Gerber-Shiu Theory

A. E. Kyprianou[1]

Department of Mathematical Sciences, University of Bath

[1] Based on several joint works with E. Biffis, R. Loeffen, C. Ott, Z. Palmowski, J.C. Pardo, J.L. Pérez, X. Zhou.

Classical Cramér-Lundberg process

Classical Cramér-Lundberg process

- The classical risk insurance ruin problem sees the wealth of an insurance problem modelled by the so-called Cramér-Lundberg process:

$$X_t := x + \mathsf{c}t - \sum_{i=1}^{N_t} \xi_i,$$

with the understanding that x is the initial wealth, c is the rate at which premiums are collected and $\{N_t : t \geq 0\}$ is a Poisson process describing the arrival of the i.i.d. claims $\{\xi_i : i \geq 0\}$.

Classical Cramér-Lundberg process

- The classical risk insurance ruin problem sees the wealth of an insurance problem modelled by the so-called Cramér-Lundberg process:

$$X_t := x + \mathsf{c}t - \sum_{i=1}^{N_t} \xi_i,$$

  with the understanding that x is the initial wealth, c is the rate at which premiums are collected and $\{N_t : t \geq 0\}$ is a Poisson process describing the arrival of the i.i.d. claims $\{\xi_i : i \geq 0\}$.
- This is nothing but a spectrally negative Lévy process (Lévy process with no positive jumps and non-monotone paths).

Classical Cramér-Lundberg process

- The classical risk insurance ruin problem sees the wealth of an insurance problem modelled by the so-called Cramér-Lundberg process:

$$X_t := x + \mathsf{c}t - \sum_{i=1}^{N_t} \xi_i,$$

with the understanding that x is the initial wealth, c is the rate at which premiums are collected and $\{N_t : t \geq 0\}$ is a Poisson process describing the arrival of the i.i.d. claims $\{\xi_i : i \geq 0\}$.
- This is nothing but a spectrally negative Lévy process (Lévy process with no positive jumps and non-monotone paths).
- Henceforth we do not distinguish between the case that X is a Cramér-Lundberg process and a general spectrally negative Lévy process. Appeal to the usual Markovian notation $\{\mathbb{P}_x : x \in \mathbb{R}\}$.

Classical Cramér-Lundberg process

- The classical risk insurance ruin problem sees the wealth of an insurance problem modelled by the so-called Cramér-Lundberg process:

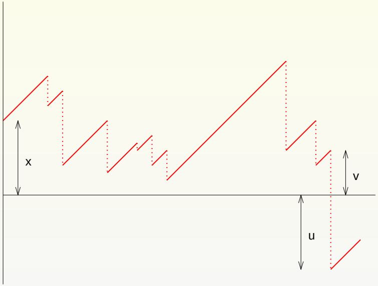
$$X_t := x + \mathsf{c}t - \sum_{i=1}^{N_t} \xi_i,$$

with the understanding that x is the initial wealth, c is the rate at which premiums are collected and $\{N_t : t \geq 0\}$ is a Poisson process describing the arrival of the i.i.d. claims $\{\xi_i : i \geq 0\}$.
- This is nothing but a spectrally negative Lévy process (Lévy process with no positive jumps and non-monotone paths).
- Henceforth we do not distinguish between the case that X is a Cramér-Lundberg process and a general spectrally negative Lévy process. Appeal to the usual Markovian notation $\{\mathbb{P}_x : x \in \mathbb{R}\}$.
- A classical field of study, so called Gerber-Shiu, theory, concerns the study of the joint law of

$$\tau_0^-, \; X_{\tau_0^-} \text{ and } X_{\tau_0^- -},$$

the time of ruin, the deficit at ruin and the wealth prior to ruin.

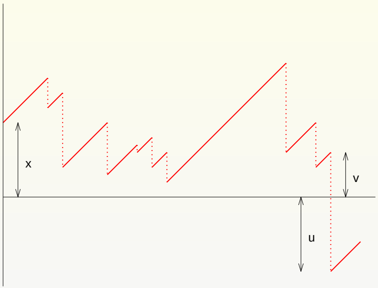
Ruin



- We are interested in (Gerber-Shiu penalty measure)

$$\mathbb{E}_x(\mathrm{e}^{-q\tau_0^-}; -X_{\tau_0^-} \in \mathrm{d}u,\, X_{\tau_0^- -} \in \mathrm{d}v).$$

Ruin



- We are interested in (Gerber-Shiu penalty measure)

$$\mathbb{E}_x(\mathrm{e}^{-q\tau_0^-}; -X_{\tau_0^-} \in \mathrm{d}u,\, X_{\tau_0^- -} \in \mathrm{d}v).$$

- More generally, one can pose the question for a general spectrally negative Lévy process.

The general solution

The general solution

- Work with Laplace exponent instead of characteristic exponent, $\theta \geq 0$

$$\mathbb{E}(\mathrm{e}^{\theta X_t}) = \mathrm{e}^{\psi(\theta)t},$$

where

$$\psi(\lambda) = -a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(0,\infty)} (\mathrm{e}^{-\lambda x} - 1 + \lambda x \mathbf{1}_{(x<1)})\nu(\mathrm{d}x),$$

$a \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a measure satisfying $\int_{(0,\infty)}(1 \wedge x^2)\nu(\mathrm{d}x) < \infty$.

The general solution

- Work with Laplace exponent instead of characteristic exponent, $\theta \geq 0$

$$\mathbb{E}(\mathrm{e}^{\theta X_t}) = \mathrm{e}^{\psi(\theta)t},$$

where

$$\psi(\lambda) = -a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(0,\infty)} (\mathrm{e}^{-\lambda x} - 1 + \lambda x \mathbf{1}_{(x<1)})\nu(\mathrm{d}x),$$

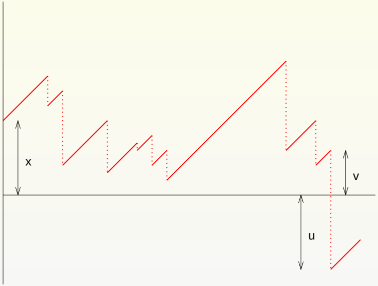
$a \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a measure satisfying $\int_{(0,\infty)}(1 \wedge x^2)\nu(\mathrm{d}x) < \infty$.

- Theorem [scale functions]: For each $q \geq 0$, there exists a continuous, non-decreasing function $W^{(q)} : [0,\infty) \to [0,\infty)$ satisfying

$$\int_0^\infty \mathrm{e}^{-\lambda x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\lambda) - q}$$

for $\lambda > \Phi(q) := \sup\{\theta : \psi(\theta) = q\}$.

The general solution



$$\mathbb{E}_x(\mathrm{e}^{-q\tau_0^-}; -X_{\tau_0^-} \in \mathrm{d}u,\, X_{\tau_0^- -} \in \mathrm{d}v) = \left\{e^{-\Phi(q)v}W^{(q)}(x) - W^{(q)}(x-y)\right\}\nu(v+\mathrm{d}u)$$

Reflection strategies

Reflection strategies

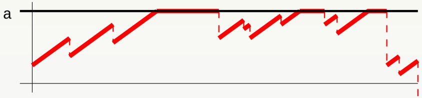
- Consider paying out dividends $\{L_t : t \geq 0\}$ from the insurance risk process $\{X_t : t \geq 0\}$.

Reflection strategies

- Consider paying out dividends $\{L_t : t \geq 0\}$ from the insurance risk process $\{X_t : t \geq 0\}$.
- A particular dividend strategy takes the form

$$L_t = a \vee (\sup_{s \leq t} X_s) - a$$

for some $a > 0$, a so-called reflection strategy.

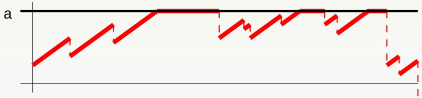


Reflection strategies

- Consider paying out dividends $\{L_t : t \geq 0\}$ from the insurance risk process $\{X_t : t \geq 0\}$.
- A particular dividend strategy takes the form

$$L_t = a \vee (\sup_{s \leq t} X_s) - a$$

for some $a > 0$, a so-called reflection strategy.

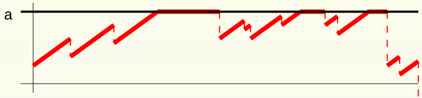


- Net present value of dividends paid until ruin

$$\mathbb{E}_x \left(\int_0^{\sigma^a} e^{-qt} \mathrm{d}L_t \right)$$

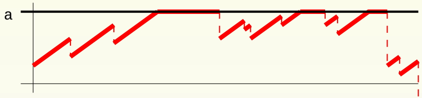
for $x, q \geq 0$, where $\sigma^a = \inf\{t > 0 : X_t - L_t < 0\}$.

Reflection strategies



[2]Question to the zealots: can fractional moments be computed explicitly?

Reflection strategies

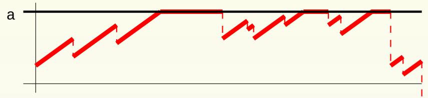


- $$I = \int_0^{\sigma^a} e^{-qt} \mathrm{d}L_t = \int_0^{\mathbf{e}_p} e^{-qL_s^{-1}} \mathrm{d}s$$

  where $\mathbf{e}_p$ is an independent and exponentially distributed random variable with some parameter $p > 0$.

[2]Question to the zealots: can fractional moments be computed explicitly?

Reflection strategies



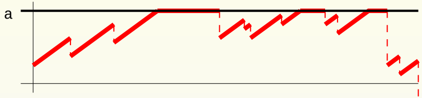
- $$I = \int_0^{\sigma^a} e^{-qt} \mathrm{d}L_t = \int_0^{\mathbf{e}_p} e^{-qL_s^{-1}} \mathrm{d}s$$

  where $\mathbf{e}_p$ is an independent and exponentially distributed random variable with some parameter $p > 0$.
- Remarkably all integer moments[2] of I can be computed under $\mathbb{P}_a$. Specifically

$$\mathbb{E}_a \left[\left(\int_0^{\sigma^a} e^{-qt} \mathrm{d}L_t \right)^n \right] = n! \prod_{k=1}^{n} \frac{W^{(kq)}(a)}{W^{(kq)\prime}(a)}$$

[2] Question to the zealots: can fractional moments be computed explicitly?

Reflection strategies



- $$I=\int_0^{\sigma^a} e^{-qt}\mathrm{d}L_t=\int_0^{\mathbf{e}_p} e^{-qL_s^{-1}}\mathrm{d}s$$
  where $\mathbf{e}_p$ is an independent and exponentially distributed random variable with some parameter $p>0$.
- Remarkably all integer moments[3] of I can be computed under $\mathbb{P}_a$. Specifically

$$\mathbb{E}_x\left[\left(\int_0^{\sigma^a} e^{-qt}\mathrm{d}L_t\right)^n\right]=n!\frac{W^{(qn)}(x)}{W^{(qn)}(a)}\prod_{k=1}^{n}\frac{W^{(kq)}(a)}{W^{(kq)\prime}(a)}$$

[3] Question to the zealots: can fractional moments be computed explicitly?

Refraction strategies

- Consider a strategy, L, which pays out dividends at a linear rate δ provided the aggregate process, $X - L$, exceeds a threshold $b > 0$.

Refraction strategies

- Consider a strategy, L, which pays out dividends at a linear rate δ provided the aggregate process, $X - L$, exceeds a threshold $b > 0$.
- Mathematically speaking, this can only be expressed through the stochastic differential equation

$$U_t = X_t - \int_0^t \delta \mathbf{1}_{(U_s > b)} \mathrm{d}s.$$

Refraction strategies

- Consider a strategy, L, which pays out dividends at a linear rate δ provided the aggregate process, $X - L$, exceeds a threshold $b > 0$.
- Mathematically speaking, this can only be expressed through the stochastic differential equation

$$U_t = X_t - \int_0^t \delta \mathbf{1}_{(U_s > b)} \mathrm{d}s.$$

- Surprisingly difficult to show that this SDE has a unique strong solution.

Refraction strategies

- Consider a strategy, L, which pays out dividends at a linear rate δ provided the aggregate process, $X - L$, exceeds a threshold $b > 0$.
- Mathematically speaking, this can only be expressed through the stochastic differential equation

$$U_t = X_t - \int_0^t \delta \mathbf{1}_{(U_s > b)} \mathrm{d}s.$$

- Surprisingly difficult to show that this SDE has a unique strong solution.
- Many desired quantities can again be computed through the use of scale functions.

Refraction strategies

- Consider a strategy, L, which pays out dividends at a linear rate δ provided the aggregate process, $X - L$, exceeds a threshold $b > 0$.
- Mathematically speaking, this can only be expressed through the stochastic differential equation

$$U_t = X_t - \int_0^t \delta \mathbf{1}_{(U_s > b)} \mathrm{d}s.$$

- Surprisingly difficult to show that this SDE has a unique strong solution.
- Many desired quantities can again be computed through the use of scale functions.
- Suppose that $\lim_{t\uparrow\infty} X_t = \infty$ and $\delta < \mathbb{E}(X_1)$ so that $\lim_{t\uparrow\infty} U_t = \infty$. Allowing for U to persist beyond ruin, we are interested in the total time spent without dividends being paid, $\int_0^\infty \mathbf{1}_{(U_t < b)} \mathrm{d}t$.

Refraction strategies

- Consider a strategy, L, which pays out dividends at a linear rate δ provided the aggregate process, $X - L$, exceeds a threshold $b > 0$.
- Mathematically speaking, this can only be expressed through the stochastic differential equation

$$U_t = X_t - \int_0^t \delta \mathbf{1}_{(U_s > b)} \mathrm{d}s.$$

- Surprisingly difficult to show that this SDE has a unique strong solution.
- Many desired quantities can again be computed through the use of scale functions.
- Suppose that $\lim_{t\uparrow\infty} X_t = \infty$ and $\delta < \mathbb{E}(X_1)$ so that $\lim_{t\uparrow\infty} U_t = \infty$. Allowing for U to persist beyond ruin, we are interested in the total time spent without dividends being paid, $\int_0^\infty \mathbf{1}_{(U_t < b)} \mathrm{d}t$.
- Starting at the barrier

$$\mathbb{E}_b \left[\exp\left\{ -q \int_0^\infty \mathbf{1}_{(U_t < b)} \mathrm{d}t \right\} \right] = \frac{(\mathbb{E}(X_1) - \delta)\Phi(q)}{q - \delta\Phi(q)}.$$

Refraction strategies

- Consider a strategy, L, which pays out dividends at a linear rate δ provided the aggregate process, $X - L$, exceeds a threshold $b > 0$.
- Mathematically speaking, this can only be expressed through the stochastic differential equation

$$U_t = X_t - \int_0^t \delta \mathbf{1}_{(U_s > b)} \mathrm{d}s.$$

- Surprisingly difficult to show that this SDE has a unique strong solution.
- Many desired quantities can again be computed through the use of scale functions.
- Suppose that $\lim_{t\uparrow\infty} X_t = \infty$ and $\delta < \mathbb{E}(X_1)$ so that $\lim_{t\uparrow\infty} U_t = \infty$. Allowing for U to persist beyond ruin, we are interested in the total time spent without dividends being paid, $\int_0^\infty \mathbf{1}_{(U_t < b)} \mathrm{d}t$.
- Starting at the barrier

$$\mathbb{E}_b\left[\exp\left\{-q\int_0^\infty \mathbf{1}_{(U_t < b)} \mathrm{d}t\right\}\right] = \frac{(\mathbb{E}(X_1) - \delta)\Phi(q)}{q - \delta\Phi(q)}.$$

- When $b = 0$, one minus this quantity gives a Parisian-type ruin probability.

Perturbation through Heavy and Light tax

Perturbation through Heavy and Light tax

- A cumulative tax is paid proportional to the maximum wealth seen to date by the insurance firm, leaving an aggregate

$$U_t = X_t - \gamma \sup_{s \le t} X_s \qquad t \geq 0,$$

for $\gamma \in (0, 1)$.

Perturbation through Heavy and Light tax

- A cumulative tax is paid proportional to the maximum wealth seen to date by the insurance firm, leaving an aggregate

$$U_t = X_t - \gamma \sup_{s \le t} X_s \qquad t \ge 0,$$

  for $\gamma \in (0,1)$.

- More generally, define $\overline{X}_t = \sup_{s \le t} X_s$ and

$$U_t = X_t - \int_{(0,t]} \gamma(\overline{X}_u) \mathrm{d}\overline{X}_u, \qquad t \ge 0.$$

  where $\gamma : \mathbb{R} \to [0, \infty)$.

Perturbation through Heavy and Light tax

- A cumulative tax is paid proportional to the maximum wealth seen to date by the insurance firm, leaving an aggregate

$$U_t = X_t - \gamma \sup_{s \le t} X_s \qquad t \ge 0,$$

for $\gamma \in (0, 1)$.

- More generally, define $\overline{X}_t = \sup_{s \le t} X_s$ and

$$U_t = X_t - \int_{(0,t]} \gamma(\overline{X}_u) \mathrm{d}\overline{X}_u, \qquad t \ge 0.$$

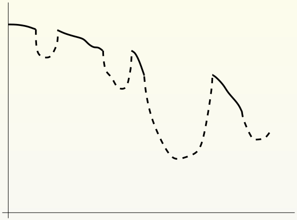
where $\gamma : \mathbb{R} \to [0, \infty)$.

- We consider two regimes

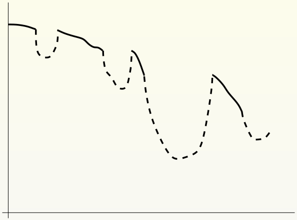
Heavy tax: $\gamma : \mathbb{R} \to (1, \infty)$

Light tax: $\gamma : \mathbb{R} \to [0, 1)$

Perturbation through Heavy and Light tax

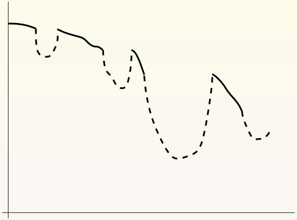


Perturbation through Heavy and Light tax



- $$U_t = X_t - \int_{(0,t]} \gamma(\overline{X}_u)\mathrm{d}\overline{X}_u = \int_{(0,t]} (1-\gamma(\overline{X}_u))\mathrm{d}\overline{X}_u + (X_t - \overline{X}_t)$$

Perturbation through Heavy and Light tax



- $$U_t = X_t - \int_{(0,t]} \gamma(\overline{X}_u)\mathrm{d}\overline{X}_u = \int_{(0,t]} (1-\gamma(\overline{X}_u))\mathrm{d}\overline{X}_u + (X_t - \overline{X}_t)$$

- This offers the following path decomposition: The process U, with $U_0 = x > 0$, follows the deterministic **and monotone** curve

$$\bar{\gamma}(s) = x + \int_x^s (1-\gamma(s))\mathrm{d}s, \qquad s \geq x$$

interlaced with excursions of X from its maximum $\overline{X}$.

Probability of Ruin

Probability of Ruin

- Consider the ruin time for the insurance risk process with tax,

$$T_0^- = \inf\{t > 0 : U_t < 0\}.$$

Probability of Ruin

- Consider the ruin time for the insurance risk process with tax,

$$T_0^- = \inf\{t > 0 : U_t < 0\}.$$

- **Theorem:** Take the light tax regime. Fix $x > 0$.

$$\mathbb{P}_x(T_0^- < \infty) = 1 - \exp\left(- \int_x^\infty \frac{W'(\bar{\gamma}(s))}{W(\bar{\gamma}(s))} \, \mathrm{d}s \right).$$

Here we write W in place of $W^{(0)}$ for convenience.

Probability of Ruin

- Consider the ruin time for the insurance risk process with tax,

$$T_0^- = \inf\{t > 0 : U_t < 0\}.$$

- **Theorem:** Take the light tax regime. Fix $x > 0$.

$$\mathbb{P}_x(T_0^- < \infty) = 1 - \exp\left(- \int_x^\infty \frac{W'(\bar{\gamma}(s))}{W(\bar{\gamma}(s))} \,\mathrm{d}s \right).$$

  Here we write W in place of $W^{(0)}$ for convenience.

- In the heavy tax regime it is (intuitively) trivial to deduce that

$$\mathbb{P}_x(T_0^- < \infty) = 1$$

Net present value of tax paid until ruin

Suppose that $U_0 = x$ and define

$$a^*(x) = \inf\{s \geq x : \bar{\gamma}(s) < 0\} \in (0, \infty].$$

Theorem: Take either light or heavy tax. For $q \geq 0$,

$$\mathbb{E}_x\left[\int_0^{T_0^-} \mathrm{e}^{-qu}\gamma(\overline{X}_u)\,\mathrm{d}\overline{X}_u\right] = \int_x^{a^*(x)} \exp\left(-\int_x^t \frac{W^{(q)\prime}(\bar{\gamma}(s))}{W^{(q)}(\bar{\gamma}(s))}\,\mathrm{d}s\right)\gamma(t)\,\mathrm{d}t.$$

Type II creeping for heavy tax

Type II creeping for heavy tax

- For a general Lévy process we say that it **creeps downwards** if for some (and hence for all) $x > 0$, $\mathbb{P}_x(X_{\tau_0^-} = 0) > 0$, where

$$\tau_0^- = \inf\{t > 0 : X_t < 0\}.$$

Type II creeping for heavy tax

- For a general Lévy process we say that it **creeps downwards** if for some (and hence for all) $x > 0$, $\mathbb{P}_x(X_{\tau_0^-} = 0) > 0$, where

$$\tau_0^- = \inf\{t > 0 : X_t < 0\}.$$

- Spectrally negative Lévy processes creep downwards if and only if a Gaussian component is present.

Type II creeping for heavy tax

- For a general Lévy process we say that it **creeps downwards** if for some (and hence for all) $x > 0$, $\mathbb{P}_x(X_{\tau_0^-} = 0) > 0$, where

$$\tau_0^- = \inf\{t > 0 : X_t < 0\}.$$

- Spectrally negative Lévy processes creep downwards if and only if a Gaussian component is present.
- When does the Lévy insurance risk process X with tax creep downwards?

Type II creeping for heavy tax

- For a general Lévy process we say that it **creeps downwards** if for some (and hence for all) $x > 0$, $\mathbb{P}_x(X_{\tau_0^-} = 0) > 0$, where

$$\tau_0^- = \inf\{t > 0 : X_t < 0\}.$$

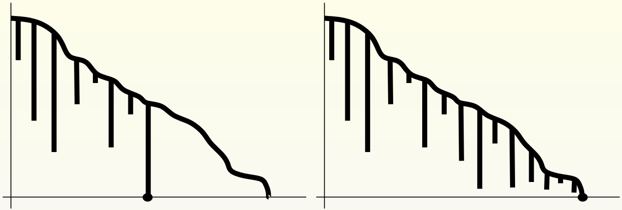
- Spectrally negative Lévy processes creep downwards if and only if a Gaussian component is present.
- When does the Lévy insurance risk process X with tax creep downwards?
- In the case of light tax, creeping occurs by the same mechanism as for a pure Lévy process during an excursion of U from the increasing curve $\bar{\gamma}$. Hence there is creeping if and only if a Gaussian component is present in X.

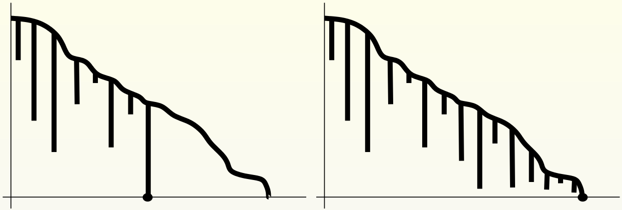
Type II creeping for heavy tax

- For a general Lévy process we say that it **creeps downwards** if for some (and hence for all) $x > 0$, $\mathbb{P}_x(X_{\tau_0^-} = 0) > 0$, where

$$\tau_0^- = \inf\{t > 0 : X_t < 0\}.$$

- Spectrally negative Lévy processes creep downwards if and only if a Gaussian component is present.
- When does the Lévy insurance risk process X with tax creep downwards?
- In the case of light tax, creeping occurs by the same mechanism as for a pure Lévy process during an excursion of U from the increasing curve $\bar{\gamma}$. Hence there is creeping if and only if a Gaussian component is present in X.
- In the case of heavy tax, creeping can occur during an excursion of U from $\bar{\gamma}$ (in which case a Gaussian component is needed), OR, if $\bar{\gamma}$ decreases sharply enough to the origin, then U can meet the origin continuously whilst moving along the curve $\bar{\gamma}$.



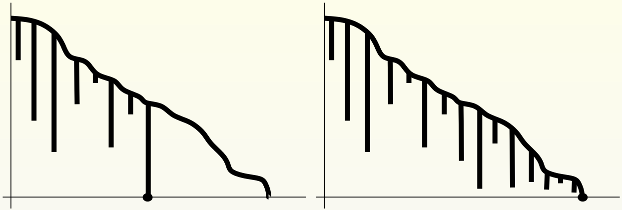


- **Theorem:** In the case of heavy tax, for $x > 0$, assume that

$$a^*(x) = \inf\{s \geq x : \bar{\gamma}(s) = 0\} < \infty.$$

Then

$$\mathbb{P}_x(\text{type II creeping at 0}) = \exp\left(- \int_x^{a^*(x)} \frac{W'(\bar{\gamma}(s))}{W(\bar{\gamma}(s))} \, \mathrm{d}s \right).$$



- **Theorem:** In the case of heavy tax, for $x > 0$, assume that

$$a^*(x) = \inf\{s \geq x : \bar{\gamma}(s) = 0\} < \infty.$$

Then

$$\mathbb{P}_x(\text{type II creeping at 0}) = \exp\left(- \int_x^{a^*(x)} \frac{W'(\bar{\gamma}(s))}{W(\bar{\gamma}(s))} \, \mathrm{d}s \right).$$

- **Corollary:** If we choose γ is continuous then $\mathbb{P}_x(\text{type II creeping at 0}) > 0$ if and only if X is a Lévy process with bounded variation paths.

Conclusion:

Gerber-Shiu theory is applied excursion theory