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Quasi-stationary distributions for Lévy processes

A.E. KYPRIANOU[1] and Z. PALMOWSKI[2,3]

[1]*School of Mathematical and Computer Sciences, Heriot-Watt University, Edinburgh EH14 4AS, UK. E-mail: kyprianou@ma.hw.ac.uk*
[2]*Mathematical Institute, University of Wroclaw, pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland. E-mail: zpalma@math.uni.wroc.pl*
[3]*Mathematical Institute, Utrecht University, P.O. Box 80 010, 3508 TA Utrecht, Netherlands. E-mail: palmowski@math.uu.nl*

In recent years there has been some focus in work by Bertoin, Chaumont and Doney on the behaviour of one-dimensional Lévy processes and random walks conditioned to stay positive. The resulting conditioned process is transient. In earlier literature, however, one encounters for special classes of random walks and Lévy processes a similar, but nonetheless different, type of asymptotic conditioning to stay positive which results in a limiting quasi-stationary distribution. We extend this theme into the general setting of a Lévy process fulfilling certain types of conditions which are analogues of known classes in the random walk literature. Our results generalize those of E.K. Kyprianou for special types of one-sided compound Poisson processes with drift and of Martínez and San Martín for Brownian motion with drift, and complement the results due to Iglehart, Doney, and Bertoin and Doney for random walks.

Keywords: conditioning; fluctuation theory; Lévy processes; quasi-stationary distribution

1. Introduction

Denote by $X = \{X_t : t \geqslant 0\}$ a Lévy process defined on the filtered space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where the filtration $\mathbb{F} = \{\mathcal{F}_t : t \geqslant 0\}$ is assumed to satisfy the usual assumptions of right continuity and completion. We work with the probabilities $\{P_x : x \in \mathbb{R}\}$ such that $P_x(X_0 = x) = 1$ and $P_0 = P$. The probabilities $\{\hat{P}_x : x \in \mathbb{R}\}$ are defined in a similar sense for the dual process, $-X$.

Define the first passage time into the lower half-line $(-\infty, 0)$ by

$$\tau = \inf\{t > 0 : X_t < 0\}.$$

In this short paper, the principal object of interest is the existence and characterization of the so-called limiting quasi-stationary distribution (or Yaglom's limit)

$$\lim_{t\uparrow\infty} P_x(X_t \in B|\tau > t) = \mu(B) \tag{1}$$

for $B \in \mathcal{B}([0, \infty))$ where, in particular, μ does not depend on the initial state $x \geqslant 0$. The sense in which this limit is quasi-stationary follows the classical interpretations of works such as Seneta and Vere-Jones (1966), Tweedie (1974), Jacka and Roberts (1995) and other references therein. We consider the limit (1) for $x \geqslant 0$ in the case of Lévy processes for which $(-\infty, 0)$ is irregular for 0, and for $x > 0$ in the case of Lévy processes for which $(-\infty, 0)$ is regular for 0. Note that the asymptotic conditioning in (1) is different but nonetheless closely related to the recent work of Bertoin (1993, 1996), Bertoin and Doney (1994), Chaumont (1996) and Chaumont and Doney (2005).

The existence and characterization result we obtain improves on the same result for spectrally positive compound Poisson processes with negative drift and jump distribution whose characteristic function is a rational function proved by Kyprianou (1971) (within the context of the $M/G/1$ queue), as well as on the same result for Brownian motion with drift proved by Martínez and San Martín (1994), and complements an analogous result for random walks obtained by Iglehart (1974) as well as related results due to Doney (1989).

In the next section we state the main result and give some special examples. Following that, in Section 3 we give two preparatory lemmas and then the proof of the main result.

2. Main result

In order to state and prove the main result it is necessary to introduce some notation. The method we appeal to requires the use of the classical exponential change of measure. Hence, it is necessary to introduce the Laplace exponent of X given by the relation

$$\mathrm{E}(\mathrm{e}^{\theta X_t}) = \mathrm{e}^{\psi(\theta)t}$$

whenever it is well defined. Note that in that case $\psi(\theta) = -\Psi(-\mathrm{i}\theta)$, where $\Psi(\lambda) = -\log \mathrm{E}(\mathrm{e}^{\mathrm{i}\lambda X_1})$ is the Lévy–Khinchine exponent. Whenever $|\psi(\theta)|$ is well defined and finite, the exponential change of measure referred to above takes the form

$$\left.\frac{\mathrm{d}P_x^\theta}{\mathrm{d}P_x}\right|_{\mathcal{F}_t} = \mathrm{e}^{\theta(X_t - x) - \psi(\theta)t}.$$

It is well known that under this change of measure X is still a Lévy process and its new Laplace exponent satisfies

$$\psi_\theta(\beta) = \psi(\theta + \beta) - \psi(\theta) \tag{2}$$

whenever $\psi(\theta + \beta)$ is well defined. One also sees from the latter equality that under the change of measure, the new Lévy measure satisfies $\Pi_\theta(\mathrm{d}x) = \mathrm{e}^{\theta x}\Pi(\mathrm{d}x)$.

We also need to recall some standard notation from the theory of fluctuations of Lévy processes; see, for example, Bertoin (1996). The process $(L^{-1}, H) = \{(L_t^{-1}, H_t)\colon t \geqslant 0\}$ denotes the ladder process of X and is characterized by its Laplace exponent $\kappa(\alpha, \beta)$ satisfying

$$\mathrm{E}(\mathrm{e}^{-\alpha L_t^{-1} - \beta H_t}) = \mathrm{e}^{-\kappa(\alpha,\beta)t}$$

for $\alpha, \beta \geqslant 0$. The latter exponent and the analogous quantity $\hat{\kappa}(\alpha, \beta)$ defined for the dual $-X$ are related to the renewal function

$$V(x) = \mathrm{E}\left(\int_0^\infty 1_{(H_t \leqslant x)}\, \mathrm{d}t\right)$$

via the Laplace transform

$$\lambda \int_0^\infty \mathrm{e}^{-\lambda x} V(x)\mathrm{d}x = \frac{1}{\kappa(0, \lambda)}. \tag{3}$$

The quantities $\hat{H}$, $\hat{V}$ and $\hat{\kappa}$ associated with $-X$ can also be identified in the obvious way. Further, κ_θ, $\hat{\kappa}_\theta$, V_θ and $\hat{V}_\theta$ all correspond to the aforementioned when working under the measure P^θ.

The main result will apply to two classes of Lévy process. We call them class A and class B, and the definitions are given below. The choice of name for each class allows us to keep some consistency with the existing literature. Analogues of classes A and B were defined for random walks in Bertoin and Doney (1996).

Before formally giving the definitions, recall that it can easily be checked from the Lévy–Khinchine formula that when $\theta_r = \sup\{\theta > 0 : |\psi(\theta)| < \infty\}$, ψ is strictly convex on $(0, \theta_r)$ and by monotonicity $\psi(\theta_r) = \psi(\theta_r -)$ and $\psi'(\theta_r) = \psi'(\theta_r -)$ are well defined. Also the right derivative at zero, $\psi'(0) = \psi'(0+) = \mathrm{E}(X_1)$.

Definition 1. *A Lévy process belongs to class A if*

(i) *there exists a Laplace exponent* ψ *for* $0 < \theta < \theta_r$*, where* $\theta_r > 0$*;*
(ii) $\psi(\theta)$ *attains its negative infimum at* $0 < \theta_0 \leqslant \theta_r$*, with* $\psi'(\theta_0) = 0$*;*
(iii) X_1 *is non-lattice; and*
(iv) (X, P^{θ_0}) *is in the domain of attraction of a stable law with index* $1 < \gamma \leqslant 2$.

Definition 2. *A Lévy process belongs to class B if*

(i) *there exists a Laplace exponent* ψ *for* $0 \leqslant \theta \leqslant \theta_r$*, where* $\theta_r \in (0, \infty)$*;*
(ii) $\psi(\theta)$ *attains its negative infimum at* $\theta_0 = \theta_r$*;*
(iii) $-\mathrm{E}^{\theta_0}(X_1) = -\psi'(\theta_0) \in (0, \infty)$*;*
(iv) X_1 *is non-lattice; and*
(v) *the function* $x \mapsto \Pi_{\theta_0}(x, \infty)$ *is regularly varying at infinity with index* $-\beta$*, where* $\beta \in (2, \infty)$.

A number of important points that follow from these assumptions are now discussed. Firstly, the assumption that X_1 is non-lattice is not absolutely necessary for the results in this paper. Analogues of the required analysis of the Lévy processes in classes A and B can in principle be reconstructed. To understand how, the reader is referred to the analogous definitions for random walks given in Bertoin and Doney (1996) as well as the respective results which are taken from that paper in the analysis here.

Secondly, in class A, it is implicit in the definition that $\mathrm{E}(X_1) = \psi'(0+) < 0$ and hence the process drifts to $-\infty$. Under the change of measure P^{θ_0}, the process X oscillates as, according to (2) and the definition of a_0, ${\psi'}_{\theta_0}(0-) = \psi'(\theta_0 -) = 0$. Note also that if (X, P) belongs to class A then so does (X, P^α) for any $\alpha \in [0, \theta_0)$. This is easy to check, remembering that under P^{α} the negative infimum of ψ_{α} is obtained at $\theta_0^{\alpha} = \theta_0 - \alpha$ (a picture of ψ will help to make this obvious).

Thirdly, in class B it is also implicit in the definition that $\mathrm{E}(X_1) = \psi'(0) < 0$. Like class A, it is again true that if (X, P) belongs to class B then for any $\alpha \in [0, \theta_0)$ the process (X, P^{α}) also belongs to the same class. To deal with the fifth condition, recall again that $\theta_0^{\alpha} = \theta_0 - \alpha$ plays the role of θ_0 under P^{α}. From the earlier-mentioned effect of the change of measure $\Pi^{\alpha}(\mathrm{d}x) = \mathrm{e}^{\alpha x}\Pi(\mathrm{d}x)$, it is now easily checked that (v) holds. Essentially it is a statement about (X, P^{θ_0}) and hence independent of α.

Finally, recall from the Wiener–Hopf factorization that (up to a multiplicative constant) for all $\theta \in \mathbb{R}$,

$$\Psi(\theta) = \kappa(0, -\mathrm{i}\theta)\hat{\kappa}(0, \mathrm{i}\theta). \tag{4}$$

In both class A and B, since $\Psi(-\mathrm{i}\theta)/\hat{\kappa}(0, \theta)$ is well defined for all $\theta \in [0, \theta_r)$, it follows that $\kappa(0, \theta)$ may be analytically extended to $(-\theta_r, \infty)$.

We now state the main result of this paper.

Theorem 1. *If X is a Lévy process of class A or B then there exists a quasi-stationary distribution* (1) *satisfying*

$$\mu(\mathrm{d}y) = \theta_0\kappa_{\theta_0}(0, \theta_0)\mathrm{e}^{-\theta_0 y}V_{\theta_0}(y)\mathrm{d}y$$

on $[0, \infty)$. *In particular,* $\mu(\{0\}) = 0$. *The existence of* (1) *holds for all initial states* $x > 0$ *when* 0 *is regular for* $(-\infty, 0)$ *and for all* $[0, \infty)$ *when* 0 *is irregular for* $(-\infty, 0)$.

Below we give three special cases of this theorem.

Example 1 *Spectrally negative processes*. Suppose that X is spectrally negative and drifts to $-\infty$. (Note that we exclude negative subordinators from the definition of spectral negativity.) It falls into class A as $|\psi(\theta)|$ is finite for all $\theta \geqslant 0$ (cf. Chapter VII of Bertoin 1996). In this case it is well known that when the process oscillates the ascending ladder height process is a linear drift with unit rate. We thus have that $\kappa_{\theta_0}(0, \theta_0) = \theta_0$ and $V_{\theta_0}(x) = x$. It therefore follows that

$$\mu(\mathrm{d}x) = \theta_0^2 x\mathrm{e}^{-\theta_0 x}\,\mathrm{d}x.$$

Example 2 *Brownian motion with drift.* This example is a continuation of the previous one as Brownian motion with drift, having no jumps at all, is spectrally negative. Necessarily, to be in class A the drift must be negative. Hence $\psi(\theta) = \frac{1}{2}\theta^2 - \rho\theta$ for some $\rho > 0$. A straightforward exercise shows that $\theta_0 = \rho$ and, hence,

$$\mu(\mathrm{d}x) = \rho^2 x\mathrm{e}^{-\rho x}\,\mathrm{d}x.$$

This result was first established in Martínez and San Martín (1994).

Example 3 *Spectrally positive processes*. In the case of spectral positivity one may consider either class A or class B. We see that in both cases, (X, P^{θ_0}) either drifts to $-\infty$ or oscillates.

This implies that $\hat{\kappa}_{\theta_0}(0, \theta) = \theta$ for all $\theta \geqslant 0$. Hence, from the Wiener–Hopf factorization (4) applied to (X, P^{θ_0}), it follows that

$$\psi_{\theta_0}(\theta) = -\theta\kappa_{\theta_0}(0, -\theta) \tag{5}$$

for all $\theta \leqslant \theta_r - \theta_0$. As will transpire from the proof, the result is established by proving that for $0 < \alpha < \theta_0$,

$$\int_{[0,\infty)} \mathrm{e}^{\alpha x}\mu(\mathrm{d}x) = \frac{\theta_0\kappa_{\theta_0}(0, \theta_0)}{(\theta_0 - \alpha)\kappa_{\theta_0}(0, \theta_0 - \alpha)}, \tag{6}$$

from which the density given in Theorem 1 can be identified. Note that since ψ is finite on $(-\infty, 0]$, equality (6) can be analytically extended to negative values of α. Using (2) and (5) we see that the left-hand side of (6) is also equal to

$$\frac{\psi_{\theta_0}(-\theta_0)}{\psi_{\theta_0}(\alpha - \theta_0)} = \frac{-\psi(\theta_0)}{\psi(\alpha) - \psi(\theta_0)}.$$

Kyprianou (1971) found the Laplace transform of the quasi-stationary distribution for the workload process of the stable $M/G/1$ queue whose service times were distributed with a rational moment generating function. In this case, it is not difficult to show that X belongs to class A or B. The workload process whilst the queue is busy is just a compound Poisson process with negative drift having Laplace exponent equal to

$$\psi(\theta) = -\theta + \lambda(\hat{m}_B(\theta) - 1),$$

where B is the distribution of the service times and $\hat{m}_B(\theta) = \mathrm{E}(\mathrm{e}^{\theta B})$.

Suppose now that X belongs to class A. Note that $\theta_0 > 0$ solves $\lambda\hat{m}'_B(\theta_0) = 1$. The Laplace transform for this process is

$$\int_{[0,\infty)} \mathrm{e}^{-\beta x}\mu(\mathrm{d}x) = \frac{-\theta_0 + \lambda(\hat{m}_B(\theta_0) - 1)}{-\theta_0 - \beta + \lambda(\hat{m}_B(\theta_0) - \hat{m}_B(\beta))}.$$

This agrees with the expression given in Theorem 2a(ii) of Kyprianou (1971) but now for a more general class of Lévy processes.

3. Proof of main result

The proof of Theorem 1 requires two preparatory lemmas.

Lemma 2. *For all sufficiently large $\vartheta > \theta_0$ the following hold.*

(i) *If X belongs to class A, then*

$$P(X_t \geqslant 0) + \mathrm{E}[\mathrm{e}^{\vartheta X_t}; X_t < 0] \sim \frac{\vartheta}{\vartheta - \theta_0}\frac{g_\gamma(0)}{\theta_0} t^{-1/\gamma} L_{\mathrm{A}}(t)\mathrm{e}^{\psi(\theta_0)t} \tag{7}$$

as $t \uparrow \infty$ where L_{A} is slowly varying and g_γ is a continuous version of the density of the stable law mentioned in the definition of class A.

(ii) *If X belongs to class B, then*

$$P(X_t \geqslant 0) + \mathrm{E}[\mathrm{e}^{\vartheta X_t}; X_t < 0] \sim \frac{\vartheta}{(\vartheta - \theta_0)} \frac{|\psi'(\theta_0)|^{-\beta}}{\theta_0} t^{-(\beta-1)} L_{\mathrm{B}}(t) \mathrm{e}^{\psi(\theta_0)t}, \tag{8}$$

where L_{B} is slowly varying as $t \uparrow \infty$.

Proof. For each $h > 0$ and within class A or B we consider a random walk $S^h = \{S_n^h : n \geqslant 0\}$ where

$$S_n^h = \sum_{i=1}^{n} X_i, \qquad n \geqslant 0,$$

and $X_i = X(ih) - X((i-1)h)$ are obviously independent and identically distributed random variables having moment generating function

$$f_h(\theta) = \mathrm{E}\mathrm{e}^{\theta S_1^h} = \mathrm{E}(\mathrm{e}^{\theta X_h}) = \mathrm{e}^{\psi(\theta)h}$$

for $\theta \in [0, \theta_r)$.

(i) When X belongs to class A, it is easy to check that for each $h > 0$, S^h belongs to class A for random walks defined in Bertoin and Doney (1996). Following the arguments in Lemma 2.6 of Getoor and Sharpe (1994), we have that $P(X_t \geqslant 0) + \mathrm{E}(\mathrm{e}^{\vartheta X_t}; X_t < 0)$ is always a right or left continuous function (in fact it is continuous if X is not a compound Poisson process with drift). Hence, we may apply Croft's lemma (see Croft 1957; and Corollary 2 of Kingman 1963); that is to say, if there exists a continuous slowly varying function L_{A} which does not depend on h such that

$$P(S_n^h \geqslant 0) + \mathrm{E}(\mathrm{e}^{\vartheta S_n^h}; S_n^h < 0) \sim \frac{\vartheta}{\vartheta - \theta_0} \frac{g_\gamma(0)}{\theta_0} (nh)^{-1/\gamma} L_{\mathrm{A}}(nh) \mathrm{e}^{\psi(\theta_0)nh} \tag{9}$$

as $n \uparrow \infty$ for each $h > 0$, then (7) holds.

Note that by virtue of the fact that X is in class A, there exists a slowly varying function, L_{A}, such that $X_t/(t^{1/\gamma} L_{\mathrm{A}}(t))$ converges in distribution under P^{θ_0} as $t \uparrow \infty$ to a stable random variable with density g_γ. Hence, the same limit holds on any subsequence. This in turn implies that the random walk S^h belongs to class A of random walks as defined in Bertoin and Doney (1996). The limit in (9) follows directly from Lemma 4(i) of Bertoin and Doney (1996). According to Theorem 1.3.3 of Bingham *et al.* (1987), we may choose L_{A} to be continuous.

(ii) To see that S^h also belongs to class B for random walks as defined in Bertoin and Doney (1996) for each $h > 0$ when X belongs to class B, we argue as follows. From a classic result of Embrechts *et al.* (1979) it now follows from the assumed regular variation of $\Pi_{\theta_0}(\cdot, \infty)$ that $P^{\theta_0}(X_h > x) \sim h\Pi_{\theta_0}(x, \infty)$ as $x \uparrow \infty$. Now from Lemma 4(ii) of Bertoin and Doney (1996) we infer that

$$P(S_n^h \geqslant 0) + \mathrm{E}(\mathrm{e}^{\vartheta S_n^h}; S_n^h < 0) \sim \frac{\vartheta}{\vartheta - \theta_0} \frac{|\psi'(\theta_0)|^{-\beta}}{\theta_0} (nh)^{-(\beta-1)} L_{\mathrm{B}}(nh) \mathrm{e}^{\psi(\theta_0)nh}$$

as $n \uparrow \infty$. The remaining part of the proof essentially mimics the proof of (i) using Croft's lemma. $\square$

Lemma 3. *For $\alpha \in [0, \theta_0)$ the following two cases hold.*

(i) *If X belongs to class A then we have that $|\kappa(\psi(\theta_0), -\alpha)|$ is finite and, for all $x \geqslant 0$,*

$$\mathrm{E}_x(\mathrm{e}^{\alpha X_t}; \tau > t) \sim \mathrm{e}^{\theta_0 x} \hat{V}_{\theta_0}(x) \frac{1}{\kappa(\psi(\theta_0), -\alpha)} \frac{g_\gamma(0)}{\theta_0 - \alpha} t^{-(1/\gamma+1)} L_{\mathrm{A}}(t) \mathrm{e}^{\psi(\theta_0)t}$$

as $t \uparrow \infty$, where L_A is the same as in Lemma 2(i).

(ii) *If X belongs to class B then we have that $|\kappa(\psi(\theta_0), -\alpha)|$ is finite and, for all $x \geqslant 0$,*

$$\mathrm{E}_x(\mathrm{e}^{\alpha X_t}; \tau > t) \sim \mathrm{e}^{\theta_0 x} \hat{V}_{\theta_0}(x) \frac{1}{\kappa(\psi(\theta_0), -\alpha)} \frac{|\psi'(\theta_0)|^{-\beta}}{\theta_0 - \alpha} t^{-\beta} L_{\mathrm{B}}(t) \mathrm{e}^{\psi(\theta_0)t}$$

as $t \uparrow \infty$, where L_{B} is the same as in Lemma 2(ii).

Note that if 0 is regular for $(-\infty, 0)$ then the descending ladder height process is not a compound poisson process, and hence $\hat{V}_{\theta_0}(0) = 0$ consistently with the expectations mentioned in the above lemma for $x = 0$. As indicated earlier, in the regular case, we only consider quasi-stationary laws for $x > 0$. If, on the other hand, 0 is irregular for $(-\infty, 0)$ then the descending ladder height process is a compound Poisson process and hence $\hat{V}_{\theta_0}(0) > 0$, so that the statement of the above lemma for $x = 0$ is non-trivial.

Proof of Lemma 3. Let $\mathbf{e}_\eta$ be an exponentially distributed random variable with parameter $\eta = q + \psi(\theta_0) > 0$ which is independent of X. Denote $\overline{X}_t = \sup_{s \leqslant t} X_s$. From the Wiener–Hopf factorization (which gives the independence of $\overline{X}_{\mathbf{e}_\eta}$ and $X_{\mathbf{e}_\eta} - \overline{X}_{\mathbf{e}_\eta}$) and duality (which gives the equality in distribution of $\overline{X}_{\mathbf{e}_\eta}$ under $\hat{P}$ and $\overline{X}_{\mathbf{e}_\eta} - X_{\mathbf{e}_\eta}$ under P) we obtain

$$\int_0^\infty \mathrm{e}^{-\eta t} \mathrm{E}_x[\mathrm{e}^{\alpha X_t}; \tau > t] \mathrm{d}t$$

$$= \frac{1}{\eta} \int_{[0,\infty)} \mathrm{e}^{\alpha y} P(\overline{X}_{\mathbf{e}_\eta} \in \mathrm{d}y) \cdot \mathrm{e}^{\alpha x} \int_{[0,x]} \mathrm{e}^{-\alpha z} \hat{P}(\overline{X}_{\mathbf{e}_\eta} \in \mathrm{d}z).$$

(The last calculation is essentially taken from the proof of Theorem VI.20 of Bertoin 1996: 176). From Theorem VI.5 of Bertoin (1996: 160), we then have that

$$\int_0^\infty \mathrm{e}^{-\eta t} \mathrm{E}_x[\mathrm{e}^{\alpha X_t}; \tau > t] \mathrm{d}t = \frac{1}{\eta} \frac{\kappa(\eta, 0)}{\kappa(\eta, -\alpha)} \mathrm{e}^{\alpha x} \int_{[0,x]} \mathrm{e}^{-\alpha z} \hat{P}(\overline{X}_{\mathbf{e}_\eta} \in \mathrm{d}z). \tag{10}$$

Taking the Laplace transforms of both sides of (10) with respect to x, we obtain with the help of Fubini's theorem that

$$\int_0^\infty \int_0^\infty \mathrm{e}^{-\eta t} \mathrm{e}^{-\theta x} \mathrm{E}_x[\mathrm{e}^{\alpha X_t}; \tau > t] \mathrm{d}x \, \mathrm{d}t = \frac{1}{\eta} \frac{1}{\theta - \alpha} \frac{\kappa(\eta, 0)}{\kappa(\eta, -\alpha)} \frac{\hat{\kappa}(\eta, 0)}{\hat{\kappa}(\eta, \theta)} \tag{11}$$

for $\theta > \alpha$.

It is known that one can put $\exp\{\int_0^\infty (\mathrm{e}^{-t} - \mathrm{e}^{-\eta t}) t^{-1} P(X_t = 0) \mathrm{d}t\}$ into the normalization constant of the product $\hat{\kappa}(\eta, -\alpha)\hat{\kappa}(\eta, \theta)$ (which itself is present as a consequence of an arbitrary normalization of local time at the maximum) and hence, with the help of the expressions for κ and $\hat{\kappa}$ given by Bertoin 1996: 166), we may write $s = \kappa(s, 0)\hat{\kappa}(s, 0)$ and

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-\eta t}\left(\int_0^\infty \mathrm{e}^{-\theta x}\mathrm{E}_x[\mathrm{e}^{\alpha X_t}; \tau > t]\mathrm{d}x\right)\mathrm{d}t \\
&= \frac{1}{\theta-\alpha}\frac{1}{\kappa(\eta, -\alpha)\hat{\kappa}(\eta, \theta)} \\
&= \frac{1}{\theta-\alpha}\exp\left\{-\int_0^\infty \frac{\mathrm{e}^{-t} - a_t(\theta)\mathrm{e}^{-(q+\psi(\theta_0))t}}{t}\,\mathrm{d}t\right\},
\end{aligned} \tag{12}$$

where

$$a_t(\theta) = \mathrm{E}[\mathrm{e}^{\alpha X_t}; X_t \geqslant 0] + \mathrm{E}[\mathrm{e}^{\theta X_t}; X_t < 0].$$

For the right-hand side of (12) we have used the integral representations of κ and $\hat{\kappa}$ in terms of the distribution of X; see Corollary VI.10 in Bertoin (1996: 165). By exponential change of measure we have

$$a_t(\theta) = \mathrm{e}^{\psi(\alpha)t}\big(P^\alpha(X_t \geqslant 0) + \mathrm{E}^\alpha[\mathrm{e}^{(\theta-\alpha)X_t}; X_t < 0]\big).$$

(i) Now assume that (X, P) belongs to class A and recall that for $\alpha \in [0, \theta_0)$ the Lévy process (X, P^α) remains in the same class as assumed, and hence Lemma 2(i) may be applied to the latter process. In that case one should replace ψ by ψ_α and θ_0 by $\theta_0^\alpha := \theta_0 - \alpha$. Importantly, one should also note that the limiting density g_γ appearing in the definition of class A applied to (X, P^α) does not depend on $\alpha \in [0, \theta_0)$ as it concerns the distribution of (X, P^{θ_0}).

Taking note of (2), it follows from Lemma 2(i) that for all sufficiently large $\theta > \theta_0$,

$$a_t(\theta) \sim \frac{\theta-\alpha}{\theta-\theta_0}\frac{g_\gamma(0)}{\theta_0-\alpha}t^{-1/\gamma}L_{\mathrm{A}}(t)\mathrm{e}^{\psi(\theta_0)t}. \tag{13}$$

as $t \uparrow \infty$. This implies that the right-hand side of (12) is finite for $q \geqslant 0$. In particular, when $q = 0$ one may check that in the integral $\int_0^\infty(\mathrm{e}^{-t} - a_t(\theta)\mathrm{e}^{-(q+\psi(\theta_0))t})t^{-1}\,\mathrm{d}t$ the integrand behaves like $O(1)$ as $t \downarrow 0$ and $O(t^{-(1+1/\gamma)})$ as $t \uparrow \infty$. It is also the case that $|\hat{\kappa}(\psi(\theta_0), \theta)| < \infty$. To see this, note that $\liminf_{t\uparrow\infty} X_t = -\infty$ and hence $\hat{L}_1^{-1}$ is an almost surely finite stopping time. We may use Doob's optional stopping theorem (cf. Jacod and Shiryaev 2003, Theorem III.3.4) and the fact that $\hat{\kappa}(0, \theta-\theta_0) < \infty$ to deduce that

$$\begin{aligned}
\mathrm{e}^{-\hat{\kappa}_{\theta_0}(0,\theta-\theta_0)} &= \mathrm{E}^{\theta_0}(\mathrm{e}^{-(\theta-\theta_0)\hat{H}_1}) \\
&= \mathrm{E}(\mathrm{e}^{\theta_0 X_{\hat{L}_1^{-1}} - \psi(\theta_0)\hat{L}_1^{-1}}\mathrm{e}^{-(\theta-\theta_0)\hat{H}_1}) \\
&= \mathrm{E}(\mathrm{e}^{-\psi(\theta_0)\hat{L}_1^{-1} - \theta\hat{H}_1}) \\
&= \mathrm{e}^{-\hat{\kappa}(\psi(\theta_0),\theta)},
\end{aligned} \tag{14}$$

proving the last claim. Since the right-hand side of (12) for $q = 0$ is finite and, from Corollary VI.10 of Bertoin (1996), is equal to $\lim_{q\downarrow 0}\kappa(q+\psi(\theta_0), -\alpha)\hat{\kappa}(q+\psi(\theta_0), \theta)$, and since $|\hat{\kappa}(\psi(\theta_0), \theta)|$ is finite, it follows that $|\kappa(\psi(\theta_0), -\alpha)| < \infty$.

Now since both sides of (12) are analytical functions in the variable $q > 0$, the identity can be extended for this parameter range. Differentiating equation (12) with respect to q yields

$$\int_0^\infty e^{-qt}e^{-\psi(\theta_0)t}t\left(\int_0^\infty e^{-\theta x}E_x[e^{\alpha X_t}; \tau > t]dx\right)dt$$

$$= \frac{1}{\theta-\alpha}\exp\left\{-\int_0^\infty \frac{e^{-t} - a_t(\theta)e^{-(q+\psi(\theta_0))t}}{t}dt\right\}\int_0^\infty e^{-qt}e^{-\psi(\theta_0)t}a_t(\theta)dt.$$

Now taking $q \downarrow 0$, we obtain from (13) and the converse to the monotone density theorem – which does not require a monotone density (see Feller 1971, Section XIII.5: 446) – that

$$\int_0^\infty e^{-qt}e^{-\psi(\theta_0)t}t\left(\int_0^\infty e^{-\theta x}E_x[e^{\alpha X_t}; \tau > t]dx\right)dt$$

$$\sim \frac{1}{\theta-\alpha}\frac{1}{\kappa(\psi(\theta_0), -\alpha)\hat{\kappa}(\psi(\theta_0), \theta)}\Gamma\left(1-\frac{1}{\gamma}\right)\frac{\theta-\alpha}{\theta-\theta_0}\frac{1}{\theta_0-\alpha}g_\gamma(0)q^{1/\gamma-1}L_A(1/q),$$

and hence

$$\int_0^\infty e^{-\theta x}E_x[e^{\alpha X_t}; \tau > t]dx$$

$$\sim \frac{1}{(\theta-\theta_0)(\theta_0-\alpha)}\frac{1}{\kappa(\psi(\theta_0), -\alpha)\hat{\kappa}(\psi(\theta_0), \theta)}g_\gamma(0)t^{-(1/\gamma+1)}L_A(t)e^{\psi(\theta_0)t}. \tag{15}$$

From (3) we have

$$\int_0^\infty e^{-\theta x}e^{\theta_0 x}\hat{V}_{\theta_0}(x)dx = \frac{1}{\theta-\theta_0}\frac{1}{\hat{\kappa}_{\theta_0}(0, \theta-\theta_0)},$$

$$= \frac{1}{\theta-\theta_0}\frac{1}{\hat{\kappa}(\psi(\theta_0), \theta)}, \tag{16}$$

where the second equality follows from (14).

To conclude, let $\underline{X}_t = \inf_{s\leqslant t}X_s$ and observe that $E_x(e^{\alpha X_t}; \tau > t) = E_x(e^{\alpha X_t}; \underline{X}_t \geqslant 0)$ is right continuous in x on $[0, \infty)$ by temporal right continuity of the bivariate process $(X, \underline{X})$. In addition, from Proposition I.15 of Bertoin (1996), $\hat{V}_{\theta_0}(x)$ is continuous on $(0, \infty)$. (Note that, in the case where 0 is irregular for $(-\infty, 0)$, it is known that $\hat{H}$ is also a compound Poisson subordinator and the assumptions imply that its jump distribution is diffuse and hence $\hat{V}$ has no atoms except at zero.) Comparing (16) with (15) then, the statement of the theorem follows.

(ii) The proof for the case that (X, P) belongs to class B is essentially the same as in (i) with obvious changes using Lemma 2(i). □

We have the following corollary corresponding to the case $\alpha = 0$.

Corollary 4. *The following asymptotic results hold as $t \uparrow \infty$.*

(i) *If X belongs to class A then, for all $x \geqslant 0$,*

$$P_x(\tau > t) \sim e^{\theta_0 x}\hat{V}_{\theta_0}(x)\frac{1}{\kappa(\psi(\theta_0), 0)}\frac{g_\gamma(0)}{\theta_0}t^{-(1/\gamma+1)}L_A(t)e^{\psi(\theta_0)t}$$

as $t \uparrow \infty$.

(ii) *If X belongs to class B then, for all $x \geqslant 0$,*

$$P_x(\tau > t) \sim e^{\theta_0 x}\hat{V}_{\theta_0}(x)\frac{1}{\kappa(\psi(\theta_0), 0)}\frac{|\psi'(\theta_0)|^{-\beta}}{\theta_0}t^{-\beta}L_B(t)e^{\psi(\theta_0)t}$$

as $t \uparrow \infty$.

We are ready now to prove the main result.

Proof of Theorem 1. From Lemma 2 and Corollary 4, we obtain that, for $0 < \alpha < \theta_0$,

$$\int_{[0,\infty)} e^{\alpha y}\mu(dy) = \lim_{t\uparrow\infty}\frac{E_x(e^{\alpha X_t}; \tau > t)}{P_x(\tau > t)} = \frac{\theta_0\kappa(\psi(\theta_0), 0)}{(\theta_0 - \alpha)\kappa(\psi(\theta_0), -\alpha)}.$$

However, with a similar calculation to the one at the end of the proof of Lemma 3 we can show that

$$\kappa(\psi(\theta_0), -\alpha) = \kappa_{\theta_0}(0, \theta_0 - \alpha).$$

Further, from (3),

$$\int_0^\infty e^{\alpha x}e^{-\theta_0 x}V_{\theta_0}(x)dx = \frac{1}{\theta_0 - \alpha}\frac{1}{\kappa_{\theta_0}(0, \theta_0 - \alpha)}.$$

The statement of the main theorem now follows. □

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