbf{N} \times S \to [0, \infty)$.

Example (Cox process)

Let Π_μ denote the distribution of a Poisson process with intensity measure μ. Let ξ be a random measure on S. Then a point process η on S is a Cox process driven by ξ if

$$\mathbb{P}(\eta \in \cdot \mid \xi) = \Pi_\xi \quad \mathbb{P}-\text{a.s.}$$

4. Stationarity

Setting

G is a locally compact second countable Abelian group with Haar measure λ.

Definition (Shift)

For $g \in G$ define $\theta_g : \mathbf{M}(G) \to \mathbf{M}(G)$, by

$$\theta_g \mu(B) := \mu(B + g), \quad B \in \mathcal{G}.$$

Remark

The group G operates on $\mathbf{M}(G)$, that is:

- $\theta_0 = \mathrm{id}_{\mathbf{M}(G)}$,
- We have the flow property

$$\theta_g \circ \theta_h = \theta_{g+h}, \quad g, h \in G.$$

- The mapping $(\mu, g) \mapsto \theta_g \mu$ is measurable.

Definition

A random measure ξ on G is stationary if

$$\theta_g \xi \overset{d}{=} \xi, \quad g \in G.$$

Example

A Poisson process on G is stationary if and only if its intensity measure is a multiple of Haar measure.

Example

Let η be a Cox process on G driven by the random measure ξ. Then η is stationary if and only if ξ is stationary.

Setting

Let $\{\theta_g : g \in G\}$ be a measurable flow on a measurable space $(\mathbf{W}, \mathcal{W})$.

Example

For $\mu \in \mathbf{M}(G \times S)$ define

$$\theta_g \mu(B \times C) := \mu((B + g) \times C), \quad B \in \mathcal{G}, C \in \mathcal{S}.$$

Then $\{\theta_g : g \in G\}$ is a measurable flow on $\mathbf{M}(G \times S)$.

Example

Let $x : G \to S$ and $g \in G$. Define $\theta_g x : G \to S$ by

$$\theta_g x(h) := x(g + h), \quad h \in G.$$

Let $\mathbf{W}$ be a subset of the path space S^G that is invariant under the above shifts equipped with the σ-field $\mathcal{W}$ rendering all mappings $x \mapsto x(g)$, $g \in G$, measurable and such that $(g, x) \mapsto \theta_g x$ is measurable.

Definition

- A random element X in $\mathbf{W}$ is stationary if

$$\theta_g X \stackrel{d}{=} X, \quad g \in G.$$

- A random measure ξ on G and a random element X in $\mathbf{W}$ are jointly stationary if

$$(\theta_g \xi, \theta_g X) \stackrel{d}{=} (\xi, X), \quad g \in G.$$

Example

Let η be a stationary Cox process on G driven by the random measure ξ. Then η and ξ are jointly stationary.

Definition

Let ξ be a random measure on G and X a random element in $\mathbf{W}$. Assume that ξ and X are jointly stationary. The measure

$$\mathbb{P}_{\xi,X}(A) := \lambda(B)^{-1}\mathbb{E}\int_B \mathbf{1}\{(\theta_g\xi, \theta_g X) \in A\}\xi(dg), \quad A \in \mathcal{M} \otimes \mathcal{W},$$

is called the Palm measure of (ξ, X). Here $B \in \mathcal{G}$ is a Borel set with $0 < \lambda(B) < \infty$. If the intensity

$$\gamma_\xi := \lambda(B)^{-1}\mathbb{E}\xi(B)$$

of ξ is positive and finite, then the normalized Palm measure $\mathbb{P}^0_{\xi,X} := \gamma_\xi^{-1}\mathbb{P}_\xi$ is called Palm distribution of (ξ, X).

Theorem (refined Campbell theorem)

Let ξ and X be jointly stationary. Then

$$\mathbb{E}\int f(\theta_g\xi,\theta_g X,g)\xi(dg)=\iint f(\mu,x,g)\mathbb{P}_{\xi,X}(d(\mu,x))\lambda(dg)$$

for all measurable $f:\mathbf{M}(G)\times\mathbf{W}\times G\to[0,\infty)$.

Corollary

The intensity measure of a stationary random measure ξ is a multiple of the Haar measure λ. The multiple is the intensity γ_ξ of ξ.

Example

The Palm distribution of a stationary Poisson process η is obtained from the stationary distribution by adding an atom at 0, that is:

$$\mathbb{P}^0_\eta = \mathbb{P}(\eta + \delta_0 \in \cdot).$$

Example

Let η be a Cox process on G driven by a stationary ξ with a finite intensity. Then

$$\mathbb{P}^0_{\eta,\xi} = \iint \mathbf{1}\{(\mu + \delta_0, \alpha) \in \cdot\}\Pi_\alpha(d\mu)\mathbb{P}^0_\xi(d\alpha).$$

Hence, up to an additional atom at 0, the Palm distribution of η is again that of a Cox process, driven by a random measure with the Palm distribution of ξ.

Theorem (Neveu's exchange formula)

Let ξ and η be random measures on G and X a random element in W. Assume that ξ, η, and X are jointly stationary. Then

$$\iint f(\theta_g\mu, \theta_g\mu', \theta_g x, -g)\mu'(dg)\mathbb{P}_{\xi,\eta,X}(d(\mu,\mu',x)) = \iint f(\mu,\mu',x,g)\mu(dg)\mathbb{P}_{\eta,\xi,X}(d(\mu,\mu',x))$$

for all measurable $f : \mathbf{M}(G) \times \mathbf{M}(G) \times \mathbf{W} \to [0,\infty]$.

Corollary (Mass transport principle)

Let $\kappa : \mathbf{M}(G) \times \mathbf{M}(G) \times G \times G \to [0, \infty)$ *be measurable and invariant under joint shifts of all arguments:*

$$\kappa(\theta_r\mu, \theta_r\mu', g-r, h-r) = \kappa(\mu, \mu', g, h), \quad \mu, \mu' \in \mathbf{M}(G),\ r, g, h \in G.$$

Then

$$\mathbb{E} \iint \mathbf{1}_B(h)\kappa(g, h)\,\eta(dg)\xi(dh) = \mathbb{E} \iint \mathbf{1}_{B'}(g)\kappa(g, h)\,\eta(dg)\xi(dh)$$

for all $B, B' \in \mathcal{G}$ *with finite and equal Haar measure.*

Proof.

Apply Neveu's exchange formula with

$$f(\mu, \mu', g) := \kappa(\mu, \mu', 0, g).$$

5. References

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