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Conditioned real self-similar Markov processes

Andreas E. Kyprianou[a,*], Víctor M. Rivero[b], Weerapat Satitkanitkul[a]

[a] Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, UK
[b] CIMAT A. C., Calle Jalisco s/n, Col. Valenciana, A. P. 402, C.P. 36000, Guanajuato, Gto., Mexico

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Abstract

In recent work, Chaumont et al. (2013) showed that is possible to condition a stable process with index $\alpha \in (1, 2)$ to avoid the origin. Specifically, they describe a new Markov process which is the Doob h-transform of a stable process and which arises from a limiting procedure in which the stable process is conditioned to have avoided the origin at later and later times. A stable process is a particular example of a real self-similar Markov process (rssMp) and we develop the idea of such conditionings further to the class of rssMp. Under appropriate conditions, we show that the specific case of conditioning to avoid the origin corresponds to a classical Cramér–Esscher-type transform to the Markov Additive Process (MAP) that underlies the Lamperti–Kiu representation of a rssMp. In the same spirit, we show that the notion of conditioning a rssMp to continuously absorb at the origin also fits the same mathematical framework. In particular, we characterise the stable process conditioned to continuously absorb at the origin when $\alpha \in (0, 1)$. Our results also complement related work for positive self-similar Markov processes in Chaumont and Rivero (2007).

1. Introduction

This work concerns conditionings of real self-similar Markov processes (rssMp) and so we start by characterising this class of stochastic processes.

* Corresponding author.
E-mail addresses: a.kyprianou@bath.ac.uk (A.E. Kyprianou), rivero@cimat.mx (V.M. Rivero), ws250@bath.ac.uk (W. Satitkanitkul).

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A rssMp with *index of self-similarity* $\alpha > 0$ is a standard Markov process $X = (X_t)_{t\geq 0}$ (in the sense of [6]) with probability laws $(\mathbb{P}_x)_{x\in\mathbb{R}}$ and filtration $(\mathcal{F}_t)_{t\geq 0}$, which satisfies the *scaling property* that for all $x \in \mathbb{R}\setminus\{0\}$ and $c > 0$,

$$\text{the law of } (cX_{tc^{-\alpha}})_{t\geq 0} \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{cx}.$$

In the language of the classical paper by Lamperti [24], where self-similar Markov processes were first analysed at depth, this corresponds to the class of semi-stable Markov process with order (or Hurst index) $1/\alpha$. The structure of real self-similar Markov processes has been investigated by [11] in the symmetric case, and [10] in general. Here, we give an interpretation of these authors' results in terms of Markov additive process (MAP) with a two-state modulating Markov chain and therefore we make an immediate digression to introduce such processes.

1.1. Markov additive processes

Let E be a finite state space and $(\mathcal{G}_t)_{t\geq 0}$ a standard filtration. A càdlàg process (ξ, J) in $\mathbb{R}\times E$ with law $\mathbf{P}$ is called a *Markov additive process (MAP)* with respect to $(\mathcal{G}_t)_{t\geq 0}$ if $(J(t))_{t\geq 0}$ is a continuous-time Markov chain in E, and the following property is satisfied, for any $i \in E$, $s, t \geq 0$:

$$\begin{aligned}&\text{given } \{J(t) = i\}, \text{ the pair } (\xi(t+s) - \xi(t), J(t+s)) \text{ is independent of } \mathcal{G}_t,\\ &\quad \text{and has the same distribution as } (\xi(s) - \xi(0), J(s)) \text{ given } \{J(0) = i\}.\end{aligned} \tag{1}$$

Aspects of the theory of Markov additive processes are covered in a number of texts, among them [5] and [4]. More classical work includes [12,13,3] amongst others. We will mainly use the notation of [15], where it was principally assumed that ξ is spectrally negative; the results which we quote are valid without this hypothesis, however.

Let us introduce some notation. For $x \in \mathbb{R}$, write $\mathbf{P}_{x,i} = \mathbf{P}(\cdot\,|\,\xi(0) = x, J(0) = i)$. If μ is a probability distribution on E, we write $\mathbf{P}_{x,\mu} = \sum_{i\in E}\mu_i\mathbf{P}_{x,i}$. We adopt a similar convention for expectations.

It is well-known that a Markov additive process (ξ, J) also satisfies (1) with t replaced by a stopping time, albeit on the event that the stopping time is finite. The following proposition gives a characterisation of MAPs in terms of a mixture of Lévy processes, a Markov chain and a family of additional jump distributions; see [4, §XI.2a] and [15, Proposition 2.5].

Proposition 1.1. *The pair (ξ, J) is a MAP (as described above) if and only if, J is a continuous-time Markov chain in E, for each $i, j \in E$, there exist a sequence of iid Lévy processes $(\xi_i^n)_{n\geq 0}$ and a sequence of iid random variables $(\Delta_{i,j}^n)_{n\geq 0}$, independent of the chain J, such that, if $\sigma_0 = 0$ and $(\sigma_n)_{n\geq 1}$ are the jump times of J, then the process ξ has the representation*

$$\xi(t) = \mathbf{1}_{(n>0)}(\xi(\sigma_n-) + \Delta^n_{J(\sigma_n-),J(\sigma_n)}) + \xi^n_{J(\sigma_n)}(t - \sigma_n), \qquad t \in [\sigma_n, \sigma_{n+1}),\ n \geq 0.$$

For each $i \in E$, it will be convenient to define ξ_i as a Lévy process whose distribution is the common law of the ξ_i^n processes in the above representation; and similarly, for each $i, j \in E$, define $\Delta_{i,j}$ to be a random variable having the common law of the $\Delta_{i,j}^n$ variables.

Henceforth, we confine ourselves to irreducible (and hence ergodic) Markov chains J. Let the state space E be the finite set $\{1, \ldots, N\}$, for some $N \in \mathbb{N}$. Denote the transition rate matrix of the chain J by $\boldsymbol{Q} = (q_{i,j})_{i,j\in E}$. For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i. To be more precise, for all $z \in \mathbb{C}$ for which it exists,

$$\psi(z) := \log \int_{\mathbb{R}} \mathrm{e}^{zx}\mathrm{P}(\xi(1) \in \mathrm{d}x).$$

For each pair of $i, j \in E$, define the Laplace transform $G_{i,j}(z) = \mathrm{E}[\mathrm{e}^{z\Delta_{i,j}}]$ of the jump distribution $\Delta_{i,j}$, whenever this exists. Write $\boldsymbol{G}(z)$ for the $N \times N$ matrix whose (i, j)th element is $G_{i,j}(z)$. We will adopt the convention that $\Delta_{i,j} = 0$ if $q_{i,j} = 0, i \neq j$, and also set $\Delta_{ii} = 0$ for each $i \in E$.

The multidimensional analogue of the Laplace exponent of a Lévy process is provided by the matrix-valued function

$$\boldsymbol{F}(z) = \mathrm{diag}(\psi_1(z), \ldots, \psi_N(z)) + \boldsymbol{Q} \circ \boldsymbol{G}(z), \tag{2}$$

for all $z \in \mathbb{C}$ such that the elements on the right are defined, where $\circ$ indicates elementwise multiplication, also called Hadamard multiplication. It is then known that

$$\mathbf{E}_{0,i}(\mathrm{e}^{z\xi(t)}; J(t) = j) = \left(\mathrm{e}^{\boldsymbol{F}(z)t}\right)_{i,j}, \qquad i, j \in E, t \geq 0,$$

such that the right-hand side of the equality is defined. For this reason, $\boldsymbol{F}$ is called the *matrix exponent* of the MAP (ξ, J). Note, using standard convexity properties of regular Laplace transforms, if we can guarantee, for $a, b \in \mathbb{R}$ with $a < b$, that $\boldsymbol{F}(a), \boldsymbol{F}(b)$ are defined and finite (element wise), then, $\boldsymbol{F}(z)$ is well defined and finite (element wise) for $\mathrm{Re}(z) \in (a, b)$.

The role of $\boldsymbol{F}$ is analogous to the role of the Laplace exponent of a Lévy process. Similarly in this respect, one might also regard the *leading eigenvalue* associated to $\boldsymbol{F}$ (sometimes referred to as the *Perron–Frobenius eigenvalue*, see [4, §XI.2c] and [15, Proposition 2.12]) as also playing this role.

Proposition 1.2. *Suppose that $z \in \mathbb{R}$ is such that $\boldsymbol{F}(z)$ is defined. Then, the matrix $\boldsymbol{F}(z)$ has a real simple eigenvalue $\chi(z)$, which is larger than the real part of all its other eigenvalues. Furthermore, the corresponding right-eigenvector $\boldsymbol{v} = (v_1(z), \ldots, v_N(z))$ may be chosen so that $v_i(z) > 0$ for every $i = 1, \ldots, N$, and normalised such that*

$$\boldsymbol{\pi} \cdot \boldsymbol{v}(z) = 1 \tag{3}$$

where $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_N)$ is the equilibrium distribution of the chain J.

One sees the leading eigenvalue appearing in a number of key results. We give two such below that will be of pertinence later on. The first one is the strong law of large numbers for (ξ, J), in which the leading eigenvalue plays the same role as the Laplace exponent of a Lévy process does in analogous result for that setting. The following result is taken from [4, Proposition 2.10].

Proposition 1.3. *If $\chi'(0)$ is well defined (either as a left or right derivative), then we have*

$$\lim_{t\to\infty} \frac{\xi(t)}{t} = \chi'(0) = \mathbf{E}_{0,\boldsymbol{\pi}}[\xi(1)] := \sum_{i\in E} \boldsymbol{\pi}_i \mathbf{E}_{0,i}[\xi(1)] \tag{4}$$

almost surely. In that case, there is a trichotomy which dictates whether $\lim_{t\to\infty}\xi(t) = \infty$ almost surely, $\lim_{t\to\infty}\xi(t) = -\infty$ almost surely or $\limsup_{t\to\infty}\xi(t) = -\liminf_{t\to\infty}\xi(t) = \infty$ accordingly as $\chi'(0) > 0$, < 0 or $= 0$, respectively.

The leading eigenvalue also features in the following probabilistic result, which identifies a martingale (also known as the generalised Wald martingale) and associated exponential change of measure corresponding to an Esscher-type transformation of a Lévy process; cf. [4, Proposition XI.2.4, Theorem XIII.8.1].

Proposition 1.4. *Let $\mathcal{G}_t = \sigma\{(\xi(s), J(s)) : s \leq t\}$, $t \geq 0$, and*

$$M(t,\gamma) = \mathrm{e}^{\gamma(\xi(t)-\xi(0))-\chi(\gamma)t}\frac{v_{J(t)}(\gamma)}{v_{J(0)}(\gamma)}, \qquad t \geq 0, \tag{5}$$

for some γ such that $\chi(\gamma)$ is defined. Then, $M(\cdot,\gamma)$ is a unit-mean martingale with respect to $(\mathcal{G}_t)_{t\geq 0}$. Moreover, under the change of measure

$$\left.\frac{\mathrm{d}\mathbf{P}^{\gamma}_{x,i}}{\mathrm{d}\mathbf{P}_{x,i}}\right|_{\mathcal{G}_t} = M(t,\gamma), \qquad t \geq 0,$$

the process (ξ, J) remains in the class of MAPs and, where defined, its characteristic exponent is given by

$$\boldsymbol{F}_{\gamma}(z) = \boldsymbol{\Delta}_{\boldsymbol{v}}(\gamma)^{-1}\boldsymbol{F}(z+\gamma)\boldsymbol{\Delta}_{\boldsymbol{v}}(\gamma) - \chi(\gamma)\mathbf{I}, \tag{6}$$

where $\mathbf{I}$ is the identity matrix and $\boldsymbol{\Delta}_{\boldsymbol{v}}(\gamma) = \mathrm{diag}(\boldsymbol{v}(\gamma))$. It is straightforward to deduce that, when it exists, the associated leading eigenvalue associated to $\boldsymbol{F}_{\gamma}(z)$ is given by $\chi_{\gamma}(z) = \chi(z+\gamma) - \chi(\gamma)$.

The following properties of χ, lifted from [19, Proposition 3.4], will also prove useful in relating the last two results together.

Proposition 1.5. *Suppose that $\boldsymbol{F}$ is defined in some open interval D of $\mathbb{R}$. Then, the leading eigenvalue χ of $\boldsymbol{F}$ is smooth and convex on D.*

On account of the fact that $\boldsymbol{F}(0) = \boldsymbol{Q}$, it is easy to see that we always have $\chi(0) = 0$. If we assume that there exists $\theta \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{F}$ is defined on $D = \{t\theta : t \in (0,1)\}$ with

$$\chi(\theta) = 0, \tag{7}$$

then by the proposition above, we can conclude that χ is defined and convex on the interval D. Henceforth denoted by θ and referred to as the *Cramér number*.

If $\theta > 0$, then $\chi'(0+)$ is well defined and convexity dictates that it must be negative. In that case $\lim_{t\to\infty}\xi(t) = -\infty$ almost surely. Moreover, if we take $\gamma = \theta$ in Proposition 1.4, then, as $\chi'_{\theta}(0-) = \chi'(\theta-) > 0$, under the associated change of measure, $\lim_{t\to\infty}\xi(t) = \infty$ almost surely.

Conversely, if $\theta < 0$, then $\chi'(0-)$ is well defined and convexity dictates that it must be positive. In that case $\lim_{t\to\infty}\xi(t) = \infty$ almost surely. Again, if we take $\gamma = \theta$ in Proposition 1.4, then $\chi'_{\theta}(0+) = \chi'(\theta+) < 0$, under the associated change of measure, $\lim_{t\to\infty}\xi(t) = -\infty$ almost surely. In both cases, the change of measure (5) using $\gamma = \theta$ exchanges the long-term drift of the underlying MAP from $\pm\infty$ to $\mp\infty$.

1.2. Real self-similar Markov processes

In [10] the authors confine their attention to rssMp in 'class **C.4**'. A rssMp X is in **C.4** if, for all $x \neq 0$, $\mathbb{P}_x(\exists t > 0 : X_t X_{t-} < 0) = 1$; that is, with probability one, the process X changes sign infinitely often. The reason behind this is to ensure that the chain J in the Lamperti–Kiu representation is recurrent. Otherwise, suppose $\{+1\}$ is an absorbing state, $(X, \mathbb{P}_x)$ can be considered as a positive self-similar Markov process once it crosses to a positive value. In particular, if it starts with a negative value X will cross the origin once and remain positive. If it starts with a positive value, it will remain positive. Henceforth we will rename the class **C.4** as the class of *infinite crossing rssMp*.

Such a process may be identified with a MAP via a deformation of space and time which we call the *Lamperti–Kiu representation* of X. The following result is a simple consequence of [10, Theorem 6]. In it, we will use the notation

$$\tau^{\{0\}} = \inf\{t \geq 0 : X_t = 0\},$$

for the time to absorption at the origin.

Proposition 1.6. *Let X be an infinite crossing rssMp and fix $x \neq 0$. Then there exists a time-change σ, adapted to the filtration of X, such that, under the law $\mathbb{P}_x$, the process*

$$(\xi(t), J(t)) = (\log|X_{\sigma(t)}|, \operatorname{sign}(X_{\sigma(t)})), \qquad t \geq 0,$$

is a MAP with state space $E = \{-1, 1\}$ under the law $\mathbf{P}_{\log|x|,\operatorname{sign}(x)}$. Furthermore, the process X under $\mathbb{P}_x$ has the representation

$$X_t = J(\varphi(t))\mathrm{e}^{\xi(\varphi(t))}, \qquad 0 \leq t < \tau^{\{0\}},$$

where φ is the inverse of the time-change σ, and may be given by

$$\varphi(t) = \inf\left\{s > 0 : \int_0^s \exp(\alpha\xi(u))\,\mathrm{d}u > t\right\}, \qquad t < \tau^{\{0\}}. \tag{8}$$

In short, up to an endogenous time change, a rssMp has a polar decomposition in which $\exp\{\xi\}$ describes the radial distance from the origin and J describes its orientation (positive or negative).

To make the connection with the previous subsection, let us understand how the existence of a Cramér number for the underlying MAP to a rssMp affects path behaviour of the latter. Revisiting the discussion at the end of the previous subsection, we see that if $\theta > 0$ then $\lim_{t\to\infty}\xi(t) = -\infty$. In that case, we deduce from the strong law of large numbers for ξ and the Lamperti–Kiu transform, that

$$\tau^{\{0\}} = \int_0^\infty \mathrm{e}^{\alpha\xi(t)}\mathrm{d}t < \infty \qquad \text{and} \qquad X_{\tau^{\{0\}}-} = 0$$

almost surely (irrespective of the point of issue of X). Said another way, the rssMp will be continuously absorbed in the origin after an almost surely finite time. Moreover, this implies that $\varphi(t) < \infty$ if and only if $t < \int_0^\infty \mathrm{e}^{\alpha\xi(s)}\mathrm{d}s$. In the case that there is a Cramér number which satisfies $\theta < 0$, then, again referring to the limiting behaviour of ξ and the Lamperti–Kiu transform, we have

$$\tau^{\{0\}} = \int_0^\infty \mathrm{e}^{\alpha\xi(t)}\mathrm{d}t = \infty \tag{9}$$

almost surely (irrespective of the point of issue of X). Hence, the associated rssMp never touches the origin. Moreover, $\varphi(t) < \infty$ for all $t \geq 0$.

We can also reinterpret Proposition 1.4 in light of the Lamperti–Kiu representation and the fact that the quantity $\varphi(t)$ in (8) is also a stopping time, as well as the fact that $(\mathcal{F}_t)_{t\geq 0} = (\mathcal{G}_{\varphi(t)})_{t\geq 0}$. Theorem III.3.4 of [16] states that a martingale change of measure remains valid at a given stopping time, providing one restricts measurement to the set that the stopping time is finite. Accordingly we have that when $\theta > 0$, respectively $\theta < 0$,

$$M(\varphi(t), \theta) = \frac{v_{J(\varphi(t))}(\theta)}{v_{\operatorname{sign}(x)}(\theta)}\mathrm{e}^{\theta(\xi(\varphi(t))-\log|x|)}\mathbf{1}_{(\varphi(t)<\infty)} = \frac{v_{\operatorname{sign}(X_t)}(\theta)}{v_{\operatorname{sign}(x)}(\theta)}\frac{|X_t|^\theta}{|x|^\theta}\mathbf{1}_{(t<\tau^{\{0\}})}, \qquad t \geq 0 \tag{10}$$

is a $\mathbb{P}_x$-martingale, respectively, a $\mathbb{P}_x$-supermartingale.

2. Main results

Throughout the remainder of the paper we make following assumption.

(A) The process X is a rssMp whose underlying MAP does not have lattice support and has a leading eigenvalue χ with Cramér number $\theta \neq 0$ such that $\chi'(\theta)$ exists in $\mathbb{R}$.

Under this assumption, our objective is to construct conditioned versions of X. When $\theta > 0$, through a limiting procedure, we will build the process X conditioned to avoid the origin. Similarly when $\theta < 0$, we will build the process X conditioned to continuously absorb at the origin. Accordingly, in both cases, we shall show the existence of a harmonic function for the process X which is used to make a Doob h-transform in the representation of the conditioned processes.

In this respect, our work is reminiscent of density transforms which have been considered in the setting of positive self-similar Markov processes (pssMp); see [27]. In that case, the density transform plays a crucial role in the construction of an entrance law or recurrent extension from 0. Similar ideas appear in [25] when constructing a Bessel-3 process from a Brownian motion killed upon hitting 0.

Theorem 2.1. *Suppose that X is a rssMp under assumption (A) and $\mathcal{F}_t := \sigma(X_s : s \le t)$, $t \ge 0$ is its natural filtration. Define*

$$h_\theta(x) := v_{\mathrm{sign}(x)}(\theta)|x|^\theta, \qquad x \in \mathbb{R},$$

and, for Borel set D, let $\tau^D := \inf\{s \ge 0 : X_s \in D\}$.

(a) *If $\theta > 0$, then*

$$\mathbb{P}^\circ_x(A) := \mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)}\mathbf{1}_{(A,\, t<\tau^{\{0\}})}\right], \tag{11}$$

for $t > 0$, $x \neq 0$ and $A \in \mathcal{F}_t$, defines a probability measure on the canonical space of X such that $(X, \mathbb{P}^\circ_x)$, $x \in \mathbb{R} \setminus \{0\}$, is a rssMp. Moreover, for all $A \in \mathcal{F}_t$,

$$\mathbb{P}^\circ_x(A) = \lim_{a\to\infty} \mathbb{P}_x(A \cap \{t < \tau^{(-a,a)^c}\} \mid \tau^{(-a,a)^c} < \tau^{\{0\}}). \tag{12}$$

(b) *If $\theta < 0$, then,*

$$\mathbb{P}^\circ_x(A,\, t < \tau^{\{0\}}) := \mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)}\mathbf{1}_A\right],$$

for all $t > 0$, $x \neq 0$ and $A \in \mathcal{F}_t$, defines a probability measure on the canonical space of X with cemetery state at 0 such that $(X, \mathbb{P}^\circ_x)$, $x \in \mathbb{R} \setminus \{0\}$, is a rssMp. Moreover, for all $t > 0$ and $A \in \mathcal{F}_t$

$$\mathbb{P}^\circ_x(A,\, t < \tau^{\{0\}}) = \lim_{a\to 0} \mathbb{P}_x(A \cap \{t < \tau^{(-a,a)}\} \mid \tau^{(-a,a)} < \infty). \tag{13}$$

In case (a) of the above theorem, as $\theta > 0$, the Doob h-transform rewards paths that drift far from the origin. Indeed the limiting procedure (12) conditions the paths of the rssMp to explore further and further distances from the origin before being absorbed at the origin. In this sense, we refer to the process described in part (a) as the rssMp *conditioned to avoid the origin*. In case (b) of the theorem above, the Doob h-transform rewards paths that stay close to the origin. Moreover, the limiting procedure (13) conditions the paths of the rssMp to ultimately visit smaller and smaller balls centred around the origin. We therefore refer to the process described in part (b) as the rssMp *conditioned to absorb continuously at the origin*.

The above theorem constructs the conditioned processes via limiting spatial requirements. For the case of conditioning to avoid the origin, we can give a second sense in which the Doob h-transform emerges as the result of a conditioning procedure. The latter is done by conditioning the first visit to the origin to occur later and later in time.

Theorem 2.2. *Suppose that X is a rssMp under assumption (A) and $\theta > 0$. Then for $x \in \mathbb{R}\setminus\{0\}$ $t > 0$, and $A \in \mathcal{F}_t$, we have*

$$\mathbb{P}^\circ_x(A) = \lim_{s\to\infty} \mathbb{P}(A \mid \tau^{\{0\}} > t + s), \tag{14}$$

where $\mathbb{P}^\circ_x$, $x \in \mathbb{R}\setminus\{0\}$, is given by (11).

In order to approach the asymptotic conditioning in Theorem 2.2, we need to understand the tail behaviour of the probabilities $\mathbb{P}_x(\tau^{\{0\}} > t)$, as $t \to \infty$, for all $x \neq 0$. Indeed, the Markov property tells us that, for any $t \geq 0$, $A \in \mathcal{F}_t$, and $x \in \mathbb{R}\setminus\{0\}$, we have

$$\lim_{s\to\infty} \mathbb{P}_x(A \mid \tau^{\{0\}} > t + s) = \lim_{s\to\infty} \mathbb{E}_x\left[\mathbf{1}(A,\ t < \tau^{\{0\}})\frac{\mathbb{P}_{X_t}(\tau^{\{0\}} > s)}{\mathbb{P}_x(\tau^{\{0\}} > t + s)}\right]. \tag{15}$$

We are thus compelled to consider the asymptotic behaviour of $\mathbb{P}_x(\tau^{\{0\}} > t)$ as $t \to \infty$. To that end, we recall that by the Lamperti–Kiu representation of X, under $\mathbb{P}_x$, $x \neq 0$, we can identify $\tau^{\{0\}}$ in terms of the MAP (ξ, J), under $\mathbf{P}_{0,\mathrm{sgn}(x)}$, via the relation

$$\tau^{\{0\}} = |x|^\alpha \int_0^\infty \mathrm{e}^{\alpha\xi(s)}\mathrm{d}s. \tag{16}$$

More precisely, the dependency of the law of $\tau^{\{0\}}$ on $|x|$, where x is the point of issue, can be seen directly in (16) through a simple multiplicative factor of $|x|^\alpha$. Hence, we should determine the asymptotic behaviour of the right tail distribution of the exponential functional of ξ, $I := \int_0^\infty \mathrm{e}^{\alpha\xi(s)}\mathrm{d}s$. In particular, we will prove the following result, which is more general than needed and of intrinsic interest.

Theorem 2.3. *Let E be a finite state space and (ξ, J) a MAP with values in $\mathbb{R}\times E$. Assume that (ξ, J) does not have lattice support and has a leading eigenvalue χ with Cramér number $\theta > 0$ such that $\chi'(\theta)$ exists in $\mathbb{R}$. Define.*

$$I = \int_0^\infty \mathrm{e}^{\alpha\xi(s)}\mathrm{d}s.$$

We have that $\mathbf{E}_{0,k}[I^{\theta/\alpha-1}] < \infty$, $k \in E$, and

$$\mathbf{P}_{0,k}(I > t) \sim v_k(\theta)t^{-\theta/\alpha}\sum_{j\in E}\pi^\theta_j\frac{\mathbf{E}_{0,j}[I^{\theta/\alpha-1}]}{\mu_\theta|\alpha-\theta|v_j(\theta)},\quad \text{as } t\to\infty,$$

where $\mu_\theta = \sum_{j\in E}\pi^\theta_j\mathbf{E}^\theta_{0,j}[\xi(1)]$ and $\boldsymbol{\pi}^\theta = (\pi^\theta_j,\ j \in E)$ is the stationary distribution of J under $\mathbf{P}^\theta_{x,i}$, $x \in \mathbb{R}$, $i \in E$.

The above result specialised to the setting in Theorem 2.2 gives that

$$\mathbb{P}_x(\tau^{\{0\}} > t) \sim v_{\mathrm{sign}(x)}(\theta)|x|^\theta t^{-\theta/\alpha}\sum_{j=\pm1}\pi^\theta_j\frac{\mathbf{E}_{0,j}[I^{\theta/\alpha-1}]}{\mu_\theta|\alpha-\theta|v_j(\theta)},\quad \text{as } t\to\infty.$$

This fact, together with the argument in (15) easily leads to the proof of Theorem 2.2. Indeed, to end the proof one should justify that it is possible to pass the limit through the expectation on the right-hand side of (15). This is done reasoning as in the proof of Theorem 2.1.

3. Remarks

We have a number of remarks pertaining to the suite of results in the previous section.

Unconditioning: It is natural to ask what happens if one takes a self-similar Markov process conditioned e.g. to continuously absorb at the origin, and condition it to avoid the origin. Does this reverse the effect of the original conditioning?

Suppose that X under $\mathbb{P}_x$, $x \neq 0$, is a self-similar process satisfying (A), with underlying MAP (ξ, J) with probabilities $\mathbf{P}_{x,i}$, $x \in \mathbb{R}$, $i \in \{-1, 1\}$. Now consider the matrix exponent of (ξ, J) under $\mathbf{P}^{\theta}_{x,i}$, $x \in \mathbb{R}$, $i \in \{-1, 1\}$, is given by

$$\boldsymbol{F}_\theta(z) = \boldsymbol{\Delta}_{\boldsymbol{v}}(\theta)^{-1}\boldsymbol{F}(z+\theta)\boldsymbol{\Delta}_{\boldsymbol{v}}(\theta).$$

This has leading eigenvector $\boldsymbol{\Delta}_{\boldsymbol{v}}(\theta)^{-1}\boldsymbol{v}(z+\theta)$ with eigenvalue $\chi_\theta(z) = \chi(z+\theta)$. Because we have assumed (A), it also follows that $\chi_\theta(-\theta) = 0$. To show that the MAP (ξ, J) under $\mathbf{P}^{\theta}_{x,i}$, $x \in \mathbb{R}$, $i \in \{-1, 1\}$, satisfies assumption (A), we need a further assumption that $\chi'_\theta(-\theta) = \chi'(0)$ exists and takes a finite value. In that case, if e.g. $\theta > 0$, then necessarily $\chi'(0) < 0$ and if we condition $(X, \mathbb{P}^{\circ}_x)$, $x \neq 0$, to be continuously absorbed at 0, then from Theorem 2.1(b), the resulting MAP representing X can be identified via the Doob h-transform. For $t \geq 0$ and $A \in \mathcal{F}_t$,

$$\mathbb{P}^{\circ\circ}_x(A,\ t < \tau^{\{0\}}) := \mathbb{E}^{\circ}_x\left[\frac{h^{\circ}_\theta(X_t)}{h^{\circ}_\theta(x)}\mathbf{1}_A\right],$$

with

$$h^{\circ}_\theta(x) := \frac{v_{\mathrm{sign(x)}}(0)}{v_{\mathrm{sign}(x)}(\theta)}|x|^{-\theta} = \frac{1}{h_\theta(x)}, \qquad x \in \mathbb{R},$$

where the second equality holds because $\boldsymbol{F}(0) = \boldsymbol{Q}$ and hence $v_1(0) = v_{1-}(0) = 1$. As a consequence, we see that, changing measure in the style of part (b) of Theorem 2.1 after a change of measure in the style of part (a), we get

$$\begin{aligned}
\mathbb{P}^{\circ\circ}_x(A,\ t < \tau^{\{0\}}) &:= \mathbb{E}^{\circ}_x\left[\frac{h^{\circ}_\theta(X_t)}{h^{\circ}_\theta(x)}\mathbf{1}_A\right] \\
&= \mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)}\frac{h^{\circ}_\theta(X_t)}{h^{\circ}_\theta(x)}\mathbf{1}_{(A,\,t<\tau^{\{0\}})}\right] \\
&= \mathbb{P}_x(A,\ t < \tau^{\{0\}}), \qquad A \in \mathcal{F}_t, t \geq 0.
\end{aligned}$$

That is to say that the resulting process agrees with the process $(X, \mathbb{P}_x)$, $x \neq 0$, up to first hitting of the origin.

This is also apparent when we consider that the effect on the Esscher transform of the underlying MAP clearly reverses the effect of the initial conditioning. Indeed,

$$\boldsymbol{\Delta}_{\boldsymbol{v}}^{-1}(-\theta+\theta)\boldsymbol{\Delta}_{\boldsymbol{v}}(\theta)\boldsymbol{F}_\theta(z-\theta)\boldsymbol{\Delta}_{\boldsymbol{v}}(\theta)^{-1}\boldsymbol{\Delta}_{\boldsymbol{v}}(-\theta+\theta) = \boldsymbol{F}(z).$$

Similar calculations show the same reversal if we had assumed $\theta < 0$.

Degenerate MAPs: We deliberately excluded the setting that $\{-1, 1\}$ is irreducible for the underlying MAP. In many cases, aside from, at most, a single crossing of the origin, we note the conditionings considered here reduce to known conditionings of Lévy processes. In particular, these are the cases of conditioning a Lévy process to stay positive, cf, [8,9], conditioning a Lévy process to continuously absorb at the origin, cf. [8] and conditioning a subordinator to stay in a strip, [23].

Stable processes: The central family of examples which fits the setting of the two main theorems above is that of a (strictly) stable process with index $\alpha \in (0, 2)$, which is killed on first hitting the origin. Recall that the latter processes are those rssMp which do not have continuous paths and which are also in the class of Lévy processes. As a Lévy process, a stable process has characteristic exponent $\Psi(\theta) := -t^{-1}\log \mathbb{E}_0[\mathrm{e}^{\mathrm{i}\theta X_t}]$, $\theta \in \mathbb{R}$, $t > 0$, given by

$$\Psi(\theta) = |\theta|^{\alpha}(\mathrm{e}^{\pi\mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta>0)} + \mathrm{e}^{-\pi\mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta<0)}), \qquad \theta \in \mathbb{R},$$

where $\rho := \mathbb{P}_0(X_1 > 0)$. For convenience, we assume throughout this section that $\alpha\rho \in (0, 1)$, which is to say that the stable process has path with discontinuities of both signs.

For such processes, the matrix exponent of the underlying MAP in the Lamperti–Kiu representation has been computed in [19], with the help of computations in [10], and takes the form

$$\boldsymbol{F}(z) = \begin{bmatrix} -\dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho}-z)\Gamma(1-\alpha\hat{\rho}+z)} & \dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\[2ex] \dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & -\dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\rho-z)\Gamma(1-\alpha\rho+z)} \end{bmatrix}, \tag{17}$$

for $\mathrm{Re}(z) \in (-1, \alpha)$, where $\hat{\rho} = 1 - \rho$. Note, the domain $(-1, \alpha)$ is a specific to the stable process and will not be the case for all rssMps.

A straightforward computation shows that, for $\mathrm{Re}(z) \in (-1, \alpha)$,

$$\det \boldsymbol{F}(z) = \frac{\Gamma(\alpha-z)^2\Gamma(1+z)^2}{\pi^2}\left\{\sin(\pi(\alpha\rho-z))\sin(\pi(\alpha\hat{\rho}-z)) - \sin(\pi\alpha\rho)\sin(\pi\alpha\hat{\rho})\right\},$$

which has a root at $z = \alpha - 1$. In turn, this implies that $\chi(\alpha-1) = 0$. One also easily checks with the help of the reflection formula for gamma functions that

$$\boldsymbol{v}(\alpha-1) \propto \begin{bmatrix} \sin(\pi\alpha\hat{\rho}) \\ \sin(\pi\alpha\rho) \end{bmatrix}.$$

In that case, we see that Theorems 2.1 and 2.2 justify the claim that the family of measures $(\mathbb{P}^{\circ}_x, x \in \mathbb{R})$ defined via the relation

$$\left.\frac{\mathrm{d}\mathbb{P}^{\circ}_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} := \frac{\sin(\pi\alpha\hat{\rho})\mathbf{1}_{(X_t>0)} + \sin(\pi\alpha\rho)\mathbf{1}_{(X_t<0)}}{\sin(\pi\alpha\hat{\rho})\mathbf{1}_{(x>0)} + \sin(\pi\alpha\rho)\mathbf{1}_{(x<0)}}\left|\frac{X_t}{x}\right|^{\alpha-1}\mathbf{1}_{(t<\tau^{\{0\}})}, \qquad t \geq 0,$$

is the Doob h-transform corresponding to the stable process conditioned to avoid the origin when $\alpha \in (1, 2)$, and the stable process conditioned to be continuously absorbed at the origin when $\alpha \in (0, 1)$. The former of these two conditionings has already been observed in [10], the latter is a new observation. Note that, when $\theta = \alpha - 1 = 0$, the Doob h-transform corresponds to no change of measure at all, as the density is equal to unity and $\tau^{\{0\}} = \infty$ almost surely under $\mathbb{P}_x$, $x \in \mathbb{R}$. This is precisely the case of a Cauchy process. It is less clear in this case how to condition it to hit the origin.

One may prove Theorems 2.1 and 2.2 for stable processes by appealing to a direct form of reasoning using Bayes formula, scaling, dominated convergence using the fact that $\mathbb{E}_x[|X_t|^{\alpha-\varepsilon}] < \infty$, $x \in \mathbb{R}$, $t > 0$, $0 < \varepsilon < \alpha$, and the representation of the probabilities:

$$\mathbb{P}_x(\tau^{(-1,1)^c} < \tau^{\{0\}}) = (\alpha-1)x^{\alpha-1}\int_1^{1/x}(t-1)^{\alpha\rho-1}(t+1)^{\alpha\hat{\rho}-1}\mathrm{d}t, \qquad x \in (0,1)$$

for $\alpha \in (1,2)$, and

$$\mathbb{P}_x(\tau^{(-1,1)} < \infty) = \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha)}\int_{\frac{x-1}{x+1}}^{1} t^{\alpha\hat{\rho}-1}(1-t)^{-\alpha}\,\mathrm{d}t, \qquad x > 1$$

for $\alpha \in (0,1)$. The first of these probabilities is taken from Corollary 1 of [26] and the second from Corollary 1.2 of [22].

For the general case, no such detailed formulae are available and a different approach is needed. The main point of interest is in understanding the asymptotic probabilities of the conditioning event in Theorems 2.1 and 2.2 by appealing to a Cramér-type result for the decay of the probabilities $\mathbb{P}_x(\tau^{(-a,a)} < \infty)$ and $\mathbb{P}_x(\tau^{(-a,a)^c} < \infty)$ as $a \to \infty$.

Interpreting the Riesz–Bogdan–Żak transform: An additional point of interest in the case of stable processes pertains to the setting of the so-called Riesz–Bogdan–Żak transform, which was first proved in [7] for isotropic stable processes and [20] for anisotropic stable processes; see also [21]. The understanding of $\mathbb{P}^\circ_x$, $x \in \mathbb{R}\setminus\{0\}$ as a conditioning, gives context to the transformation which states that transforming the range of a stable process through the mapping $-1/x$, and then making an additional change of time, results in a new process which is the Doob h-transform of the stable process. We now see that the latter is nothing more than one of the two conditionings discussed in Theorem 2.1.

Theorem 3.1 (*Riesz–Bogdan–Żak Transform*)**.** *Suppose that X is a stable process with $\alpha \in (0,2)$ satisfying $\alpha\rho \in (0,1)$. Define*

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha}\mathrm{d}u > t\}, \qquad t \geq 0.$$

Then, for all $x \in \mathbb{R}\setminus\{0\}$, $(-1/X_{\eta(t)})_{t\geq 0}$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}^\circ_{-1/x})$. Moreover, the process $(X, \mathbb{P}^\circ_x)$, $x \in \mathbb{R}\setminus\{0\}$ is a self-similar Markov process with underlying MAP via The Lamperti–Kiu representation whose Matrix exponent satisfies, for $\mathrm{Re}(z) \in (-\alpha, 1)$,

$$\boldsymbol{F}^\circ(z) = \begin{bmatrix} -\dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\rho-z)\Gamma(\alpha\rho+z)} & \dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} \\[2ex] \dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} & -\dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\hat{\rho}-z)\Gamma(\alpha\hat{\rho}+z)} \end{bmatrix}. \tag{18}$$

4. Cramér-type results for MAPs and the proof of Theorem 2.1

Appealing to the Lamperti–Kiu process, we note that, for $|x| < a$

$$\mathbb{P}_x(\tau^{(-a,a)^c} < \tau^{\{0\}}) = \mathbf{P}_{\log|x|,\mathrm{sign}(x)}(T^+_{\log a} < \infty) = \mathbf{P}_{0,\mathrm{sign}(x)}(T^+_{\log(a/|x|)} < \infty)$$

where $T^+_y = \inf\{t > 0 : \xi(t) > y\}$. A similar result may be written for $\mathbb{P}_x(\tau^{(-a,a)} < \infty)$, albeit using $T^-_y := \inf\{t > 0 : \xi(t) < y\}$. This suggests that the asymptotic behaviour of the two probabilities of interest can be studied through the behaviour of the underlying MAP. In fact, it turns out that, in both cases, a Cramér-type result in the MAP context provides the desired asymptotics.

Proposition 4.1. *Suppose that X is a rssMp under assumption (A).*

(a) *When $\theta > 0$, there exists a constant $C_\theta \in (0, \infty)$ such that, for $|y| > 0$*

$$\lim_{a\to\infty} a^\theta \mathbb{P}_y(\tau^{(-a,a)^c} < \tau^{\{0\}}) = \upsilon_{\mathrm{sign}(y)}(\theta) C_\theta |y|^\theta.$$

In particular,

$$\lim_{a\to\infty} \frac{\mathbb{P}_y(\tau^{(-a,a)^c} < \tau^{\{0\}})}{\mathbb{P}_x(\tau^{(-a,a)^c} < \tau^{\{0\}})} = \lim_{a\to\infty} \frac{\mathbf{P}_{0,\mathrm{sign}(y)}(T^+_{\log(a/|y|)} < \infty)}{\mathbf{P}_{0,\mathrm{sign}(x)}(T^+_{\log(a/|x|)} < \infty)}$$
$$= \frac{\upsilon_{\mathrm{sign}(y)}(\theta)}{\upsilon_{\mathrm{sign}(x)}(\theta)} \left|\frac{y}{x}\right|^\theta, \qquad x, y \in \mathbb{R}. \tag{19}$$

(b) *When $\theta < 0$, there exists a constant $\tilde{C}_\theta \in (0, \infty)$ such that, for $|y| > 0$*

$$\lim_{a\to 0} a^\theta \mathbb{P}_y(\tau^{(-a,a)} < \infty) = \upsilon_{\mathrm{sign}(y)}(\theta) \tilde{C}_\theta |y|^\theta.$$

In particular,

$$\lim_{a\to 0} \frac{\mathbb{P}_y(\tau^{(-a,a)} < \infty)}{\mathbb{P}_x(\tau^{(-a,a)} < \infty)} = \lim_{a\to\infty} \frac{\mathbf{P}_{0,\mathrm{sign}(x)}(T^-_{\log(a/|y|)} < \infty)}{\mathbf{P}_{0,\mathrm{sign}(x)}(T^-_{\log(a/|x|)} < \infty)}$$
$$= \frac{\upsilon_{\mathrm{sign}(y)}(\theta)}{\upsilon_{\mathrm{sign}(x)}(\theta)} \left|\frac{y}{x}\right|^\theta, \qquad x, y \in \mathbb{R}. \tag{20}$$

This result will be proved below after some preliminary lemmas. Recalling the discussion from [20], an excursion theory for MAPs reflected in their running maxima exists with strong similarities to the case of Lévy processes. Specifically, there is a MAP, say $(H^+(t), J^+(t))_{t\geq 0}$, with the property that H^+ is non-decreasing with the same range as the running maximum process $\sup_{s\leq t} \xi(s)$, $t \geq 0$. Moreover, the trajectory of the associated Markov chain J^+ agrees with the chain J on the times of increase of the running maximum. We also refer to the Appendix in [14] for further information on classical excursion theory for MAPs.

As an increasing MAP, the process (H^+, J^+) has associated to it a number of characteristics. When $J^+ = \pm 1$, the process H^+ has the increments of a subordinator with drift $\delta_{\pm 1}$ and Lévy measure $\Upsilon_{\pm 1}$ and is sent to a cemetery state $\{+\infty\}$ at rate $q_{\pm 1}$. When J^+ jumps from i to j with $i, j \in \{-1, 1\}$ and $i \neq j$, the process H^+ experiences an independent jump with distribution $F^+_{i,j}$ at rate $\Lambda_{i,j}$. For convenience, we will introduce the Laplace matrix exponent $\boldsymbol{\kappa}$ in the form

$$\mathbf{E}_{0,i}[\mathrm{e}^{-\lambda H^+(t)}; J^+(t) = j] = [\mathrm{e}^{-\boldsymbol{\kappa}(\lambda)t}]_{i,j}, \qquad \lambda \geq 0,$$

where,

$$\boldsymbol{\kappa}(\lambda) = \mathrm{diag}(\Phi_1(\lambda), \Phi_{-1}(\lambda)) - \boldsymbol{\Lambda} \circ \boldsymbol{K}(\lambda), \qquad \lambda \geq 0,$$

where for $i = \pm 1$, $\Phi_i(\lambda)$ is the Laplace exponent of the subordinator encoding the dynamics of H when $J^+ = i$, $\boldsymbol{\Lambda}$ is the intensity matrix of J^+ and $\boldsymbol{K}(\lambda)_{i,j} = \int_{(0,\infty)} \mathrm{e}^{-\lambda x} F^+_{i,j}(\mathrm{d}x)$ for $i, j = \pm 1$ with $i \neq j$ and otherwise $\boldsymbol{K}(\lambda)_{i,i} = 1$, for $i = \pm 1$.

In the next lemma we write the crossing probability of interest in terms of the potential measures

$$U^+_{i,j}(\mathrm{d}x) = \int_0^\infty \mathbf{P}_{0,i}(H^+(t) \in \mathrm{d}x, J^+(t) = j)\mathrm{d}s, \qquad x \geq 0, i, j \in \{-1, 1\}.$$

Lemma 4.1. *The probability of first passage over a threshold can be decomposed into the probability of creeping and the probability of jumping over it.*

(a) *For $y > 0$,*

$$\mathbf{P}_{0,i}(T_y^+ < \infty, H^+(T_y^+) > y) = \sum_{j,k=\pm 1} \int_0^y U_{i,j}^+(\mathrm{d}z) \Big[\mathbf{1}_{(k\neq j)} \Lambda_{j,k} \overline{F}_{j,k}^+(y-z) + \mathbf{1}_{(k=j)} \overline{\Upsilon}_j(y-z) \Big]. \tag{21}$$

(b) *If $\delta_j > 0$ for some $j = \pm 1$, then $U_{i,j}^+$ has a density on $[0,\infty)$ for $i = \pm 1$, which has a continuous version, say $u_{i,j}^+$. Moreover, for $y > 0$,*

$$p_i(y) := \mathbf{P}_i(T_y^+ < \infty, H^+(T_y^+) = y) = \sum_{j=\pm 1} \delta_j u_{i,j}^+(y), \qquad y > 0, i = \pm 1,$$

where we understand $u_{i,j}^+ \equiv 0$ if $\delta_j = 0$. If $\delta_j = 0$ for both $j = \pm 1$, then $p_i(y) = 0$ for all $y > 0$.

Proof. (a) Appealing to the compensation formula for the Cox process that describes the jumps in H^+, we may write for $y > 0$,

$$\begin{aligned}
&\mathbf{P}_{0,i}(T_y^+ < \infty) \\
&= \mathbf{E}_{0,i}\left[\sum_{0<s<\infty} \mathbf{1}_{(y-H^+(s-)>0)} \mathbf{1}_{(y-H^+(s)<0)} \right] \\
&= \sum_{j,k=\pm 1} \mathbf{E}_{0,i}\left[\sum_{0<s<\infty} \mathbf{1}_{(y-H^+(s-)>0)} \mathbf{1}_{(\Delta H^+(s)>y-H^+(s-))} \mathbf{1}_{(J^+(s-)=j, J^+(s)=k)} \right] \\
&= \sum_{j,k=\pm 1} \mathbf{1}_{(j\neq k)} \int_0^\infty \mathbf{P}_{0,i}(H^+(s-) < y, J^+(s-) = j) \Lambda_{j,k} \overline{F}_{j,k}^+(y - H^+(s-)) \mathrm{d}s \\
&\quad + \sum_{j=\pm 1} \int_0^\infty \mathbf{P}_{0,i}(H^+(s-) < y, J^+(s-) = j) \overline{\Upsilon}_j(y - H^+(s-)) \mathrm{d}s,
\end{aligned} \tag{22}$$

where $\Delta H^+(s) = H^+(s) - H^+(s-)$, $\overline{F}_{j,k}^+(x) = 1 - F_{j,k}^+(x)$ and $\overline{\Upsilon}_j(x) = \Upsilon_j(x,\infty)$. When we express the right-hand side of (22) in terms of the potential measure we get (21).

(b) We first define, for $a > 0$,

$$M_i(a) := \int_0^a \mathbf{P}_{0,i}(H^+(T_y^+) = y, T_y^+ < \infty) dy = \int_0^a p_i(y) dy. \tag{23}$$

The analogue of the Lévy–Itô decomposition for subordinators tells us that, up to killing at rate $q_{\pm 1}$, when J^+ is in state ± 1,

$$H^+(t) = \int_0^t \delta_{J^+(t)} dt + \sum_{0<s<t} \Delta H^+(s), \qquad t \geq 0.$$

Then,

$$M_i(a) = \mathbf{E}_{0,i}\left[H^+(T_a^+-) - \sum_{0<s<T_a^+}(H^+(s) - H^+(s-))\,;\; T_a^+ < \infty\right]$$
$$= \mathbf{E}_{0,i}\left[\int_0^{T_a^+} \delta_{J^+(t)} dt\,;\; T_a^+ < \infty\right].$$

Hence, for $a > 0$,

$$M_i(a) = \mathbf{E}_i\left[\int_0^\infty \mathbf{1}_{(0\le H^+(t)\le a)}\delta_{J(t)}\mathrm{d}t\right] = \sum_{j=\pm1}\delta_j U^+_{i,j}[0,a].$$

Noting from (23) that M_i is almost everywhere differentiable on $(0,\infty)$, the above equality tells us that, for j such that $\delta_j \ne 0$ the potential measure $U^+_{i,j}$ has a density. Otherwise, if $\delta_j = 0$ for both $j = \pm 1$, then $p_i(y) = 0$ for Lebesgue almost every $y > 0$.

We define, for each i, $j = \pm 1$ and $x > 0$,

$$p_{i,j}(x) = \mathbf{P}_{0,i}(T_x^+ < \infty, H^+(T_x^+) = x, J^+(T_x^+) = j) \text{ such that } p_i(x) = \sum_{j=\pm1} p_{i,j}(x).$$

Fix $i \in \{-1, 1\}$. We want to show that $p_i(x)$ is continuous. For that, we shall use the fact that

$$\lim_{\epsilon\downarrow 0} p_{i,j}(\epsilon) = \mathbf{1}(\delta_i > 0)\mathbf{1}(i = j). \tag{24}$$

This is due to the fact that the stopping time $T := \inf\{s > 0 : J^+(s) \ne i \text{ or } H^+(s) = +\infty\}$ is exponentially distributed while the time $T_\epsilon^+ \downarrow 0$ as $\epsilon \downarrow 0$. Hence, on $\{t < T\}$, $H^+(t)$ behaves as a (killed) Lévy subordinator and so $T_\epsilon^+ < T$ with increasing probability, tending to 1 as $\epsilon \downarrow 0$. Hence, the result follows from the classical case of Lévy subordinators; see [18].

By the Markov property we have the lower bound

$$p_i(x+\epsilon) \ge \mathbf{P}_{0,i}(H^+(T_x^+) = x, H^+(T^+_{x+\epsilon}) = x+\epsilon, T^+_{x+\epsilon} < \infty)$$
$$= \sum_{j=\pm1} p_{i,j}(x)p_j(\epsilon). \tag{25}$$

If we take the limit $\epsilon \downarrow 0$ and use (24), then we have that

$$\lim_{\epsilon\downarrow 0} p_i(x+\epsilon) \ge \sum_{j=\pm1} p_{i,j}(x)\mathbf{1}(\delta_j > 0) = \sum_{j=\pm1} p_{i,j}(x) = p_i(x).$$

On the other hand, we can split the behaviour of creeping over $x + \epsilon$ into two types

$$p_i(x+\epsilon) = \mathbf{P}_{0,i}(H^+(T_x^+) = x, H^+(T^+_{x+\epsilon}) = x+\epsilon, T^+_{x+\epsilon} < \infty)$$
$$+\mathbf{P}_{0,i}(H^+(T_x^+) > x, H^+(T^+_{x+\epsilon}) = x+\epsilon, T^+_{x+\epsilon} < \infty).$$

The first probability on the right-hand side above corresponds to the right-hand side of (25) and we can bound the second term by the event that $\{0 < O_x \le \epsilon\}$, where we define the overshoot $O_x := H^+(T_x^+) - x$. Hence, we deduce that

$$p_i(x+\epsilon) \le \sum_{j=\pm1} p_{i,j}(\epsilon)p_j(x) + \mathbf{P}_{0,i}(O_x \in (0,\epsilon]).$$

The second probability on the right-hand side above goes to zero as $\epsilon \to 0$. If we now combine this inequality with (25) and take the limit $\epsilon \downarrow 0$, then we can then show that

$$\lim_{\epsilon \downarrow 0} p_i(x+\epsilon) = p_i(x) = \sum_{j=\pm 1} p_{i,j}(x).$$

We can also show in a similar way that $\lim_{\epsilon\downarrow 0} p_i(x-\epsilon) = p_i(x)$ and hence p_i is continuous. Note that the preceding reasoning is valid without discrimination for the case that p_i is almost everywhere equal to zero. The proof is now complete. □

Understanding the asymptotic of $\mathbf{P}_{0,i}(T_y^+ < \infty)$ is now a matter of Markov additive renewal theory. In this respect, let us say some more words about the Markov additive renewal measure $U_{i,j}$.

We will restrict the forthcoming discussion to the setting that $\theta > 0$. Recall from the discussion at the end of Section 1.1 that this implies $\lim_{t\to\infty}\xi(t) = -\infty$, where ξ is the MAP underlying the rssMp. A consequence of this observation is that the process H^+ experiences killing. To be more precise it has killing rates which we previously denoted by $q_{\pm 1} > 0$. This makes the measures $U^+_{i,j}$ finite. As with classical renewal theory, we can use the existence of the Cramér number θ to renormalise the measures $U^+_{i,j}$ so that they are appropriate for use with asymptotic Markov additive renewal theory.

Appealing to the exponential change of measure described in Proposition 1.4, we note that the law of (H^+, J^+) under $\mathbf{P}^\theta_{0,i}$ satisfies

$$\mathbf{P}^\theta_{0,i}(H^+(t)\in \mathrm{d}x,\, J^+(t)=j) = \frac{v_j(\theta)}{v_i(\theta)} \mathrm{e}^{\theta x}\mathbf{P}_{0,i}(H^+(t)\in \mathrm{d}x,\, J^+(t)=j),$$
$$i, j = \pm 1, x \geq 0.$$

In particular, the role of $\boldsymbol{\kappa}$ for (H^+, J^+) under $\mathbb{P}^\theta_{0,i}$, $i=\pm 1$ is played by

$$\boldsymbol{\kappa}_\theta(\lambda) = \boldsymbol{\kappa}(\lambda-\theta), \qquad \lambda \geq 0.$$

Hence, we have that

$$U^{\theta,+}_{i,j}(\mathrm{d}x) := \int_0^\infty \mathbf{P}^\theta_{0,i}(H^+(t)\in \mathrm{d}x,\, J^+(t)=j)\mathrm{d}t = \frac{v_j(\theta)}{v_i(\theta)}\mathrm{e}^{\theta x} U^+_{i,j}(\mathrm{d}x), \qquad x\geq 0.$$

Again, referring to the discussion at the end of Section 1.1, since $\lim_{t\to\infty}\xi(t) = \infty$ almost surely under $\mathbf{P}^\theta_{0,i}$, we may now claim that the adjusted Markov additive renewal measure $U^{\theta,+}_{i,j}(\mathrm{d}x)$ is that of an unkilled subordinator MAP.

Lemma 4.2. *Suppose that $\theta > 0$. There exists a constant $C_\theta > 0$, such that, as $y \to \infty$,*

$$\mathrm{e}^{\theta y}\mathbf{P}_{0,i}(T^+_y < \infty) \to v_i(\theta) C_\theta.$$

Proof. Picking up Eq. (21), we have, for $i = \pm 1$,

$$\begin{aligned} &\mathrm{e}^{\theta y}\mathbf{P}_{0,i}(T^+_y<\infty,\ H^+(T^+_y) > y) \\ &\quad = v_i(\theta)\sum_{j,k=\pm 1}\int_0^y \mathrm{e}^{\theta(y-z)}\frac{1}{v_j(\theta)} U^{\theta,+}_{i,j}(\mathrm{d}z) \\ &\qquad \times \left[\mathbf{1}_{(k\neq j)}\Lambda_{j,k}\overline{F}^+_{j,k}(y-z) + \mathbf{1}_{(k=j)}\overline{\Upsilon}_j(y-z)\right]. \end{aligned} \tag{26}$$

Our aim is to convert this into a form that we can apply the discrete-time Markov Additive Renewal Theorem A.1 in the Appendix.

To this end, we define the sequence of random times $\Theta_1, \Theta_2, \dots$ such that $\Theta_{i+1} - \Theta_i$ are independent and exponentially distributed with parameter 1. For convenience, define $\Theta_0 = 0$. We want to relate (H^+, J^+) to a discrete-time Markov additive renewal process (Ξ_n, M_n), $n \geq 0$, such that

$$\Delta_n := \Xi_{n+1} - \Xi_n = H^+(\Theta_{n+1}) - H^+(\Theta_n) \text{ and } M_n = J^+(\Theta_n), \qquad n \geq 0.$$

A future quantity of interest is the stationary mean increment $\mu^+_\theta := \mathbf{E}^\theta_{0,\boldsymbol{\pi}^\theta}[H_1(\Theta_1)]$, where $\boldsymbol{\pi}^\theta = (\pi^\theta_1, \pi^\theta_{-1})$ is the stationary distribution of J (and hence of J^+ since it is described pathwise by J sampled at a sequence of stopping times) under $\mathbf{P}^\theta$. In this respect, we note from Corollary 2.5 in Chapter XI of [4] that,

$$\begin{aligned}
\mu^+_\theta &= \int_0^\infty \mathrm{e}^{-t} \mathbf{E}^\theta_{0,\boldsymbol{\pi}^\theta}[H^+(t)]\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-t}[\chi^+_\theta(0)t + \boldsymbol{\pi}^\theta \cdot \boldsymbol{k}^\theta - \boldsymbol{\pi}^\theta \cdot \mathrm{e}^{\boldsymbol{\Lambda}^\theta t}\boldsymbol{k}^\theta]\mathrm{d}t \\
&= \chi^+_\theta(0) + \boldsymbol{\pi}^\theta \cdot \boldsymbol{k}^\theta - \boldsymbol{\pi}^\theta \cdot (\boldsymbol{\Lambda}^\theta - \boldsymbol{I})^{-1}\boldsymbol{k}^\theta,
\end{aligned} \tag{27}$$

where $\chi^+_\theta(0)$ is the leading eigenvalue of $\boldsymbol{\kappa}_\theta(0)$, $\boldsymbol{k}^\theta = \boldsymbol{v}'(\theta)$ and $\boldsymbol{\Lambda}^\theta = \boldsymbol{\kappa}_\theta(0)$. All of these quantities are guaranteed to exist thanks to the assumption (A); see for example Section 2 of Chapter XI in [4].

Note, moreover, that

$$\begin{aligned}
U^{\theta,+}_{i,j}(\mathrm{d}x) &= \int_0^\infty \mathbf{P}^\theta_{0,i}(H^+_t \in \mathrm{d}x,\ J^+(t) = j)\mathrm{d}t \\
&= \sum_{n=1}^\infty \int_0^\infty \mathrm{e}^{-t} \frac{t^{n-1}}{(n-1)!} \mathbf{P}^\theta_{0,i}(H^+_t \in \mathrm{d}x,\ J^+(t) = j)\mathrm{d}t \\
&= \sum_{n=1}^\infty \mathbf{P}^\theta_{0,i}(H_{\Theta_n} \in \mathrm{d}x, J_{\Theta_n} = j) \\
&=: R^\theta_{i,j}(\mathrm{d}x) - \delta_0(\mathrm{d}x)\mathbf{1}(i = j),
\end{aligned} \tag{28}$$

where, on the right-hand side, we have used the notation of the discrete-time Markov additive renewal measure in the Appendix.

Turning back to (26), if we define

$$g_j(x) = \sum_{k=\pm 1} \frac{1}{v_j(\theta)} \mathrm{e}^{\theta x} \left[\mathbf{1}(k \neq j)\Lambda_{j,k}\overline{F}_{j,k}(x) + \mathbf{1}(k = j)\overline{\Upsilon}_j(x)\right], \qquad x \geq 0, \tag{29}$$

for $j = \pm 1$, then, as soon as we can verify that these functions are directly Riemann integrable, then we can apply the conclusion of Theorem A.1 in the Appendix and conclude that

$$\begin{aligned}
&\lim_{y\to\infty} \mathrm{e}^{\theta y}\mathbf{P}_{0,i}(T^+_y < \infty,\ H^+(T^+_y) > y) \\
&= v_i(\theta) \sum_{j,k=\pm 1} \frac{\pi^\theta_j}{v_j(\theta)\mu^+_\theta} \int_0^\infty \mathrm{e}^{\theta s} \left[\mathbf{1}_{(k\neq j)}\Lambda_{j,k}\overline{F}^+_{j,k}(s) + \mathbf{1}_{(k=j)}\overline{\Upsilon}_j(s)\right] \mathrm{d}s,
\end{aligned}$$

where π^{θ}_j, $j = \pm 1$ is the stationary distribution of the chain J^+ under $\mathbf{P}^{\theta}_{x,i}$, $x \in \mathbb{R}$, $i = \pm 1$. Note, moreover that, from Lemma 4.1, together with Theorem 1.2 of [1],

$$e^{\theta y}\mathbb{P}_i(T^+_y < \infty, H^+(T^+_y) = y) = v_i(\theta)\sum_{j=\pm 1}\frac{1}{v_j(\theta)}\delta_j u^{\theta,+}_{i,j}(y) \to v_i(\theta)\sum_{j=\pm 1}\delta_j\frac{\pi^{\theta}_j}{v_j(\theta)\mu^+_{\theta}},$$

as $y \to \infty$.

To finish the proof we must thus verify the direct Riemann integrability of $g_j(x)$, $j = \pm 1$ in (29). Note however, that $g_j(x)$ is the product of $e^{\theta x}$ and a monotone decreasing function, hence it suffices to check that $\int_0^\infty g_j(x)\mathrm{d}x < \infty$, $j = \pm 1$. To this end, remark that, for λ in the domain where $\boldsymbol{\kappa}$ is defined,

$$(\boldsymbol{\kappa}(\lambda)\mathbf{1})_j = q_j + \delta_j\lambda + \int_0^\infty (1 - e^{-\lambda x})\Upsilon_j(\mathrm{d}x) + \sum_{k=\pm 1}\mathbf{1}_{(j\neq k)}\Lambda_{j,k}\int_0^\infty e^{-\lambda x}F_{j,k}(\mathrm{d}x),$$
$$j = \pm 1.$$

In particular, with an integration by parts, we have

$$\frac{q_j - (\boldsymbol{\kappa}(-\theta)\mathbf{1})_j}{\theta} = \delta_j + \int_0^\infty e^{\theta s}\left[\sum_{k=\pm 1}\mathbf{1}_{(k\neq j)}\Lambda_{j,k}\overline{F}^{+}_{j,k}(s) + \mathbf{1}_{(k=j)}\overline{\Upsilon}_j(s)\right]\mathrm{d}s,$$
$$j = \pm 1,$$

where the left-hand side is finite thanks to the assumption (A). This completes the proof, albeit to note that

$$\lim_{y\to\infty} e^{\theta y}\mathbb{P}_i(T^+_y < \infty) = v_i(\theta)\sum_{j=\pm 1}\frac{\pi^{\theta}_j[q_j - (\boldsymbol{\kappa}(-\theta)\mathbf{1})_j]}{\theta v_j(\theta)\mu^+_{\theta}},$$

which identifies explicitly the constant C_θ in the statement of the lemma. □

Proof of Proposition 4.1. First assume that $\theta > 0$. A particular consequence of Lemma 4.2 is that

$$\lim_{a\to\infty}\frac{\mathbb{P}_y(\tau^{(-a,a)^c} < \tau^{\{0\}})}{\mathbb{P}_x(\tau^{(-a,a)^c} < \tau^{\{0\}})} = \lim_{a\to\infty}\frac{\mathbf{P}_{0,\mathrm{sign}(y)}(T^+_{\log(a/|y|)} < \infty)}{\mathbf{P}_{0,\mathrm{sign}(x)}(T^+_{\log(a/|x|)} < \infty)} = \frac{v_{\mathrm{sign}(y)}(\theta)}{v_{\mathrm{sign}(x)}(\theta)}\left|\frac{y}{x}\right|^{\theta},$$
$$x, y \in \mathbb{R}.$$

Now we turn our attention to the case that $\theta < 0$. We appeal to duality and write

$$\mathbb{P}_x(\tau^{(-a,a)} < \infty) = \mathbf{P}_{(\log|x|,\mathrm{sign}(x))}(T^-_{\log a} < \infty) = \tilde{\mathbf{P}}_{(-\log|x|,\mathrm{sign}(x))}(T^+_{-\log a} < \infty),$$

where under $\tilde{\mathbf{P}}_{x,i}$, $x \in \mathbb{R}$, $i = \pm 1$, is the law of $(-\xi, J)$. Note, the associated matrix exponent of this process is $\tilde{\boldsymbol{F}}(z) := \boldsymbol{F}(-z)$, whenever the right-hand side is well defined. In particular, we note that $\tilde{\boldsymbol{F}}(-\theta) = 0$, which is to say that $-\theta > 0$ is the Cramér number for the process $(-\xi, J)$. Moreover, $\tilde{\boldsymbol{F}}(-\theta)\boldsymbol{v}(\theta) := \boldsymbol{F}(\theta)\boldsymbol{v}(\theta) = 0$, which is to say that $\tilde{\boldsymbol{v}}(-\theta) = \boldsymbol{v}(\theta)$. The first part of the proof can now be re-cycled to deduce the conclusions in part (b) of the statement of the proposition. □

Proof of Theorem 2.1. The (super)martingale (10) applies an exponential change of measure to (ξ, J), albeit on the sequence of stopping times $\varphi(t)$, for $t < \tau^{\{0\}}$. As the change of measure (5) keeps (ξ, J) in the class of MAPs, thanks to Proposition 1.4, it follows that $\mathbb{P}^{\circ}_x$, $x \in \mathbb{R}\setminus\{0\}$, corresponds to the law of a rssMp whose underlying MAP is that of the Esscher transform of (ξ, J).

In the case of (a), recalling the discussion preceding Section 1.2, the underlying MAP for $(X, \mathbb{P}^\circ_x)$, $x \in \mathbb{R} \setminus \{0\}$ drifts to $+\infty$. This means that under the change of measure, X is a rssMp that never touches the origin, i.e. it is a conservative process. In the case of (b), the underlying MAP drifts to $-\infty$ and hence, under the change of measure X is continuously absorbed at the origin, so it is non-conservative.

For the proof of (a), we follow a standard line of reasoning that can be found, for example, in [8]. Appealing to the Markov property, self-similarity, Fatou's Lemma and (19), we have, for $A \in \mathcal{F}_t$,

$$\begin{aligned}
&\liminf_{a\to\infty} \mathbb{P}_x(A \cap \{t < \tau^{(-a,a)^{\mathrm{c}}}\} \mid \tau^{(-a,a)^{\mathrm{c}}} < \tau^{\{0\}}) \\
&= \liminf_{a\to\infty} \mathbb{E}_x\left[\mathbf{1}_{(A,\, t<\tau^{\{0\}}\wedge\tau^{(-a,a)^{\mathrm{c}}})} \frac{\mathbb{P}_{X_t}(\tau^{(-a,a)^{\mathrm{c}}} < \tau^{\{0\}})}{\mathbb{P}_x(\tau^{(-a,a)^{\mathrm{c}}} < \tau^{\{0\}})}\right] \\
&\geq \mathbb{E}_x\left[\mathbf{1}_{(A,\, t<\tau^{\{0\}})} \liminf_{a\to\infty} \frac{\mathbb{P}_{a^{-1}X_t}(\tau^{(-1,1)^{\mathrm{c}}} < \tau^{\{0\}})}{\mathbb{P}_{a^{-1}x}(\tau^{(-1,1)^{\mathrm{c}}} < \tau^{\{0\}})}\right] \\
&= \mathbb{E}_x\left[\mathbf{1}_{(A,\, t<\tau^{\{0\}})} \frac{h_\theta(X_t)}{h_\theta(x)}\right].
\end{aligned}$$

Recalling the martingale property from (10) together with the above inequality, but now applied to the event A^c, tells us that

$$\begin{aligned}
&\limsup_{a\to\infty} \mathbb{P}_x(A \cap \{t < \tau^{(-a,a)^{\mathrm{c}}}\} \mid \tau^{(-a,a)^{\mathrm{c}}} < \tau^{\{0\}}) \\
&\leq 1 - \liminf_{a\to\infty} \mathbb{P}_x(A^c \cap \{t < \tau^{(-a,a)^{\mathrm{c}}}\} \mid \tau^{(-a,a)^{\mathrm{c}}} < \tau^{\{0\}}) \\
&\leq \mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)} \mathbf{1}_{(t<\tau^{\{0\}})}\right] - \mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)} \mathbf{1}_{(A^c,\, t<\tau^{\{0\}})}\right] \\
&= \mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)} \mathbf{1}_{(A,\, t<\tau^{\{0\}})}\right],
\end{aligned}$$

where the final equality follows as we have used the martingale property of the chance of measure for which recall the discussion around (10). The required limiting identity follows.

The proof of (b) is similar to that of (a) except that in this case (10) ensures that X_t^θ is a super-martingale only and hence the final part of the argument above does not extend to this setting. To overcome this difficulty we proceed as follows. Notice $\tau^{(-a,a)} \to \tau^{\{0\}}$ as $a \to 0$. As before for $A \in \mathcal{F}_t$, we have

$$\begin{aligned}
&\liminf_{a\to 0} \mathbb{P}_x(A \cap \{t < \tau^{(-a,a)}\} \mid \tau^{(-a,a)} < \infty) \\
&= \liminf_{a\to 0} \mathbb{E}_x\left[\mathbf{1}_{(A,\, t<\tau^{(-a,a)})} \frac{\mathbb{P}_{X_t}(\tau^{(-a,a)} < \infty)}{\mathbb{P}_x(\tau^{(-a,a)} < \infty)}\right] \\
&\geq \mathbb{E}_x\left[\mathbf{1}_{(A,\, t<\tau^{\{0\}})} \liminf_{a\to 0} \frac{\mathbb{P}_{a^{-1}X_t}(\tau^{(-1,1)} < \infty)}{\mathbb{P}_{a^{-1}x}(\tau^{(-1,1)} < \infty)}\right] \\
&= \mathbb{E}_x\left[\mathbf{1}_{(A,\, t<\tau^{\{0\}})} \frac{h_\theta(X_t)}{h_\theta(x)}\right] \\
&= \mathbb{E}_x\left[\mathbf{1}_A \frac{h_\theta(X_t)}{h_\theta(x)}\right],
\end{aligned}$$

where, recalling the discussion around (9), in the final equality we have used the fact that $\theta < 0$ implies that $\tau^{\{0\}} = \infty$ almost surely (irrespective of the point of issue of X). Now, the second half of the argument in (a) extends to this setting if the following equation holds true

$$\lim_{a\to 0} \mathbb{P}_x(t < \tau^{(-a,a)} \mid \tau^{(-a,a)} < \infty) = \mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)}\right].$$

On the one hand, the Markov property, Fatou's lemma and the estimate (20) imply that

$$\begin{aligned}\liminf_{a\to 0} \mathbb{P}_x(t < \tau^{(-a,a)} \mid \tau^{(-a,a)} < \infty) &= \liminf_{a\to 0} \mathbb{P}_x\left(\mathbf{1}_{(t<\tau^{(-a,a)})}\frac{\mathbb{P}_{X_t}(\tau^{(-a,a)} < \infty)}{\mathbb{P}_x(\tau^{(-a,a)} < \infty)}\right)\\ &\geq \mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)}\mathbf{1}_{(t<\tau^{\{0\}})}\right]\\ &= \mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)}\right].\end{aligned}$$

Now, the estimate in (b) in Proposition 4.1 implies that for $y \neq 0$

$$\lim_{a\to 0}\left(\frac{a}{|y|}\right)^\theta \mathbb{P}_y(\tau^{(-a,a)} < \infty) = \lim_{a\to 0}\left(\frac{a}{|y|}\right)^\theta \mathbb{P}_{\mathrm{sgn}(y)}(\tau^{(-\frac{a}{|y|},\frac{a}{|y|})} < \infty) = \upsilon_{\mathrm{sign}(y)}(\theta)\tilde{C}_\theta,$$

and the convergence holds uniformly in $a/|y|$ such that $a/|y| < \epsilon$, for $\epsilon > 0$. Moreover, for $a/|y| > \epsilon$ the term $(a/|y|)^\theta \mathbb{P}_y(\tau^{(-a,a)} < \infty)$ remains bounded. Thus for $x \neq 0, \epsilon > 0$, fixed we have

$$\begin{aligned}&\limsup_{a\to 0} \mathbb{P}_x\left(\mathbf{1}_{(t<\tau^{(-a,a)})}\frac{\mathbb{P}_{X_t}(\tau^{(-a,a)} < \infty)}{\mathbb{P}_x(\tau^{(-a,a)} < \infty)}\right)\\ &= \limsup_{a\to 0} \mathbb{P}_x\left(\mathbf{1}_{((a/|X_t|)<\epsilon,\ t<\tau^{(-a,a)})}\frac{\mathbb{P}_{X_t}(\tau^{(-a,a)} < \infty)}{\mathbb{P}_x(\tau^{(-a,a)} < \infty)}\right)\\ &\quad + \limsup_{a\to 0} \mathbb{P}_x\left(\mathbf{1}_{((a/|X_t|)\geq\epsilon,\ t<\tau^{(-a,a)})}\frac{\mathbb{P}_{X_t}(\tau^{(-a,a)} < \infty)}{\mathbb{P}_x(\tau^{(-a,a)} < \infty)}\right)\\ &= \mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)}\mathbf{1}_{(t<\tau^{\{0\}})}\right] + \limsup_{a\to 0} \mathbb{P}_x\left(\mathbf{1}_{((a/|X_t|)\geq\epsilon,\ t<\tau^{(-a,a)})}\frac{\mathbb{P}_{X_t}(\tau^{(-a,a)} < \infty)}{\mathbb{P}_x(\tau^{(-a,a)} < \infty)}\right).\end{aligned}$$

Finally the limsup in the above estimate is equal to zero because it can be bounded by above as follows

$$\begin{aligned}&\mathbb{P}_x\left(\mathbf{1}_{((a/|X_t|)\geq\epsilon,\ t<\tau^{(-a,a)})}\frac{\mathbb{P}_{X_t}(\tau^{(-a,a)} < \infty)}{\mathbb{P}_x(\tau^{(-a,a)} < \infty)}\right)\\ &\leq \frac{1}{a^\theta \mathbb{P}_x(\tau^{(-a,a)} < \infty)}\mathbb{P}_x\left(\mathbf{1}_{((a/|X_t|)\geq\epsilon,\ t<\tau^{(-a,a)})}|X_t|^\theta \sup_{|z|\geq\epsilon}|z|^\theta \mathbb{P}_{\mathrm{sgn}(z)}(\tau^{(-z,z)} < \infty)\right)\\ &= \frac{x^\theta \sup_{|z|\geq\epsilon}|z|^\theta \mathbb{P}_{\mathrm{sgn}(z)}(\tau^{(-z,z)} < \infty)}{a^\theta \mathbb{P}_x(\tau^{(-a,a)} < \infty)}\mathbb{P}^\circ_x\left(\mathbf{1}_{(a/|X_t|\geq\epsilon,\ t<\tau^{(-a,a)})}\frac{\upsilon_{\mathrm{sign}(x)}(\theta)}{\upsilon_{\mathrm{sign}(X_t)}(\theta)}\right),\end{aligned}$$

and by the monotone convergence theorem the rightmost term in the above inequality tends to zero when $a \to 0$. □

5. Integrated exponential MAPs, proof of Theorem 2.3

The asymptotic behaviour of the tail distribution of objects similar to I, when the process ξ is replaced by a Lévy process, has been considered in [28,2]. We will borrow some of the ideas from the second of these two papers and apply them in the Markov additive setting in establishing the estimate in Theorem 2.3. To this end, recall that in this setting J takes values in a finite state space E, and let us introduce the potential measure

$$V_{i,j}(\mathrm{d}x) = \int_0^\infty \mathbf{P}_{0,i}(\xi(s) \in \mathrm{d}x, J(s) = j)\mathrm{d}s, \quad i, j \in E.$$

Proposition 5.1. *For $t > 0$ and $i \in E$,*

$$\mathbf{P}_{0,i}(I > t)\mathrm{d}t = \sum_{j\in E}\int_{\mathbb{R}} V_{i,j}(\mathrm{d}y)\mathrm{e}^{\alpha y}\mathbf{P}_{0,j}(\mathrm{e}^{\alpha y}I \in \mathrm{d}t). \tag{30}$$

Proof. The method of proof is to show the left- and right-hand sides of (30) are equal by considering their Laplace transforms. Integration by parts shows us that, for $\lambda > 0$, we have on the one hand,

$$\mathbf{E}_{0,i}(1-\mathrm{e}^{-\lambda I}) = \lambda\int_0^\infty \mathrm{e}^{-\lambda t}\mathbf{P}_{0,i}(I > t)\mathrm{d}t. \tag{31}$$

We shall use the above equation for comparison later. On the other hand, we have for $\lambda > 0$,

$$\begin{aligned}
\mathbf{E}_{0,i}(1-\mathrm{e}^{-\lambda I}) &= \mathbf{E}_{0,i}\left[\int_0^\infty \mathrm{d}(\mathrm{e}^{-\lambda\int_s^\infty \mathrm{e}^{\alpha\xi(u)}\mathrm{d}u})\right]\\
&= \lambda\mathbf{E}_{0,i}\left[\int_0^\infty \mathrm{e}^{\alpha\xi(s)}\mathrm{e}^{-\lambda\int_s^\infty \mathrm{e}^{\alpha\xi(u)}\mathrm{d}u}\mathrm{d}s\right]\\
&= \lambda\int_0^\infty\sum_{j\in E}\mathbf{E}_{0,i}\left[\mathrm{e}^{\alpha\xi(s)}\mathrm{e}^{-\lambda\mathrm{e}^{\alpha\xi(s)}\int_s^\infty \mathrm{e}^{\alpha(\xi(u)-\xi(s))}\mathrm{d}u}; J(s)=j\right]\mathrm{d}s\\
&= \lambda\sum_{j\in E}\int_0^\infty \mathbf{E}_{0,i}\left[\mathrm{e}^{\alpha\xi(s)}\mathbf{E}_{0,j}\left[\mathrm{e}^{-\lambda\mathrm{e}^{\alpha y}I}\right]\Big|_{y=\xi(s)}\right]\mathrm{d}s\\
&= \lambda\sum_{j\in E}\int_{\mathbb{R}} V_{i,j}(\mathrm{d}y)\mathrm{e}^{\alpha y}\mathbf{E}_{0,j}[\mathrm{e}^{-\lambda\mathrm{e}^{\alpha y}I}]\\
&= \lambda\sum_{j\in E}\int_{\mathbb{R}} V_{i,j}(\mathrm{d}y)\mathrm{e}^{\alpha y}\int_0^\infty \mathbf{P}_{0,j}(\mathrm{e}^{\alpha y}I \in \mathrm{d}t)\mathrm{e}^{-\lambda t},
\end{aligned} \tag{32}$$

where we have applied the conditional stationary independent increments of (ξ, J) in the fourth equality. Now comparing (32) with (31), we see that

$$\mathbf{P}_{0,i}(I > t)\mathrm{d}t = \sum_{j\in E}\int_{\mathbb{R}} V_{i,j}(\mathrm{d}y)\mathrm{e}^{\alpha y}\mathbf{P}_{0,j}(\mathrm{e}^{\alpha y}I \in \mathrm{d}t),$$

for $t > 0$, as required. □

Now that we have expressed the tail probabilities $\mathbf{P}_{0,i}(I > t)$ in terms of the potential measure $V_{i,j}$, we may again turn to renewal theory for Markov additive random walks in order to extract the desired asymptotics as $t \to \infty$. With a view to applying Theorem A.1 in the Appendix, let us therefore introduce (M_n, Δ_n) defined as

$$M_n = J(\Theta_n) \text{ and } \Delta_n = \xi(\Theta_n), \qquad n \geq 0,$$

where, as before, $\Theta_0 = 0$ and Θ_n is the sum of an independent sequence of exponential random variables with unit mean. As in the Appendix, we write $R_{i,j}(\mathrm{d}x)$ for the renewal measure of (Ξ, M), where $\Xi_0 = 0$, $\Xi_n = \Delta_1 + \cdots \Delta_n$, $n \geq 1$. We also introduce

$$R^\theta_{i,j}(\mathrm{d}x) := \frac{v_j(\theta)}{v_i(\theta)} \mathrm{e}^{\theta x} R_{i,j}(\mathrm{d}x), \qquad x \in \mathbb{R},\ i, j \in E.$$

We note again that $V_{i,j}(\mathrm{d}x) = R_{i,j}(\mathrm{d}x) - \delta_0(\mathrm{d}x)\mathbf{1}_{(i=j)}$.

In a similar spirit to (28), we may use these Markov additive random walks to write for any interval $A \subseteq [0, \infty)$

$$\begin{aligned}
\mathrm{e}^{(\theta-\alpha)t} \int_{A\mathrm{e}^{\alpha t}} \mathbf{P}_{0,i}(I > s)\mathrm{d}s &= \sum_{j\in E} \int_{\mathbb{R}} V_{i,j}(\mathrm{d}y)\mathrm{e}^{\alpha y} \mathrm{e}^{(\theta-\alpha)t} \int_{A\mathrm{e}^{\alpha t}} \mathbf{P}_{0,j}(\mathrm{e}^{\alpha y} I \in \mathrm{d}s) \\
&= v_i(\theta) \sum_{j\in E} \frac{1}{v_j(\theta)} \int_{\mathbb{R}} R^\theta_{i,j}(\mathrm{d}y)\mathrm{e}^{(\theta-\alpha)(t-y)} \\
&\quad \times \int_{A\mathrm{e}^{\alpha t}} \mathbf{P}_{0,j}(\mathrm{e}^{\alpha y} I \in \mathrm{d}s) + \mathbf{1}_{(i=j)} \mathrm{e}^{(\theta-\alpha)t} \mathbf{P}_{0,j}(I \in A\mathrm{e}^{\alpha t}) \\
&= v_i(\theta) \sum_{j\in E} \frac{1}{v_j(\theta)} \int_{\mathbb{R}} R^\theta_{i,j}(\mathrm{d}y)\mathrm{e}^{(\theta-\alpha)(t-y)} \mathbf{P}_{0,j}(I \in A\mathrm{e}^{\alpha(t-y)}) \\
&\quad - \mathbf{1}_{(i=j)} \mathrm{e}^{(\theta-\alpha)t} \mathbf{P}_{0,j}(I \in A\mathrm{e}^{\alpha t}).
\end{aligned} \tag{33}$$

Noting that the main term on the right-hand side above is a convolution between the renewal measure $R^\theta_{i,j}$ and the function

$$g_j(z, A) := \frac{1}{v_j(\theta)} \mathrm{e}^{(\theta-\alpha)z} \mathbf{P}_{0,j}(I \in A\mathrm{e}^{\alpha z}), \qquad z \in \mathbb{R},\ j \in E,$$

we are now almost ready to apply the discrete-time Markov Additive Renewal Theorem A.1 in the Appendix. It turns out that we need to choose the interval A judiciously according to whether θ is bigger or smaller than α in order to respect the direct Riemann integrability condition in the renewal theorem. We therefore digress with an additional technical lemma before returning to the limit in (33) and the proof of Theorem 2.3.

Lemma 5.1. *When $\theta > 0$, $\mathbf{E}_{0,j}\left(I^{\theta/\alpha-1}\right) < \infty$, for all $j \in E$.*

Proof. When $\theta = \alpha$ the result is trivial. The case that $\theta/\alpha < 1$ turns out to be a direct consequence of Proposition 3.6 from [19]. To be more precise, careful inspection of the proof there shows that (in our terminology) if $0 < \alpha\beta \leq \theta$ then $\mathbf{E}_{0,i}[I^{\beta-1}] < \infty$, for all $i \in E$, in which case one takes $\beta = \theta/\alpha$.

For the final case that $\theta/\alpha > 1$, we can replicate the recurrence relation from Section 1.2 of [2]. Appealing to (30), we have, for $\beta \in (0, \theta/\alpha)$ and $k \in E$,

$$\mathbf{E}_{0,k}[I^\beta] = \beta \int_0^\infty s^{\beta-1} \mathbf{P}_{0,k}(I > s)\mathrm{d}s = \beta \int_0^\infty s^{\beta-1} \sum_{j\in E} \int_{\mathbb{R}} V_{k,j}(\mathrm{d}y)\mathrm{e}^{\alpha y} \mathbf{P}_{0,j}(\mathrm{e}^{\alpha y} I \in \mathrm{d}s).$$

Let us momentarily assume that $\mathbb{E}_{0,k}[I^{\beta-1}] < \infty$ for $k \in E$. We can use Fubini's theorem and put $s = t\mathrm{e}^{\alpha y}$, and get

$$\begin{aligned}
\mathbf{E}_{0,k}[I^\beta] &= \beta \sum_j \int_{\mathbb{R}} \mathrm{e}^{\alpha\beta y} V_{k,j}(\mathrm{d}y) \int_0^\infty t^{\beta-1} \mathbf{P}_{0,j}(I \in \mathrm{d}t) \\
&= \beta \sum_{j\in E} \mathbf{E}_{0,j}[I^{\beta-1}] \int_{\mathbb{R}} \mathrm{e}^{\alpha\beta y} V_{k,j}(\mathrm{d}y) \\
&= \beta \sum_{j\in E} \mathbf{E}_{0,j}[I^{\beta-1}] \int_0^\infty \mathrm{d}s \int_{\mathbb{R}} \mathrm{e}^{\alpha\beta y} \mathbf{E}_{0,k}[\xi(s) \in \mathrm{d}y, J(s) = j] \\
&= \beta \sum_{j\in E} \mathbf{E}_{0,j}[I^{\beta-1}] \int_0^\infty (\exp\{t F(\alpha\beta)\})_{k,j} \mathrm{d}t \\
&= \beta \sum_{j\in E} \mathbf{E}_{0,j}[I^{\beta-1}] (\boldsymbol{F}(\alpha\beta)^{-1})_{k,j}
\end{aligned}$$

where the right-hand side uses the fact that $\beta \in (0, \theta/\alpha)$. We deduce that $\mathbf{E}_{0,k}[I^{\beta-1}] < \infty$ for $k \in E$ implies that $\mathbf{E}_{0,k}[I^\beta] < \infty$ for $k \in E$.

If n is the smallest non-negative integer such that $\theta/\alpha - n \in (0, 1]$, we can use Proposition 3.6 from [19] again, to deduce that $\mathbf{E}_{0,k}[I^{\theta/\alpha-n}] < \infty$. The argument in the previous paragraph can now be used inductively to conclude that $\mathbf{E}_{0,k}[I^{\theta/\alpha-1}] < \infty$, for any $k \in E$. $\square$

Proof of Theorem 2.3. We break the proof into three cases. We start by assuming that $\theta < \alpha$. In that case, referring to (33), we have, assuming the limit exists on the right-hand side,

$$\begin{aligned}
&\lim_{t\to\infty} \mathrm{e}^{(\theta-\alpha)t} \int_0^{\mathrm{e}^{\alpha t}} \mathbf{P}_{0,i}(I > s)\mathrm{d}s \\
&= \lim_{t\to\infty} v_i(\theta) \sum_{j\in E} \frac{1}{v_j(\theta)} \int_{\mathbb{R}} R^\theta_{i,j}(\mathrm{d}y) \mathrm{e}^{(\theta-\alpha)(t-y)} \mathbf{P}_{0,j}(I \in [0, \mathrm{e}^{\alpha(t-y)}]) \\
&= \lim_{t\to\infty} v_i(\theta) \sum_{j\in E} \frac{1}{v_j(\theta)} \int_{\mathbb{R}} R^\theta_{i,j}(\mathrm{d}y) g_j(t-y) \qquad (34)
\end{aligned}$$

where

$$g_k(y) = \frac{1}{v_k(\theta)} \mathrm{e}^{(\theta-\alpha)y} \int_0^{\mathrm{e}^{\alpha y}} \mathbf{P}_{0,k}(I \in \mathrm{d}s), \qquad k \in E,\ y \in \mathbb{R}.$$

Note in particular that

$$\int_{\mathbb{R}} g_k(y)\mathrm{d}y = \frac{1}{(\alpha-\theta)v_k(\theta)} \mathbf{E}_{0,k}[I^{\theta/\alpha-1}], \qquad k \in E,$$

which is finite by Lemma 5.1. Moreover, since $g_k(x)$ is product of an exponential and a monotone function, it is a standard exercise to show that it is also directly Riemann integrable.

The discrete-time Markov Additive Renewal Theorem A.1 in the Appendix now justifies the limit in (34) so that

$$\lim_{t\to\infty} \mathrm{e}^{(\theta-\alpha)t} \int_0^{\mathrm{e}^{\alpha t}} \mathbf{P}_{0,i}(I > s)\mathrm{d}s = v_i(\theta) \sum_{j\in E} \frac{\pi^\theta_j}{\mu_\theta |\alpha-\theta| v_j(\theta)} \mathbf{E}_{0,j}[I^{\theta/\alpha-1}], \qquad (35)$$

provided $\mu_\theta < \infty$. This last condition is easily verified as a consequence of assumption (A). Indeed, according to Corollary 2.5 of Chapter XI in [4], we have

$$\mu_\theta = \chi'(\theta) + \boldsymbol{\pi}^\theta \cdot \boldsymbol{k}^\theta - \boldsymbol{\pi}^\theta \cdot (\boldsymbol{Q}^\theta - \boldsymbol{I})^{-1}\boldsymbol{k}^\theta,$$

where $\boldsymbol{Q}^\theta = \boldsymbol{F}_\theta(0)$ is the intensity matrix of J under $\mathbf{P}^\theta$. Writing the limit in (35) with a change of variables, we have

$$\lim_{u\to\infty} u^{(\theta/\alpha-1)} \int_0^u \mathbf{P}_{0,i}(I > s)\mathrm{d}s = v_i(\theta) \sum_{j\in E} \frac{\pi_j^\theta}{\mu_\theta|\alpha-\theta|v_j(\theta)} \mathbf{E}_{0,j}[I^{\theta/\alpha-1}],$$

which shows, for each i, regular variation of the integral on the left-hand side. Appealing to the monotone density theorem for regularly varying functions, we now conclude that

$$\mathbf{P}_{0,i}(I > u) \sim u^{-\theta/\alpha} v_i(\theta) \sum_{j\in E} \frac{\pi_j^\theta}{\mu_\theta|\alpha-\theta|v_j(\theta)} \mathbf{E}_{0,j}[I^{\theta/\alpha-1}], \qquad u \to \infty,$$

and the result for the case that $\theta < \alpha$ now follows from (16).

The proof for the case $\theta > \alpha$ is completed by starting the reasoning as with the case of $\theta < \alpha$ but with $A = (1, \infty)$ in (33). The desired asymptotics again comes from the first term on the right-hand side of (33) using a similar application of the Markov Additive Renewal Theorem A.1. The details are left to the reader. The second term on the right-hand side of (33) becomes negligible since

$$\lim_{t\to\infty} \mathrm{e}^{(\theta-\alpha)t}\mathbf{P}_{0,j}(I > \mathrm{e}^{\alpha t}) = 0$$

on account of the fact that $\mathbf{E}_{0,i}[I^{\theta/\alpha-1}] < \infty$.

The case that $\alpha = \theta$ is dealt with similarly by starting from (33) but now setting $A = (1, \lambda)$ for some $\lambda > 1$. In that case, the second term on the right-hand side of (33) makes no contribution to the limit in question since

$$\lim_{t\to\infty} \mathbf{P}_{0,j}(I > \mathrm{e}^{\alpha t}) = 0.$$

The integral in the first term on the right-hand side of (33) can be written in the form

$$\begin{aligned}&\int_{\mathbb{R}} R_{i,j}^\theta(\mathrm{d}y)\mathbf{P}_{0,j}(I \in A\mathrm{e}^{\alpha(t-y)})\\ &\quad = \int_{\mathbb{R}} \mathbf{P}_{0,j}(I \in \mathrm{d}v) R_{i,j}^\theta(t - \alpha^{-1}\log v, t - \alpha^{-1}\log v + \alpha^{-1}\log\lambda).\end{aligned}$$

Thanks to Lemma 3.5 of [1], we have the uniform estimate

$$\sup_{x\in\mathbb{R}} R_{i,j}^\theta(x, x+\alpha^{-1}\log\lambda) \le \pi_j^\theta R_{i,i}^\theta(-\alpha^{-1}\log\lambda, \alpha^{-1}\log\lambda).$$

This result is accompanied by the classical form of the Markov Additive Renewal Theorem (c.f Theorem 3.1 of [1]), which states that

$$\lim_{x\to\infty} R_{i,j}^\theta(x, x+\alpha^{-1}\log\lambda) = \pi_j^\theta \frac{\log\lambda}{\alpha\mu_\theta}.$$

This allows us to apply the dominated convergence and note, in conjunction with the classical form of the Markov Additive Renewal Theorem (c.f Theorem 3.1 of [1]) that

$$\lim_{t\to\infty} \int_{\mathbb{R}} R_{i,j}^\theta(\mathrm{d}y)\mathbf{P}_{0,j}(I \in A\mathrm{e}^{\alpha(t-y)}) = \pi_j^\theta \frac{\log\lambda}{\alpha\mu_\theta}.$$

Plugging this limit back into the first term on the right-hand side of (33) provides the necessary convergence to complete the proof in the same way as the previous two cases. The details are again left to the reader. □

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Appendix. Markov additive renewal theory

Consider a discrete-time stochastic process described by the pair $(\Delta, M) := ((\Delta_n, M_n))_{n\geq 0}$, where Δ_n takes real (or just positive) values and M_n takes values in the set $E := \{1, 2, \ldots, N\}$. We shall specify the law of such a process as follows.

Set $\Delta_0 = 0$. For each $i, j \in E$, there is a probability distribution $P_{i,j}(x)$ such that, conditioning on the history of (Δ, M) up to time $n-1$, the distribution of (Δ_n, M_n) is given by

$$\mathrm{P}(M_n = j, \Delta_n \leq x | (M_k, X_k), k = 0, \ldots, n-1) = P_{M_{n-1},j}(x).$$

In this sense, we have that the process $M = \{M_n : n \geq 0\}$ alone is a Markov chain on E with transition matrix $p_{i,j} := P_{i,j}(\infty)$, for $i, j \in E$. The possibility that $p_{ii} > 0$ is not excluded here.

The distribution of Δ_n only depends on the state at time $n-1$ which makes the discrete-time Markov additive process

$$\Xi_n := \sum_{k=0}^{n} \Delta_k, \qquad n \geq 0,$$

the analogue of a Markov additive random walk (or Markov additive renewal process if the increments are all positive).

To state a classical renewal result for discrete-time Markov additive processes, we need to introduce a little more notation. The mean transition is given by

$$\eta_i = \sum_{j\in E} \int_{\mathbb{R}} x P_{i,j}(\mathrm{d}x), \qquad i \in E.$$

Moreover, the measure $R_{i,j}$ denotes the occupation measure

$$R_{i,j}(x) = \sum_{n=1}^{\infty} \mathrm{P}(\Xi_n \leq x,\ M_n = j | M_0 = i).$$

The following discrete-time Markov additive renewal theorem is lifted from Proposition 9.3 in [17].

Theorem A.1 (*Markov Additive Renewal Theorem*)**.** *Given a sequence of functions $g_1, g_2, \ldots, g_N$ that are directly Riemann integrable, we have, for $j \in E$,*

$$\lim_{t\to\infty} \int_{\mathbb{R}} g_j(t-s) R_{i,j}(\mathrm{d}s) = \frac{\pi_j \int_0^\infty g_j(y)\mathrm{d}y}{\sum_{j=1}^{N} \pi_j \eta_j}, \tag{A.1}$$

as soon as $\sum_{j=1}^{N} \pi_j \eta_j \in (0, \infty)$, where π_i is the stationary distribution for the chain M.

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