On recent developments of Markov Processes and Applications

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Based on joint work with V. Rivero and W. Satitkanitkul

A more thorough set of lecture notes can be found here:
https://arxiv.org/abs/1707.04343
Other related material found here
https://arxiv.org/abs/1511.06356
https://arxiv.org/abs/1706.09924

Contents

§1. Quick review of Lévy processes

(Killed) Lévy process

- $(\xi_t, t \geq 0)$ is a (killed) Lévy process if it has stationary and independents with RCLL paths (and is sent to a cemetery state after and independent and exponentially distributed time).
- Process is entirely characterised by its one-dimensional transitions, which are coded by the Lévy–Khinchine formula [Exercise! Show that the exponent must factorise]:

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\theta\cdot\xi_t}] = \mathrm{e}^{-\Psi(\theta)t}, \qquad \theta \in \mathbb{R}^d,$$

where,

$$\Psi(\theta) = q + \mathrm{i}\mathrm{a}\cdot\theta + \frac{1}{2}\theta\cdot\mathbf{A}\theta + \int_{\mathbb{R}^d}(1 - \mathrm{e}^{\mathrm{i}\theta\cdot x} + \mathrm{i}(\theta\cdot x)\mathbf{1}_{(|x|<1)})\Pi(\mathrm{d}x),$$

where $\mathrm{a} \in \mathbb{R}$, $\mathbf{A}$ is a $d \times d$ Gaussian covariance matrix and Π is a measure satisfying $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\Pi(\mathrm{d}x) < \infty$. Think of Π as the intensity of jumps in the sense of

$$\mathbb{P}(X \text{ has jump at time } t \text{ of size } \mathrm{d}x) = \Pi(\mathrm{d}x)\mathrm{d}t + o(\mathrm{d}t).$$

- In one dimension the path of a Lévy process can be monotone, in which case it is called a *subordinator* and we work with the Laplace exponent

$$\mathbb{E}[\mathrm{e}^{-\lambda\xi_t}] = \mathrm{e}^{-\Phi(\lambda)t}, \qquad t \geq 0$$

where

$$\Phi(\lambda) = q + \delta\lambda + \int_{(0,\infty)}(1 - \mathrm{e}^{-\lambda x})\Upsilon(\mathrm{d}x), \qquad \lambda \geq 0.$$

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Lévy process: One dimension

Two examples in one dimension:

- **Stable subordinator** $(\xi_t, t \geq 0)$ is a subordinator which satisfies the additional scaling property: For $c > 0$

$$\text{under } \mathbb{P}, \text{ the law of } (c\xi_{c^{-\alpha}t}, t \geq 0) \text{ is equal to } \mathbb{P},$$

where $\alpha \in (0,1)$. We have

$$\Phi(\lambda) = \lambda^\alpha, \qquad \lambda \geq 0, \qquad \text{and} \qquad \Pi(\mathrm{d}x) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{1}{x^{1+\alpha}} \mathrm{d}x, \qquad x > 0.$$

- **Hypgergeometric Lévy process:** For $\beta \leq 1, \gamma \in (0,1), \hat{\beta} \geq 0, \hat{\gamma} \in (0,1)$

$$\Psi(\theta) = \frac{\Gamma(1-\beta+\gamma-\mathrm{i}\theta)}{\Gamma(1-\beta-\mathrm{i}\theta)} \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\mathrm{i}\theta)}{\Gamma(\hat{\beta}+\mathrm{i}\theta)} \qquad \theta \in \mathbb{R}.$$

The Lévy measure has a density with respect to Lebesgue measure which is given by

$$\pi(x) = \begin{cases} -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\hat{\gamma})\Gamma(-\gamma)} \mathrm{e}^{-(1-\beta+\gamma)x} {}_2F_1\left(1+\gamma, \eta; \eta-\hat{\gamma}; \mathrm{e}^{-x}\right), & \text{if } x > 0, \\ -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\gamma)\Gamma(-\hat{\gamma})} \mathrm{e}^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1\left(1+\hat{\gamma}, \eta; \eta-\gamma; \mathrm{e}^{x}\right), & \text{if } x < 0, \end{cases}$$

where $\eta := 1-\beta+\gamma+\hat{\beta}+\hat{\gamma}$.

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LÉVY PROCESS: ONE DIMENSION

- If ξ has a characteristic exponent Ψ then necessarily

$$\Psi(\theta) = \kappa(-\mathrm{i}\theta)\hat{\kappa}(\mathrm{i}\theta), \qquad \theta \in \mathbb{R}.$$

where κ and $\hat{\kappa}$ are Bernstein functions, e.g.

$$\kappa(\lambda) = q + \delta\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})\Upsilon(\mathrm{d}x), \qquad \lambda \geq 0.$$

- The factorisation has a physical interpretation:
  - range of the κ-subordinator agrees with the range of $\sup_{s\leq t} \xi_s$, $t \geq 0$
  - range $\hat{\kappa}$-subordinator agrees with the range of $-\inf_{s\leq t} \xi_s$, $t \geq 0$.
- Note if $\delta > 0$, then $\mathbb{P}(\xi_{\tau_x^+} = x) > 0$, where $\tau_x^+ = \inf\{t > 0 : \xi_t = x\}$, $x > 0$.
- We have already seen the hypergeometric example

$$\Psi(\theta) = \frac{\Gamma(1-\beta+\gamma-\mathrm{i}\theta)}{\Gamma(1-\beta-\mathrm{i}\theta)} \qquad \times \qquad \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\mathrm{i}\theta)}{\Gamma(\hat{\beta}+\mathrm{i}\theta)} \qquad \theta \in \mathbb{R}.$$

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Hitting points

- We say that ξ *can hit a point* $x \in \mathbb{R}$ if

$$\mathbb{P}(\xi_t = x \text{ for at least one } t > 0) > 0.$$

- Creeping is one way to hit a point, but not the only way

Theorem (Kesten (1969)/Bretagnolle (1971))

Suppose that ξ is not a compound Poisson process. Then ξ can hit points if and only if

$$\int_{\mathbb{R}} \mathrm{Re}\left(\frac{1}{1+\Psi(z)}\right) \mathrm{d}z < \infty.$$

If the Kesten-Bretagnolle integral test is satisfied, then

$$\mathbb{P}(\tau^{\{x\}} < \infty) = \frac{u(x)}{u(0)},$$

where $\tau^{\{x\}} = \inf\{t > 0 : \xi_t = x\}$, providing we can compute the inversion

$$u(x) = \int_{c+\mathrm{i}\mathbb{R}} \frac{\mathrm{e}^{-zx}}{\Psi(-\mathrm{i}z)} \mathrm{d}z$$

for some $c \in \mathbb{R}$.

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§2. Self-similar Markov processes

Self-similar Markov processes (ssMp)

Definition

A regular strong Markov process $(Z_t : t \geq 0)$ on $\mathbb{R}^d$, with probabilities $\mathbb{P}_x$, $x \in \mathbb{R}^d$, is a rssMp if there exists an index $\alpha \in (0, \infty)$ such that for all $c > 0$ and $x \in \mathbb{R}^d$,

$$(cZ_{tc^{-\alpha}} : t \geq 0) \text{ under } \mathbb{P}_x \text{ is equal in law to } (Z_t : t \geq 0) \text{ under } \mathbb{P}_{cx}.$$

Some of your best friends are ssMp

- Write $\mathcal{N}_d(\mathbf{0}, \mathbf{\Sigma})$ for the Normal distribution with mean $\mathbf{0} \in \mathbb{R}^d$ and correlation (matrix) $\mathbf{\Sigma}$. The moment generating function of $X_t \sim \mathcal{N}_d(\mathbf{0}, \mathbf{\Sigma}t)$ satisfies, for $\theta \in \mathbb{R}^d$,
$$E[\mathrm{e}^{\theta \cdot X_t}] = \mathrm{e}^{t\theta^{\mathrm{T}}\mathbf{\Sigma}\theta/2} = \mathrm{e}^{(c^{-2}t)(c\theta)^{\mathrm{T}}\mathbf{\Sigma}(c\theta)/2} = E[\mathrm{e}^{\theta \cdot cX_{c^{-2}t}}].$$
- Thinking about the stationary and independent increments of Brownian motion, this can be used to show that $\mathbb{R}^d$**-Brownian motion:** is a ssMp with $\alpha = 2$.

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Suppose that $(X_t : t \geq 0)$ is an $\mathbb{R}$-Brownian motion:

- Write $\underline{X}_t := \inf_{s\leq t} X_s$. Then $(X_t, \underline{X}_t)$, $t \geq 0$ is a Markov process.
- For $c > 0$ and $\alpha = 2$,

$$\begin{pmatrix} c\underline{X}_{c^{-\alpha}t} \\ cX_{c^{-\alpha}t} \end{pmatrix} = \begin{pmatrix} c\inf_{s\leq c^{-\alpha}t} X_s \\ cX_{c^{-\alpha}t} \end{pmatrix} = \begin{pmatrix} \inf_{u\leq t} cX_{c^{-\alpha}u} \\ cX_{c^{-\alpha}t} \end{pmatrix}, \qquad t \geq 0,$$

  and the latter is equal in law to $(X, \underline{X})$, because of the scaling property of X.
- [Exercise!] $\Rightarrow$ Markov process $Z_t := X_t - (-x \wedge \underline{X}_t)$, $t \geq 0$ is also a ssMp on $[0, \infty)$ issued from $x > 0$ with index 2.
- [Exercise!] $\Rightarrow Z_t := X_t \mathbf{1}_{(\underline{X}_t > 0)}$, $t \geq 0$ is also a ssMp, again on $[0, \infty)$.

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SOME OF YOUR BEST FRIENDS ARE SSMP

Suppose that $(X_t : t \geq 0)$ is an $\mathbb{R}^d$-Brownian motion:

- Consider $Z_t := |X_t|$, $t \geq 0$. Because of rotational invariance, it is a Markov process. [Exercise!]
- Again the self-similarity (index 2) of Brownian motion, transfers to the case of $|X|$. Note again, this is a ssMp on $[0, \infty)$. [Exercise!]
- Note that $|X_t|$, $t \geq 0$ is a Bessel-d process. It turns out that all Bessel processes, *and* all squared Bessel processes are self-similar on $[0, \infty)$. Once can check this by e.g. considering scaling properties of their transition semi-groups.

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Suppose that $(X_t : t \geq 0)$ is an $\mathbb{R}^d$-Brownian motion:

- Note when $d = 3$, $|X_t|, t \geq 0$ is also equal in law to a Brownian motion conditioned to stay positive: i.e if we define, for a 1-d Brownian motion $(B_t : t \geq 0)$,

$$\mathbb{P}^{\uparrow}_x(A) = \lim_{s \to \infty} \mathbb{P}_x(A | \underline{B}_{t+s} > 0) = \mathbb{E}_x\left[\frac{B_t}{x}\mathbf{1}_{(\underline{B}_t > 0)}\mathbf{1}_{(A)}\right]$$

where $A \in \sigma\{B_t : u \leq t\}$, then

$$(|X_t|, t \geq 0) \text{ with } |X_0| = x \text{ is equal in law to } (B, \mathbb{P}^{\uparrow}_x).$$

- [Exercise!] Prove that

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- All of the previous examples have in common that their paths are continuous. Is this a necessary condition?
- We want to find more exotic examples as most of the previous examples have been extensively studied through existing theories (of Brownian motion and continuous semi-martingales).
- All of the previous examples are functional transforms of Brownian motion and have made use of the scaling and Markov properties and (in some cases) isotropic distributional invariance.
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SOME OF THE BEST FRIENDS OF YOUR BEST FRIENDS ARE SSMP

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α-STABLE PROCESS

Definition
A Lévy process X is called (strictly) α-stable if it is also a self-similar Markov process.

- Necessarily $\alpha \in (0, 2]$. [$\alpha = 2 \to$ BM, exclude this.]
- The characteristic exponent $\Psi(\theta) := -t^{-1} \log \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_t})$ satisfies

$$\Psi(\theta) = |\theta|^\alpha (\mathrm{e}^{\pi \mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta>0)} + \mathrm{e}^{-\pi \mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta<0)}), \qquad \theta \in \mathbb{R}.$$

  where $\rho = \mathrm{P}_0(X_t \geq 0)$ will frequently appear as will $\hat{\rho} = 1 - \rho$
- Assume jumps in both directions ($0 < \alpha\rho, \alpha\hat{\rho} < 1$), so that the Lévy **density** takes the form

$$\frac{\Gamma(1+\alpha)}{\pi} \frac{1}{|x|^{1+\alpha}} \left(\sin(\pi\alpha\rho)\mathbf{1}_{\{x>0\}} + \sin(\pi\alpha\hat{\rho})\mathbf{1}_{\{x<0\}}\right)$$

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YOUR NEW FRIENDS

Suppose $X = (X_t : t \geq 0)$ is within the assumed class of α-stable processes in one-dimension and let $\underline{X}_t = \inf_{s \leq t} X_s$.

Your new friends are:

- $Z = X$
- $Z = X - (-x \wedge \underline{X}), x > 0$.
- $Z = X\mathbf{1}_{(\underline{X}>0)}$
- $Z = |X|$ providing $\rho = 1/2$
- [Exercise!] Verify these cases!
- What about $Z =$"X conditioned to stay positive"?

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- Recall that each Lévy processes, $\xi = \{\xi_t : t \geq 0\}$, enjoys the Wiener-Hopf factorisation i.e. up to a multiplicative constant, $\Psi_\xi(\theta) := t^{-1} \log \mathbb{E}[\mathrm{e}^{\mathrm{i}\theta\xi_t}]$ respects the factorisation

$$\Psi_\xi(\theta) = \kappa(-\mathrm{i}\theta)\hat{\kappa}(\mathrm{i}\theta), \qquad \theta \in \mathbb{R},$$

  where κ and $\hat{\kappa}$ are Bernstein functions. That is e.g. κ takes the form

$$\kappa(\lambda) = q + \mathrm{a}\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})\nu(\mathrm{d}x), \qquad \lambda \geq 0$$

  where ν is a measure satisfying $\int_{(0,\infty)} (1 \wedge x)\nu(\mathrm{d}x) < \infty$.

- The probabilistic significance of these subordinators, is that their range corresponds precisely to the range of the running maximum of ξ and of $-\xi$ respectively.
- In the case of α-stable processes, up to a multiplicative constant,

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- Associated to the descending ladder subordinator $\hat{\kappa}$ is its potential measure $\hat{U}$, which satisfies
$$\int_{[0,\infty)} \mathrm{e}^{-\lambda x} \hat{U}(\mathrm{d}x) = \frac{1}{\hat{\kappa}(\lambda)}, \qquad \lambda \geq 0.$$
- It can be shown that for a Lévy process which satisfies $\limsup_{t\to\infty} \xi_t = \infty$, for $A \in \sigma(\xi_u : u \leq t)$,
$$\mathbb{P}_x^{\uparrow}(A) = \lim_{s\to\infty} \mathbb{P}_x(A | \underline{X}_{t+s} > 0) = \mathbb{E}_x \left[\frac{\hat{U}(X_t)}{\hat{U}(x)} \mathbf{1}_{(\underline{X}_t > 0)} \mathbf{1}_{(A)} \right]$$
- In the α-stable case $\hat{U}(x) \propto x^{\alpha\hat{\rho}}$
[Note in the excluded case that $\alpha = 2$ and $\rho = 1/2$, i.e. Brownian motion, $\hat{U}(x) = x$.]

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Conditioned α-stable processes

- For $c, x > 0, t \geq 0$ and appropriately bounded, measurable and non-negative f, we can write,

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&\mathbb{E}^{\uparrow}_x[f(\{cX_{c^{-\alpha}s} : s \leq t\})] \\
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\end{aligned}
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- This also makes the process $(X, \mathbb{P}^{\uparrow}_x)$, $x > 0$, a self-similar Markov process on $[0, \infty)$.
- Unlike the case of Brownian motion, the conditioned stable process does not have the law of the radial part of a 3-dimensional stable process (the analogue to the Brownian case).

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§3. Lamperti Transform

NOTATION

- Use $\xi := \{\xi_t : t \geq 0\}$ to denote a Lévy process which is killed and sent to the cemetery state $-\infty$ at an independent and exponentially distributed random time, $\mathbf{e}_q$, with rate in $q \in [0, \infty)$. The characteristic exponent of ξ is thus written

$$-\log E(\mathrm{e}^{\mathrm{i}\theta\xi_1}) = \Psi(\theta) = q + \text{Lévy–Khintchine}$$

- Define the associated integrated exponential Lévy process

$$I_t = \int_0^t \mathrm{e}^{\alpha\xi_s} \mathrm{d}s, \qquad t \geq 0. \tag{1}$$

and its limit, $I_\infty := \lim_{t\uparrow\infty} I_t$.

- Also interested in the inverse process of I:

$$\varphi(t) = \inf\{s > 0 : I_s > t\}, \qquad t \geq 0. \tag{2}$$

As usual, we work with the convention $\inf \emptyset = \infty$.

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LAMPERTI TRANSFORM FOR POSITIVE SSMP

Theorem (Part (i))

Fix $\alpha > 0$. If $Z^{(x)}$, $x > 0$, is a positive self-similar Markov process with index of self-similarity α, then up to absorption at the origin, it can be represented as follows. For $x > 0$,

$$Z_t^{(x)} \mathbf{1}_{(t<\zeta^{(x)})} = x \exp\{\xi_{\varphi(x^{-\alpha} t)}\}, \qquad t \geq 0,$$

where $\zeta^{(x)} = \inf\{t > 0 : Z_t^{(x)} = 0\}$ and either

(1) *$\zeta^{(x)} = \infty$ almost surely for all $x > 0$, in which case ξ is a Lévy process satisfying $\limsup_{t\uparrow\infty} \xi_t = \infty$,*

(2) *$\zeta^{(x)} < \infty$ and $Z^{(x)}_{\zeta^{(x)}-} = 0$ almost surely for all $x > 0$, in which case ξ is a Lévy process satisfying $\lim_{t\uparrow\infty} \xi_t = -\infty$, or*

(3) *$\zeta^{(x)} < \infty$ and $Z^{(x)}_{\zeta^{(x)}-} > 0$ almost surely for all $x > 0$, in which case ξ is a Lévy process killed at an independent and exponentially distributed random time.*

In all cases, we may identify $\zeta^{(x)} = x^\alpha I_\infty$.

LAMPERTI TRANSFORM FOR POSITIVE SSMP

Theorem (Part (ii))

Conversely, suppose that ξ is a given (killed) Lévy process. For each $x > 0$, define

$$Z_t^{(x)} = x \exp\{\xi_{\varphi(x^{-\alpha} t)}\}\mathbf{1}_{(t < x^\alpha I_\infty)}, \qquad t \geq 0.$$

Then $Z^{(x)}$ defines a positive self-similar Markov process, up to its absorption time $\zeta^{(x)} = x^\alpha I_\infty$, with index α.

LAMPERTI TRANSFORM FOR POSITIVE SSMP

$$\begin{array}{c} (Z, \mathrm{P}_x)_{x>0} \text{ pssMp} \\ Z_t = \exp(\xi_{S(t)}), \\ S \text{ a random time-change} \end{array} \quad \leftrightarrow \quad \begin{array}{c} (\xi, \mathbb{P}_y)_{y\in\mathbb{R}} \text{ killed Lévy} \\ \xi_s = \log(Z_{T(s)}), \\ T \text{ a random time-change} \end{array}$$

$$\left.\begin{array}{c} Z \text{ never hits zero} \\ Z \text{ hits zero continuously} \\ Z \text{ hits zero by a jump} \end{array}\right\} \quad \leftrightarrow \quad \left\{\begin{array}{c} \xi \to \infty \text{ or } \xi \text{ oscillates} \\ \xi \to -\infty \\ \xi \text{ is killed} \end{array}\right.$$

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§4. Positive self-similar Markov processes

STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

- The stable process cannot 'creep' downwards across the threshold 0 and so must do so with a jump.
- This puts $Z_t^* := X_t \mathbf{1}_{(\underline{X}_t > 0)}$, $t \geq 0$, in the class of pssMp for which the underlying Lévy process experiences exponential killing.
- Write $\xi^* = \{\xi_t^* : t \geq 0\}$ for the underlying Lévy process and denote its killing rate by q^*.
- Let's try and decode the characteristics of ξ^*.

Stable process killed on entry to $(-\infty, 0)$

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STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

- The stable process cannot 'creep' downwards across the threshold 0 and so must do so with a jump.
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- We know that the α-stable process experiences downward jumps at rate

$$\frac{\Gamma(1+\alpha)}{\pi}\sin(\pi\alpha\hat{\rho})\frac{1}{|x|^{1+\alpha}}\mathrm{d}x, \qquad x < 0.$$

- Given that we know the value of Z^*_{t-}, on $\{\underline{X}_t > 0\}$, the stable process will pass over the origin at rate

$$\frac{\Gamma(1+\alpha)}{\pi}\sin(\pi\alpha\hat{\rho})\left(\int_{Z^*_{t-}}^{\infty}\frac{1}{|x|^{1+\alpha}}\mathrm{d}x\right) = \frac{\Gamma(1+\alpha)}{\alpha\pi}\sin(\pi\alpha\hat{\rho})(Z^*)_{t-}^{-\alpha}.$$

- On the other hand, the Lamperti transform says that on $\{t < \zeta\}$, as a pssMp, Z is sent to the origin at rate

$$q^*\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = q^*\mathrm{e}^{-\alpha\xi^*_{\varphi(t)}} = q^*(Z^*)_t^{-\alpha}.$$

- Comparing gives us

$$q^* = \Gamma(\alpha)\sin(\pi\alpha\hat{\rho})/\pi = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}.$$

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- Referring again to the Lamperti transform, we know that, under $\mathbb{P}_1$ (so that $\mathbb{P}_1(\xi_0^* = 0) = 1$),

$$Z^*_{\zeta-} = X_{\tau_0^- -} = e^{\xi^*_{\mathbf{e}_{q^*}}},$$

where $\mathbf{e}_{q^*}$ is an exponentially distributed random variable with rate q^*.

- This motivates the computation

$$\mathbb{E}_1[(Z^*_{\zeta-})^{i\theta}] = \mathbb{E}_1[e^{i\theta\xi^*_{\mathbf{e}_{q^*}-}}] = \frac{q^*}{(\Psi^*(z) - q^*) + q^*}, \qquad \theta \in \mathbb{R},$$

where Ψ^* is the characteristic exponent of ξ^*.

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Stable process killed on entry to $(-\infty, 0)$

Setting

$$K = \frac{\sin \alpha\hat{\rho}\pi}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})},$$

Remembering the "overshoot-undershoot" distributional law at first passage (well known in the literature - see e.g. Chapter 8 of my book) and deduce that, for all $v \in [0,1]$,

$$\begin{aligned}
&\mathbb{P}_1(X_{\tau_0^- -} \in \mathrm{d}v) \\
&\quad = \hat{\mathbb{P}}_0(1 - X_{\tau_1^+ -} \in \mathrm{d}v) \\
&\quad = K\left(\int_0^\infty \int_0^\infty \mathbf{1}_{(y \le 1 \wedge v)} \frac{(1-y)^{\alpha\hat{\rho}-1}(v-y)^{\alpha\rho-1}}{(v+u)^{1+\alpha}} \mathrm{d}u\mathrm{d}y\right) \mathrm{d}v \\
&\quad = \frac{K}{\alpha}\left(\int_0^1 \mathbf{1}_{(y \le v)} v^{-\alpha}(1-y)^{\alpha\hat{\rho}-1}(v-y)^{\alpha\rho-1}\mathrm{d}y\right) \mathrm{d}v,
\end{aligned}$$

where $\hat{\mathbb{P}}_0$ is the law of $-X$ issued from 0.

STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

We are led to the conclusion that

$$
\begin{aligned}
\frac{q*}{\Psi*(\theta)} &= \frac{K}{\alpha}\int_0^1 (1-y)^{\alpha\hat{\rho}-1}\int_0^\infty \mathbf{1}_{(y\leq v)} v^{\mathrm{i}\theta-\alpha\hat{\rho}-1}\left(1-\frac{y}{v}\right)^{\alpha\rho-1} \mathrm{d}v\mathrm{d}y \\
&= \frac{K}{\alpha}\int_0^1 (1-y)^{\alpha\hat{\rho}-1} y^{\mathrm{i}\theta-\alpha\hat{\rho}}\mathrm{d}y \frac{\Gamma(\alpha\hat{\rho}-\mathrm{i}\theta)\Gamma(\alpha\rho)}{\Gamma(\alpha-\mathrm{i}\theta)} \\
&= \frac{\Gamma(\alpha\hat{\rho}-\mathrm{i}\theta)\Gamma(\alpha\rho)\Gamma(1-\alpha\hat{\rho}+\mathrm{i}\theta)\Gamma(\alpha\hat{\rho})\Gamma(\alpha+1)}{\alpha\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})\Gamma(\alpha\hat{\rho})\Gamma(1+\mathrm{i}\theta)\Gamma(\alpha-\mathrm{i}\theta)},
\end{aligned}
$$

where in the first equality Fubini's Theorem has been used, in the second equality a straightforward substitution $w = y/v$ has been used for the inner integral on the preceding line together with the classical beta integral and, finally, in the third equality, the Beta integral has been used for a second time. Inserting the respective values for the constants q^* and K, we come to rest at the following result:

STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

Theorem

For the pssMp constructed by killing a stable process on first entry to $(-\infty, 0)$, the underlying killed Lévy process, ξ^*, that appears through the Lamperti transform has characteristic exponent given by

$$\Psi^*(z) = \frac{\Gamma(\alpha - \mathrm{i}z)}{\Gamma(\alpha\hat{\rho} - \mathrm{i}z)} \frac{\Gamma(1 + \mathrm{i}z)}{\Gamma(1 - \alpha\hat{\rho} + \mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

Stable processes conditioned to stay positive

- Use the Lamperti representation of the α-stable process X to write, for $A \in \sigma(X_u : u \leq t)$,

$$\mathbb{P}_x^{\uparrow}(A) = \mathbb{E}_x\left[\frac{X_t^{\alpha\hat{\rho}}}{x^{\alpha\hat{\rho}}}\mathbf{1}_{(\underline{X}_t > 0)}\mathbf{1}_{(A)}\right] = E\left[e^{\alpha\hat{\rho}\xi^*_\tau}\mathbf{1}_{(\tau < \mathbf{e}_{q^*})}\mathbf{1}_{(A)}\right],$$

  where $\tau = \varphi(x^{-\alpha}t)$ is a stopping time in the natural filtration of ξ^*.
- Noting that $\Psi^*(-\mathrm{i}\alpha\hat{\rho}) = 0$, the change of measure constitutes an Esscher transform at the level of ξ^*.

Theorem

The underlying Lévy process, $\xi^{\uparrow}$, that appears through the Lamperti transform applied to $(X, \mathbb{P}_x^{\uparrow})$, $x > 0$, has characteristic exponent given by

$$\Psi^{\uparrow}(z) = \frac{\Gamma(\alpha\rho - \mathrm{i}z)}{\Gamma(-\mathrm{i}z)}\frac{\Gamma(1 + \alpha\hat{\rho} + \mathrm{i}z)}{\Gamma(1 + \mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

- In particular $\Psi^{\uparrow}(z) = \Psi^*(z - \mathrm{i}\alpha\hat{\rho})$, $z \in \mathbb{R}$ so that $\Psi^{\uparrow}(0) = 0$ (i.e. no killing!)
- One can also check by hand that $\Psi^{\uparrow\prime}(0+) = E[\xi_1^{\uparrow}] > 0$ so that $\lim_{t\to\infty} \xi_t^{\uparrow} = \infty$.

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STABLE PROCESSES CONDITIONED TO STAY POSITIVE

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Did you spot the other root?

- In essence, the case of the stable process conditioned to stay positive boils down to an Esscher transform in the underlying (Lamperti-transformed) Lévy process.
- It was important that we identified a root of $\Psi^*(z) = 0$ in order to avoid involving a 'time component' of the Esscher transform.
- However, there is another root of the equation

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  namely $z = -\mathrm{i}(1 - \alpha\hat{\rho})$.
- And this means that

$$\mathrm{e}^{(1-\alpha\hat{\rho})\xi^*}, \qquad t \geq 0,$$

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Reverse engineering

- What now happens if we define for $A \in \sigma(X_u : u \leq t)$,

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- In the same way we checked that $(X, \mathbb{P}_x^{\uparrow})$, $x > 0$, is a pssMp, we can also check that $(X, \mathbb{P}_x^{\downarrow})$, $x > 0$ is a pssMp.
  [Exercise!] Do it!

- In an appropriate sense, it turns out that $(X, \mathbb{P}_x^{\downarrow})$, $x > 0$ is the law of a stable process conditioned to continuously approach the origin from above.

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ξ^*, $\xi^\uparrow$ AND $\xi^\downarrow$

- The three examples of pssMp offer quite striking underlying Lévy processes
- Is this exceptional?

CENSORED STABLE PROCESSES

- Start with X, the stable process.
- Let $A_t = \int_0^t \mathbf{1}_{(X_t > 0)}\,\mathrm{d}t$.
- Let γ be the right-inverse of A, and put $\check{Z}_t := X_{\gamma(t)}$.
- Finally, make zero an absorbing state: $Z_t = \check{Z}_t \mathbf{1}_{(t < T_0)}$ where

$$T_0 = \inf\{t > 0 : X_t = 0\}.$$

  Note $T_0 < \infty$ a.s. if and only if $\alpha \in (1, 2)$ and otherwise $T_0 = \infty$ a.s.
- This is the censored stable process.

CENSORED STABLE PROCESSES

Theorem

Suppose that the underlying Lévy process for the censored stable process is denoted by $\overset{\leadsto}{\xi}$. *Then* $\overset{\leadsto}{\xi}$ *is equal in law to* $\xi^{**} \oplus \xi^{C}$, *with*

- ξ^{**} *equal in law to* ξ^{*} *with the killing removed,*
- ξ^{C} *a compound Poisson process with jump rate* $q^{*} = \Gamma(\alpha)\sin(\pi\alpha\hat{\rho})/\pi$.

Moreover, the characteristic exponent of $\overset{\leadsto}{\xi}$ *is given by*

$$\overset{\leadsto}{\Psi}(z) = \frac{\Gamma(\alpha\rho - \mathrm{i}z)}{\Gamma(-\mathrm{i}z)} \frac{\Gamma(1 - \alpha\rho + \mathrm{i}z)}{\Gamma(1 - \alpha + \mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

The radial part of a stable process

- Suppose that X is a symmetric stable process, i.e $\rho = 1/2$.
- We know that $|X|$ is a pssMp.

Theorem

Suppose that the underlying Lévy process for $|X|$ is written ξ, then it characteristic exponent is given by

$$\Psi(z) = 2^\alpha \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z+\alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z+1))}{\Gamma(\frac{1}{2}(\mathrm{i}z+1-\alpha))}, \qquad z \in \mathbb{R}.$$

[Exercise!] This is quite hard to prove for $\alpha \in (1,2)$, but could can be proved in a straightforward way for $\alpha \in (0,1]$. Try it!
[Hint!] Think about what happens after X first crosses the origin and apply the Markov property as well as symmetry. You will need to use the law of the overshoot of X below the origin given a few slides back.

HYPERGEOMETRIC LÉVY PROCESSES (REMINDER)

Definition (and Theorem)

For $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$ in

$$\left\{ \ \beta \leq 2, \ \gamma, \hat{\gamma} \in (0,1) \ \hat{\beta} \geq -1, \text{ and } 1 - \beta + \hat{\beta} + \gamma \wedge \hat{\gamma} \geq 0 \ \right\}$$

there exists a (killed) Lévy process, henceforth refered to as a hypergeometric Lévy process, having the characteristic function

$$\Psi(z) = \frac{\Gamma(1-\beta+\gamma-\mathrm{i}z)}{\Gamma(1-\beta-\mathrm{i}z)} \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\mathrm{i}z)}{\Gamma(\hat{\beta}+\mathrm{i}z)} \qquad z \in \mathbb{R}.$$

The Lévy measure of Y has a density with respect to Lebesgue measure is given by

$$\pi(x) = \begin{cases} -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\hat{\gamma})\Gamma(-\gamma)} \mathrm{e}^{-(1-\beta+\gamma)x} {}_2F_1\left(1+\gamma, \eta; \eta-\hat{\gamma}; \mathrm{e}^{-x}\right), & \text{if } x > 0, \\[2ex] -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\gamma)\Gamma(-\hat{\gamma})} \mathrm{e}^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1\left(1+\hat{\gamma}, \eta; \eta-\gamma; \mathrm{e}^{x}\right), & \text{if } x < 0, \end{cases}$$

where $\eta := 1-\beta+\gamma+\hat{\beta}+\hat{\gamma}$, for $|z|<1$, ${}_2F_1(a,b;c;z) := \sum_{k\geq 0} \frac{(a)_k (b)_k}{(c)_k k!} z^k$.

§5. Entrance Laws

Starting from zero

- We have carefully avoided the issue of talking about pssMp issued from the origin.
- This should ring alarm bells when we look at the Lamperti transform

$$Z_t^{(x)} \mathbf{1}_{(t<\zeta^{(x)})} = x \exp\{\xi_{\varphi(x^{-\alpha}t)}\} = \exp\{\xi_{\varphi(x^{-\alpha}t)} + \log x\}, \qquad t \geq 0,$$

- On the one hand $\log x \downarrow -\infty$, which is the point of issue of ξ, but

$$\varphi(x^{-\alpha}t) = \inf\{s > 0 : \int_0^s e^{\alpha(\xi_u + \log x)} du > t\},$$

  meaning that we are sampling the Lévy process over a longer and longer time horizon.
- We know that 0 is an **absorbing point**, but it might also be an **entrance point** (can it be both?).
- We know that some of our new friends have no problem using the origin as an entrance point, e.g. $|X|$, where X is an α-stable process (or Brownian motion).
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$$\mathbb{E}_0[f(Z_s : s \leq t)] := \lim_{x \to 0} \mathbb{E}_x[f(Z_s : s \leq t)]$$

for an appropriate measurable function on cadlag paths of length t.

- The right setting to discuss *distributional convergence* is with respect to so-called *Skorokhod topology*.
- ROUGHLY: There is a metric on cadlag path space which does a better job of measuring how “close" two paths are than e.g. the uniform functional metric.
- This metric induces a topology (the Skorokhod topology). From this topology, we build a measurable space around the space of cadlag paths.
- Think of $\mathbb{P}_x, x > 0$ as a family of measures on this space and we want weak convergence “$\mathbb{P}_0 := \lim_{x \to 0} \mathbb{P}_x$” on this space.

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Starting from zero

Assume that Z is a pssMp with $\zeta = \infty$ a.s. Moreover, suppose that the Lévy process ξ, associated with Z through the Lamperti transform, is not a compound Poisson process.

Theorem

Under the assumption that $\mathbb{E}(\xi_1) > 0$, for any positive measurable function f and $t > 0$,

$$\mathbb{E}_0(f(Z_t)) = \frac{1}{\alpha \mathbb{E}(\xi_1)} E\left(\frac{1}{I_\infty^-} f\left(\left(\frac{t}{I_\infty^-} \right)^{1/\alpha} \right) \right),$$

where $I_\infty^- = \int_0^\infty \exp\{-\alpha \xi_s\} \mathrm{d}s$.

Theorem

The limit $\mathbb{P}_0 := \lim_{x \to 0} \mathbb{P}_x$ exists in the sense of convergence with respect to the Skorokhod topology if and only if $\mathbb{E}(H_1^+) < \infty$ (H^+ is the ascending ladder process of ξ).

SKETCH PROOF OF THE SECOND THEOREM

- The basic idea is to give a pathwise construction of a candidate for "$(Z, \mathbb{P}_0)$" then check that there is weak convergence to it.
- Suppose we can identify ξ° which is a version of the underlying Lévy process ξ of $(Z, \mathbb{P}_x)$, $x > 0$ but now indexed by $\mathbb{R}$ rather than indexed by $[0, \infty)$, then we can identify the pathwise candidate for "$(Z, \mathbb{P}_0)$" by

$$Z_t^{(0)} = \exp\{\xi^\circ_{\varphi^\circ(t)}\}, \qquad t \geq 0,$$

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CONSTRUCTION OF ξ°

- If ξ° enters from $(-\infty, \infty)$, then it must make first passage over any level x in a "stationary" way.
- Specifically, we would need that $(\xi^\circ_{\sigma_a^+} - a, a - \xi^\circ_{\sigma_a^+ -})$ is independent of $a \in \mathbb{R}$, where $\sigma_a^+ = \inf\{t > -\infty : \xi^\circ_t > a\}$. This motivates the following construction:
- Take the stationary overshoot/undershoot law of ξ (which requires the necessary and sufficient condition $E[H_1^+] < \infty$)

$$\chi(\mathrm{d}y, \mathrm{d}z) = \frac{1}{\mathbb{E}[H_1^+]} \left(\widehat{U}_\xi(z) \Pi_\xi(z + \mathrm{d}y)\mathrm{d}z + \gamma\delta_0(\mathrm{d}y)\delta_0(\mathrm{d}z) \right), \qquad y, z \geq 0.$$

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$$\xi^\circ_t := \begin{cases} \xi_t & \text{under } P_\Delta \text{ if } t \geq 0, \\ -\xi^\uparrow_{|t|-} & \text{under } P^\uparrow_{\Delta^\uparrow} \text{ if } t < 0, \end{cases}$$

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Recurrent extension

- The previous construction has insisted that Z is a *pssMp* with $\zeta = \infty$ a.s. But what about the case that $\zeta < \infty$ a.s.
- We can say something about the case that $\zeta < \infty$ a.s. and $X_{\zeta-} = 0$.
- A cadlag strong Markov process, $\vec{Z} := \{\vec{Z}_t : t \geq 0\}$ with probabilities $\{\vec{P}_x, x \geq 0\}$, is a *recurrent extension* of Z if, for each $x > 0$, the origin is not an absorbing state $\vec{P}_x$-almost surely and $\{\vec{Z}_{t \wedge \vec{\zeta}} : t \geq 0\}$ under $\vec{P}_x$ has the same law as (Z, P_x), where

$$\vec{\zeta} = \inf\{t > 0 : \vec{Z}_t = 0\}.$$

Theorem

If $\zeta < \infty$ a.s. and $X_{\zeta-} = 0$, then there exists a unique recurrent extension of Z which leaves 0 continuously if and only if there exists a $\beta \in (0, \alpha)$ such

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§6. Real valued self-similar Markov processes

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- What can we say about $\mathbb{R}$-valued self-similar Markov processes.
- This requires us to first investigate Markov Additive (Lévy) Processes

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Markov additive processes (MAPs)

- E is a finite state space
- $(J(t))_{t\geq 0}$ is a continuous-time, irreducible Markov chain on E
- process (ξ, J) in $\mathbb{R} \times E$ is called a *Markov additive process (MAP)* with probabilities $\mathbf{P}_{x,i}$, $x \in \mathbb{R}, i \in E$, if, for any $i \in E$, $s, t \geq 0$: Given $\{J(t) = i\}$,
  $(\xi(t+s) - \xi(t), J(t+s)) \stackrel{d}{=} (\xi(s), J(s))$ with law $\mathbf{P}_{0,i}$.

PATHWISE DESCRIPTION OF A MAP

The pair (ξ, J) is a Markov additive process if and only if, for each $i, j \in E$,

- there exist a sequence of iid Lévy processes $(\xi_i^n)_{n\geq 0}$
- and a sequence of iid random variables $(U_{ij}^n)_{n\geq 0}$, independent of the chain J,
- such that if $T_0 = 0$ and $(T_n)_{n\geq 1}$ are the jump times of J, the process ξ has the representation

$$\xi(t) = \mathbf{1}_{(n>0)}(\xi(T_n-) + U^n_{J(T_n-),J(T_n)}) + \xi^n_{J(T_n)}(t - T_n),$$

  for $t \in [T_n, T_{n+1})$, $n \geq 0$.
- [Exercise!] Show that the property above implies the definition on the previous slide.

Characteristics of a MAP

- Denote the transition rate matrix of the chain J by $\mathbf{Q} = (q_{ij})_{i,j\in E}$.
- For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i (when it exists).
- For each pair of $i, j \in E$ with $i \neq j$, define the Laplace transform $G_{ij}(z) = \mathbb{E}(e^{zU_{ij}})$ of the jump distribution U_{ij} (when it exists).
- Otherwise define $U_{i,i} \equiv 0$, for each $i \in E$.
- Write $G(z)$ for the $N \times N$ matrix whose (i, j)th element is $G_{ij}(z)$.
- Let

$$\boldsymbol{\Psi}(z) = \mathrm{diag}(\psi_1(z), \ldots, \psi_N(z)) + \mathbf{Q} \circ G(z),$$

  (when it exists), where $\circ$ indicates elementwise multiplication.
- The matrix exponent of the MAP (ξ, J) is given by

$$\mathbf{E}_{0,i}(e^{z\xi(t)}; J(t) = j) = \left(e^{\boldsymbol{\Psi}(z)t}\right)_{i,j}, \qquad i, j \in E,$$

  (when it exists).

LAMPERTI-KIU TRANSFORM

- Take J to be irreducible on $E = \{1, -1\}$.
- Let
$$Z_t = |x|e^{\xi(\tau(|x|^{-\alpha}t))}J(\tau(|x|^{-\alpha}t)) \qquad 0 \le t < T_0,$$
where
$$\tau(t) = \inf\left\{s > 0 : \int_0^s \exp(\alpha\xi(u))\mathrm{d}u > t\right\}$$
and
$$T_0 = |x|^{-\alpha}\int_0^\infty e^{\alpha\xi(u)}\mathrm{d}u.$$
- Then Z_t is a real-valued self-similar Markov process in the sense that the law of $(cZ_{tc^{-\alpha}} : t \ge 0)$ under $\mathbb{P}_x$ is $\mathbb{P}_{cx}$.
- The converse (within a special class of rssMps) is also true.
- [Exercise!] Explain what happens if e.g. J is an absorbing Markov Chain on $\{1, -1\}$ with $\{1\}$ as an absorbing state?

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ENTRANCE AT ZERO

- Given the Lamperti-Kiu representation

$$Z_t = \mathrm{e}^{\xi(\tau(|x|^{-\alpha}t))+\log|x|} J(\tau(|x|^{-\alpha}t)) \qquad 0 \le t < T_0,$$

  it is clear that we can think of a similar construction from zero to the case of pssMp.
- We need to construct a stationary version of the pair (ξ, J) which is indexed by $\mathbb{R}$ and pinned at space-time point $(-\infty, \infty)$.
- Just like the theory of Lévy processes, by observing the range of the process (ξ_t, J_t) $t \ge 0$, **only** at the points of its new suprema, we see a process (H_t^+, J_t^+), $t \ge 0$, which is also a MAP, where H^+ is has increasing paths.

Theorem

Suppose that J is irreducible.
Then the limit $\mathbb{P}_0 := \lim_{|x|\to 0} \mathbb{P}_x$ exists in the sense of convergence with respect to the Skorokhod topology if and only if $\mathbb{E}_1(H_1^+) + \mathbb{E}_{-1}(H_1^+) < \infty$, and otherwise limit does not exist.

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AN α-STABLE PROCESS IS A RSSMP

- An α-stable process up to absorption in the origin is a rssMp.
- When $\alpha \in (0,1]$, the process never hits the origin a.s.
- When $\alpha \in (1,2)$, the process is absorbs at the origin a.s.
- The matrix exponent of the underlying MAP is given by:

$$\begin{bmatrix} \dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho} - z)\Gamma(1-\alpha\hat{\rho}+z)} & -\dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\ -\dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & \dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\rho - z)\Gamma(1-\alpha\rho+z)} \end{bmatrix},$$

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Esscher transform for MAPs

- If $\mathbf{\Psi}(z)$ is well defined then it has a real simple eigenvalue $\chi(z)$, which is larger than the real part of all its other eigenvalues.
- Furthermore, the corresponding right-eigenvector $\mathbf{v}(z) = (v_1(z), \cdots, v_N(z))$ has strictly positive entries and may be normalised such that $\pi \cdot \mathbf{v}(z) = 1$.

Theorem

Let $\mathcal{G}_t = \sigma\{(\xi(s), J(s)) : s \leq t\}$, $t \geq 0$, and

$$M_t := \mathrm{e}^{\gamma\xi(t) - \chi(\gamma)t} \frac{v_{J(t)}(\gamma)}{v_i(\gamma)}, \qquad t \geq 0,$$

for some $\gamma \in \mathbb{R}$ such that $\chi(\gamma)$ is defined. Then, M_t, $t \geq 0$, is a unit-mean martingale. Moreover, under the change of measure

$$\mathrm{d}\mathbf{P}^{\gamma}_{0,i}\Big|_{\mathcal{G}_t} = M_t \, \mathrm{d}\mathbf{P}_{0,i}\Big|_{\mathcal{G}_t}, \qquad t \geq 0,$$

the process (ξ, J) remains in the class of MAPs with new exponent given by

$$\mathbf{\Psi}_\gamma(z) = \mathbf{\Delta}_v(\gamma)^{-1} \mathbf{\Psi}(z+\gamma) \mathbf{\Delta}_v(\gamma) - \chi(\gamma)\mathbf{I}.$$

Here, $\mathbf{I}$ is the identity matrix and $\mathbf{\Delta}_v(\gamma) = \mathrm{diag}(\mathbf{v}(\gamma))$.

Esscher and drift

- Suppose that χ is defined in some open interval D of $\mathbb{R}$, then, it is smooth and convex on D.
- Since $\mathbf{\Psi}(0) = -\mathbf{Q}$, if, moreover, J is irreducible, we always have $\chi(0) = 0$ and $\mathbf{v}(0) = (1, \cdots, 1)$. So $0 \in D$ and $\chi'(0)$ is well defined and finite.
- With all of the above

$$\lim_{t \to \infty} \frac{\xi_t}{t} = \chi'(0) \qquad \text{a.s.}$$

- [Exercise!] Show that in the above circumstances, if $\chi'(0) < 0$, then the associated ssMp hits the origin in an almost surely finite time, independently of its point of issue $x \in \mathbb{R}$.

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ESSCHER AND THE STABLE-MAP

- For the MAP that underlies the stable process $D = (-1, \alpha)$, it can be checked that $\det\Psi(\alpha - 1) = 0$ i.e. $\chi(\alpha - 1) = 0$, which makes

$$\begin{aligned}\boldsymbol{\Psi}^\circ(z) &= \boldsymbol{\Delta}^{-1}\boldsymbol{\Psi}(z+\alpha-1)\boldsymbol{\Delta}\\ &= \begin{bmatrix} \dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\rho-z)\Gamma(\alpha\rho+z)} & -\dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} \\ -\dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\hat\rho)\Gamma(1-\alpha\hat\rho)} & \dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\hat\rho-z)\Gamma(\alpha\hat\rho+z)} \end{bmatrix},\end{aligned}$$

where $\boldsymbol{\Delta} = \mathrm{diag}(\sin(\pi\alpha\hat\rho), \sin(\pi\alpha\rho))$.

- When $\alpha \in (0,1)$, $\chi'(0) > 0$ (because the stable process never touches the origin a.s.) and $\boldsymbol{\Psi}^\circ(z)$-MAP drifts to $-\infty$
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Riesz-Bogdan-Zak transform

Theorem (Riesz–Bogdan–Zak transform)

Suppose that X is an α-stable process as outlined in the introduction. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}, \qquad t \geq 0.$$

Then, for all $x \in \mathbb{R}\backslash\{0\}$, $(-1/X_{\eta(t)})_{t\geq 0}$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}^\circ_{-1/x})$, where

$$\left.\frac{\mathrm{d}\mathbb{P}^\circ_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \left(\frac{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\mathrm{sgn}(X_t)}{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\mathrm{sgn}(x)}\right) \left|\frac{X_t}{x}\right|^{\alpha-1} \mathbf{1}_{(t<\tau^{\{0\}})}$$

and $\mathcal{F}_t := \sigma(X_s : s \leq t)$, $t \geq 0$. Moreover, the process $(X, \mathbb{P}^\circ_x)$, $x \in \mathbb{R}\backslash\{0\}$ is a self-similar Markov process with underlying MAP via the Lamperti-Kiu transform given by $\mathbf{\Psi}^\circ(z)$.

WHAT IS THE Ψ°-MAP?

Thinking of the affect on the long term behaviour of the underlying MAP of the Esscher transform

- When $\alpha \in (0,1)$, $(X, \mathbb{P}^\circ_x)$, $x \neq 0$ has the law of the the stable process conditioned to absorb continuously at the origin in the sense,

$$\mathbb{P}^\circ_y(A) = \lim_{a\to 0} \mathbb{P}_y(A, t < T_0 \,|\, \tau_{(-a,a)} < \infty),$$

  for $A \in \mathcal{F}_t = \sigma(X_s, s \leq t)$,
  $\tau_{(-a,a)} = \inf\{t > 0 : |X_t| < a\}$ and $T_0 = \inf\{t > 0 : X_t = 0\}$.
- When $\alpha \in (1,2)$, $(X, \mathbb{P}^\circ_x)$, $x \neq 0$ has the law of the stable process conditioned to avoid the origin in the sense

$$\mathbb{P}^\circ_y(A) = \lim_{s\to\infty} \mathbb{P}_y(A \,|\, T_0 > t + s),$$

  for $A \in \mathcal{F}_t = \sigma(X_s, s \leq t)$ and $T_0 = \inf\{t > 0 : X_t = 0\}$.

[Exercise!] Explain this change in behaviour heuristically.

§7. Isotropic stable processes in dimension $d \geq 2$ seen as Lévy processes

ISOTROPIC α-STABLE PROCESS IN DIMENSION $d \geq 2$

For $d \geq 2$, let $X := (X_t : t \geq 0)$ be a d-dimensional isotropic stable process.

- X has stationary and independent increments (it is a Lévy process)
- Characteristic exponent $\Psi(\theta) = -\log \mathbb{E}_0(e^{i\theta \cdot X_1})$ satisfies

$$\Psi(\theta) = |\theta|^\alpha, \qquad \theta \in \mathbb{R}.$$

- Necessarily, $\alpha \in (0, 2]$, we exclude 2 as it pertains to the setting of a Brownian motion.
- Associated Lévy measure satisfies, for $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned}\Pi(B) &= \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|} \int_B \frac{1}{|y|^{\alpha+d}} \mathrm{d}y \\ &= \frac{2^{\alpha-1}\Gamma((d+\alpha)/2)\Gamma(d/2)}{\pi^d \left|\Gamma(-\alpha/2)\right|} \int_{\mathbb{S}_{d-1}} r^{d-1}\sigma_1(\mathrm{d}\theta) \int_0^\infty \mathbf{1}_B(r\theta) \frac{1}{r^{\alpha+d}} \mathrm{d}r,\end{aligned}$$

  where $\sigma_1(\mathrm{d}\theta)$ is the surface measure on $\mathbb{S}_{d-1}$ normalised to have unit mass.
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- Stable processes are also self-similar. For $c > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$,

$$\text{under } \mathbb{P}_x, \text{ the law of } (cX_{c^{-\alpha}t}, t \geq 0) \text{ is equal to } \mathbb{P}_{cx}.$$

- Isotropy means, for all orthogonal transformations (e.g. rotations) $U : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $x \in \mathbb{R}^d$,

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$$\mathbb{E}[e^{i\theta X_t}] = \mathbb{E}\left[e^{-\theta^2 S_t}\right] = e^{-|\theta|^{\alpha} t}, \qquad \theta \in \mathbb{R}.$$

[Exercise!] Show, more generally, that if $(\Lambda_t, t \geq 0)$ is a subordinator and $(Y_t, t \geq 0)$ is a Lévy process (in $\mathbb{R}^d$), then $(Y_{\Lambda_t}, t \geq 0)$ is a Lévy process in $\mathbb{R}^d$.

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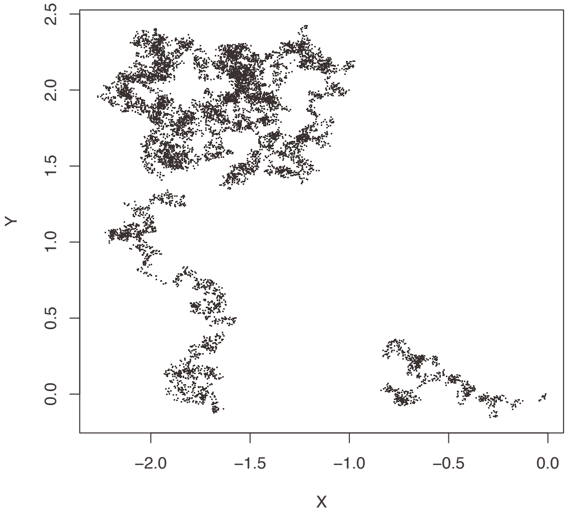
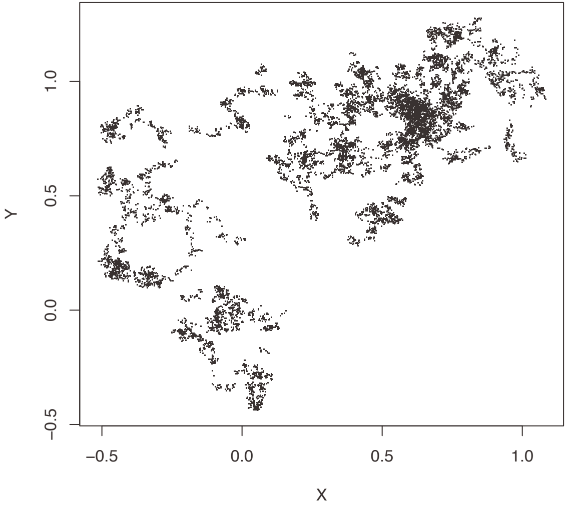
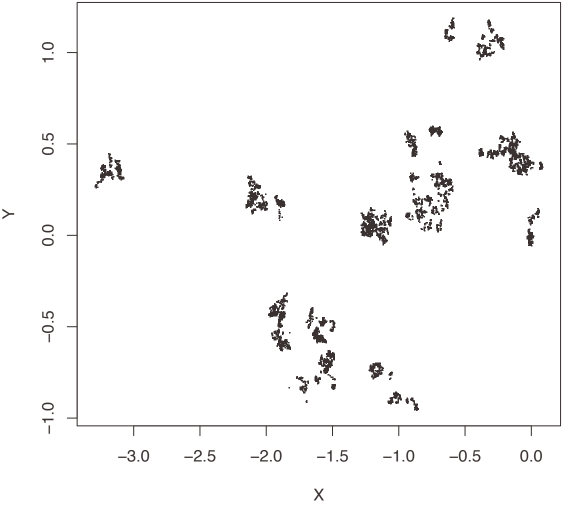
SAMPLE PATH, $\alpha = 1.9$



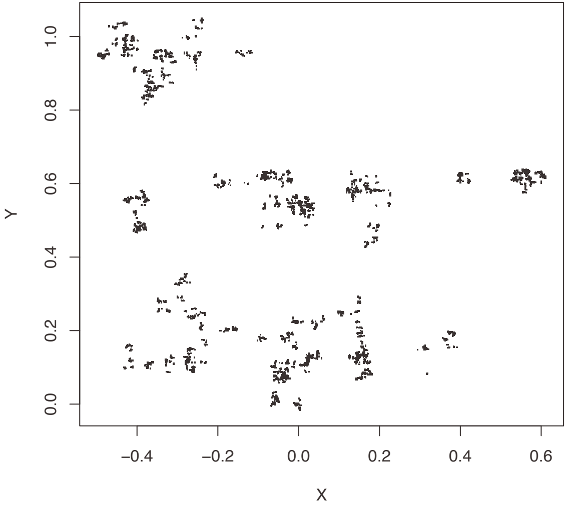
SAMPLE PATH, $\alpha = 1.7$



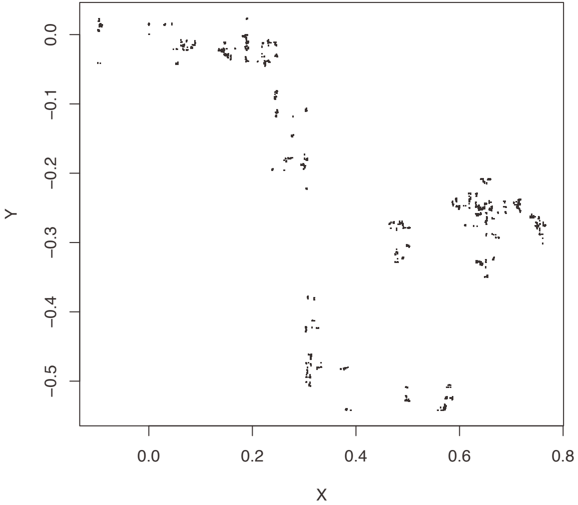
SAMPLE PATH, $\alpha = 1.5$



SAMPLE PATH, $\alpha = 1.2$



SAMPLE PATH, $\alpha = 0.9$



Some classical properties: Transience

We are interested in the potential measure

$$U(x, \mathrm{d}y) = \int_0^\infty \mathbb{P}_x(X_t \in \mathrm{d}y)\mathrm{d}t = \left(\int_0^\infty \mathrm{p}_t(y - x)\mathrm{d}t\right) \mathrm{d}y, \qquad x, y \in \mathbb{R}.$$

Note: stationary and independent increments means that it suffices to consider $U(0, \mathrm{d}y)$.

Theorem

The potential of X is absolutely continuous with respect to Lebesgue measure, in which case, its density in collaboration with spatial homogeneity satisfies $U(x, \mathrm{d}y) = u(y - x)\mathrm{d}y$, $x, y \in \mathbb{R}^d$, where

$$u(z) = 2^{-\alpha}\pi^{-d/2}\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}|z|^{\alpha-d}, \qquad z \in \mathbb{R}^d.$$

In this respect X is transient. It can be shown moreover that

$$\lim_{t\to\infty} |X_t| = \infty$$

almost surely

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PROOF OF THEOREM

Now note that, for bounded and measurable $f : \mathbb{R}^d \mapsto \mathbb{R}^d$,

$$
\begin{aligned}
\mathbb{E}\left[\int_0^\infty f(X_t)\mathrm{d}t\right] &= \mathbb{E}\left[\int_0^\infty f(\sqrt{2}B_{S_t})\mathrm{d}t\right] \\
&= \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}t\, \mathbb{P}(S_t \in \mathrm{d}s) \int_{\mathbb{R}} \mathbb{P}(B_s \in \mathrm{d}x) f(\sqrt{2}x) \\
&= \frac{1}{\Gamma(\alpha/2)\pi^{d/2}2^d} \int_{\mathbb{R}} \mathrm{d}y \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-|y|^2/4s} s^{-1+(\alpha-d)/2} f(y) \\
&= \frac{1}{2^\alpha \Gamma(\alpha/2)\pi^{d/2}} \int_{\mathbb{R}} \mathrm{d}y\, |y|^{(\alpha-d)} \int_0^\infty \mathrm{d}u\, e^{-u} u^{-1+(d-\alpha/2)} f(y) \\
&= \frac{\Gamma((d-\alpha)/2)}{2^\alpha \Gamma(\alpha/2)\pi^{d/2}} \int_{\mathbb{R}} \mathrm{d}y\, |y|^{(\alpha-d)} f(y).
\end{aligned}
$$

Some classical properties: Polarity

- Kesten-Bretagnolle integral test, in dimension $d \geq 2$,

$$\int_{\mathbb{R}} \mathrm{Re}\left(\frac{1}{1+\Psi(z)}\right) \mathrm{d}z = \int_{\mathbb{R}} \frac{1}{1+|z|^{\alpha}} \mathrm{d}z \propto \int_{\mathbb{R}} \frac{1}{1+r^{\alpha}} r^{d-1} \mathrm{d}r\, \sigma_1(\mathrm{d}\theta) = \infty.$$

- $\mathbb{P}_x(\tau^{\{y\}} < \infty) = 0$, for $x, y \in \mathbb{R}^d$.
- i.e. the stable process cannot hit individual points almost surely.

§8. Isotropic stable processes in dimension $d \geq 2$ seen as a self-similar Markov process

THE RADIAL PART OF A STABLE PROCESS

Lemma

The process $(|X_t|, t \geq 0)$ is strong Markov and self-similar.

- Temporarily write $(X_t^{(x)}, t \geq 0)$ in place of $(X, \mathbb{P}_x)$
- Markov property of X tells us that, for $s, t \geq 0$,

$$X_{t+s}^{(x)} = \tilde{X}_s^{(X_t^{(x)})},$$

  where $\tilde{X}^{(x)}$ is an independent copy of $X^{(x)}$.
- Isotropy implies that

$$|X_{t+s}^{(x)}| = |\tilde{X}_s^{(y)}|_{y=X_t^{(x)}} \stackrel{d}{=} |\tilde{X}_s^{(z)}|_{z=(|X_t^{(x)}|,0,0\cdots,0)}$$

- Hence Markov property holds, strong Markov property (and Feller property) can be developed from this argument
- Self-similarity of $|X|$ follows directly from the self-similarity of X.

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LAMPERTI-TRANSFORM OF $|X|$

Theorem (Caballero-Pardo-Perez (2011))

For the pssMp constructed using the radial part of an isotropic d-dimensional stable process, the underlying Lévy process, ξ that appears through the Lamperti has characteristic exponent given by

$$\Psi(z) = 2^{\alpha} \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z + \alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z + d))}{\Gamma(\frac{1}{2}(\mathrm{i}z + d - \alpha))}, \qquad z \in \mathbb{R}.$$

- The fact that $\lim_{t\to\infty} |X_t| = \infty$ implies that $\lim_{t\to\infty} \xi_t = \infty$
- If we write $\psi(\lambda) = -\Psi(-\mathrm{i}\lambda) = \log \mathbb{E}[\mathrm{e}^{\lambda X_1}]$ for the Laplace exponent of ξ, then it is well defined for $\lambda \in (-d, \alpha)$ with roots at $\lambda = 0$ and $\lambda = \alpha - d$.
- Note that

$$\exp((\alpha - d)\xi_t), \qquad t \geq 0,$$

is a martingale

- Recalling that $|X_t| = \exp(\xi_{\varphi_t})$ and that φ_t is an almost surely finite stopping time (because $\lim_{t\to\infty} \xi_t = \infty$) we can deduce that

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is a martingale (effectively invoking an Esscher transform to ψ).

LAMPERTI-TRANSFORM OF $|X|$

Theorem (Caballero-Pardo-Perez (2011))

For the pssMp constructed using the radial part of an isotropic d-dimensional stable process, the underlying Lévy process, ξ that appears through the Lamperti has characteristic exponent given by

$$\Psi(z) = 2^{\alpha}\frac{\Gamma(\frac{1}{2}(-\mathrm{i}z+\alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)}\frac{\Gamma(\frac{1}{2}(\mathrm{i}z+d))}{\Gamma(\frac{1}{2}(\mathrm{i}z+d-\alpha))}, \qquad z \in \mathbb{R}.$$

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CONDITIONED STABLE PROCESS

- We can define the change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}_x^\circ}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \qquad t \geq 0, x \neq 0$$

- Suppose that f is a bounded measurable function then, for all $c > 0$,

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- Markovian, isotropy and self-similarity properties pass through to $(X, \mathbb{P}_x^\circ)$, $x \neq 0$.
- Similarly $(|X|, \mathbb{P}_x^\circ)$, $x \neq 0$ is a positive self-similar Markov process.

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- It turns out that $(X, \mathbb{P}^\circ_x)$, $x \neq 0$, corresponds to the stable process conditioned to be continuously absorbed at the origin.
- More precisely, for $A \in \sigma(X_s, s \leq t)$, if we set $\{0\}$ to be 'cemetery' state and $\mathtt{k} = \inf\{t > 0 : X_t = 0\}$, then

$$\mathbb{P}^\circ_x(A, t < \mathtt{k}) = \lim_{a \downarrow 0} \mathbb{P}_x(A,\ t < \mathtt{k} | \tau^\oplus_a < \infty),$$

where $\tau^\oplus_a = \inf\{t > 0 : |X_t| < a\}$.
- In light of the associated Esscher transform on ξ, we note that the Lamperti transform of $(|X|, \mathbb{P}^\circ_x)$, $x \neq 0$, corresponds to the Lévy process with characteristic exponent

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$\mathbb{R}^d$-SELF-SIMILAR MARKOV PROCESSES

Definition

A $\mathbb{R}^d$-valued regular Feller process $Z = (Z_t, t \geq 0)$ is called a $\mathbb{R}^d$*-valued self-similar Markov process* if there exists a constant $\alpha > 0$ such that, for any $x > 0$ and $c > 0$,

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Lamperti–Kiu transform

In order to introduce the analogue of the Lamperti transform in d-dimensions, we need to remind ourselves of what we mean by a Markov additive process in this context.

Definition

An $\mathbb{R} \times E$ valued regular Feller process $(\xi, \Theta) = ((\xi_t, \Theta_t) : t \geq 0)$ with probabilities $\mathbf{P}_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in E$, and cemetery state $(-\infty, \dagger)$ is called a *Markov additive process* (MAP) if Θ is a regular Feller process on E with cemetery state $\dagger$ such that, for every bounded measurable function $f : (\mathbb{R} \cup \{-\infty\}) \times (E \cup \{\dagger\}) \to \mathbb{R}$, $t, s \geq 0$ and $(x, \theta) \in \mathbb{R} \times E$, on $\{t < \varsigma\}$,

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- Roughly speaking, one thinks of a MAP as a 'Markov modulated' Lévy process
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LAMPERTI–KIU TRANSFORM

Theorem

Fix $\alpha > 0$. The process Z is a ssMp with index α if and only if there exists a (killed) MAP, (ξ, Θ) on $\mathbb{R} \times \mathbb{S}_{d-1}$ such that

$$Z_t := \mathrm{e}^{\xi_{\varphi(t)}} \Theta_{\varphi(t)} \qquad , \qquad t \le I_\varsigma,$$

where

$$\varphi(t) = \inf\left\{ s > 0 : \int_0^s \mathrm{e}^{\alpha \xi_u}\, \mathrm{d}u > t \right\}, \qquad t \le I_\varsigma,$$

and $I_\varsigma = \int_0^\varsigma \mathrm{e}^{\alpha \xi_s} \mathrm{d}s$ is the lifetime of Z until absorption at the origin. Here, we interpret $\exp\{-\infty\} \times \dagger := 0$ and $\inf \emptyset := \infty$.

- In the representation (**??**), the time to absorption in the origin,

  $$\zeta = \inf\{t > 0 : Z_t = 0\},$$

  satisfies $\zeta = I_\varsigma$.
- Note $x \in \mathbb{R}^d$ if and only if

  $$x = (|x|, \mathrm{Arg}(x)),$$

  where $\mathrm{Arg}(x) = x/|x| \in \mathbb{S}_{d-1}$. The Lamperti–Kiu decomposition therefore gives us a d-dimensional skew product decomposition of self-similar Markov processes.

LAMPERTI-STABLE MAP

- The stable process X is an $\mathbb{R}^d$-valued self-similar Markov process and therefore fits the description above
- **How do we characterise its underlying MAP** (ξ, Θ)?
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MAP ISOTROPY

Theorem

Suppose (ξ, Θ) is the MAP underlying the stable process. Then $((\xi, U^{-1}\Theta), \mathbf{P}_{x,\theta})$ is equal in law to $((\xi, \Theta), \mathbf{P}_{x,U^{-1}\theta})$, for every orthogonal d-dimensional matrix U and $x \in \mathbb{R}^d$, $\theta \in \mathbb{S}_{d-1}$.

Proof.

First note that $\varphi(t) = \int_0^t |X_u|^{-\alpha} \mathrm{d}u$. It follows that

$$(\xi_t, \Theta_t) = (\log |X_{A(t)}|, \operatorname{Arg}(X_{A(t)})), \qquad t \geq 0,$$

where the random times $A(t) = \inf\{s > 0 : \int_0^s |X_u|^{-\alpha} \mathrm{d}u > t\}$ are stopping times in the natural filtration of X.

Now suppose that U is any orthogonal d-dimensional matrix and let $X' = U^{-1}X$. Since X is isotropic and since $|X'| = |X|$, and $\operatorname{Arg}(X') = U^{-1}\operatorname{Arg}(X)$, we see that, for $x \in \mathbb{R}$ and $\theta \in \mathbb{S}_{d-1}$

$$\begin{aligned}
((\xi, U^{-1}\Theta), \mathbf{P}_{\log|x|,\theta}) &= ((\log |X_{A(\cdot)}|, U^{-1}\operatorname{Arg}(X_{A(\cdot)})), \mathbb{P}_x) \\
&\overset{d}{=} ((\log |X_{A(\cdot)}|, \operatorname{Arg}(X_{A(\cdot)})), \mathbb{P}_{U^{-1}x}) \\
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as required. $\square$

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&= ((\xi, \Theta), \mathbf{P}_{\log|x|,U^{-1}\theta})
\end{aligned}$$

as required. □

MAP ISOTROPY

Theorem

Suppose (ξ, Θ) is the MAP underlying the stable process. Then $((\xi, U^{-1}\Theta), \mathbf{P}_{x,\theta})$ is equal in law to $((\xi, \Theta), \mathbf{P}_{x,U^{-1}\theta})$, for every orthogonal d-dimensional matrix U and $x \in \mathbb{R}^d$, $\theta \in \mathbb{S}_{d-1}$.

Proof.

First note that $\varphi(t) = \int_0^t |X_u|^{-\alpha} \mathrm{d}u$. It follows that

$$(\xi_t, \Theta_t) = (\log |X_{A(t)}|, \ \mathrm{Arg}(X_{A(t)})), \qquad t \geq 0,$$

where the random times $A(t) = \inf \left\{ s > 0 : \int_0^s |X_u|^{-\alpha} \mathrm{d}u > t \right\}$ are stopping times in the natural filtration of X.

Now suppose that U is any orthogonal d-dimensional matrix and let $X' = U^{-1}X$. Since X is isotropic and since $|X'| = |X|$, and $\mathrm{Arg}(X') = U^{-1}\mathrm{Arg}(X)$, we see that, for $x \in \mathbb{R}$ and $\theta \in \mathbb{S}_{d-1}$

$$\begin{aligned}
((\xi, U^{-1}\Theta), \mathbf{P}_{\log|x|,\theta}) &= ((\log |X_{A(\cdot)}|, \ U^{-1}\mathrm{Arg}(X_{A(\cdot)})), \mathbb{P}_x) \\
&\stackrel{d}{=} ((\log |X_{A(\cdot)}|, \ \mathrm{Arg}(X_{A(\cdot)})), \mathbb{P}_{U^{-1}x}) \\
&= ((\xi, \Theta), \mathbf{P}_{\log|x|, U^{-1}\theta})
\end{aligned}$$

as required. $\square$

MAP CORROLATION

- We will work with the increments $\Delta\xi_t = \xi_t - \xi_{t-} \in \mathbb{R}$, $t \geq 0$,

Theorem (Bo Li, Victor Rivero, Bertoin-Werner (1996))

Suppose that f is a bounded measurable function on $[0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}_{d-1} \times \mathbb{S}_{d-1}$ such that $f(\cdot,\cdot,0,\cdot,\cdot) = 0$, then, for all $\theta \in \mathbb{S}_{d-1}$,

$$
\mathbf{E}_{0,\theta}\left(\sum_{s>0} f(s, \xi_{s-}, \Delta\xi_s, \Theta_{s-}, \Theta_s)\right)
= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{S}_{d-1}} \int_{\mathbb{S}_{d-1}} \int_{\mathbb{R}} V_\theta(\mathrm{d}s, \mathrm{d}x, \mathrm{d}\vartheta)\sigma_1(\mathrm{d}\phi)\mathrm{d}y \frac{c(\alpha)\mathrm{e}^{yd}}{|\mathrm{e}^y\phi - \vartheta|^{\alpha+d}} f(s,x,y,\vartheta,\phi),
$$

where

$$
V_\theta(\mathrm{d}s, dx, \mathrm{d}\vartheta) = \mathbf{P}_{0,\theta}(\xi_s \in \mathrm{d}x, \Theta_s \in \mathrm{d}\vartheta)\mathrm{d}s, \qquad x \in \mathbb{R}, \vartheta \in \mathbb{S}_{d-1}, s \geq 0,
$$

is the space-time potential of (ξ,Θ) under $\mathbf{P}_{0,\theta}$, $\sigma_1(\phi)$ is the surface measure on $\mathbb{S}_{d-1}$ normalised to have unit mass and

$$
c(\alpha) = 2^{\alpha-1}\pi^{-d}\Gamma((d+\alpha)/2)\Gamma(d/2)/\big|\Gamma(-\alpha/2)\big|.
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MAP OF $(X, \mathbb{P}^\circ_\cdot)$

- Recall that $(|X_t|^{\alpha-d}, t \geq 0)$, is a martingale.
- Informally, we should expect $\mathcal{L}h = 0$, where $h(x) = |x|^{\alpha-d}$ and $\mathcal{L}$ is the infinitesimal generator of the stable process, which has action

$$\mathcal{L}f(x) = \mathrm{a} \cdot \nabla f(x) + \int_{\mathbb{R}^d} [f(x+y) - f(x) - \mathbf{1}_{(|y|\leq 1)} y \cdot \nabla f(x)] \Pi(\mathrm{d}y), \qquad |x| > 0,$$

for appropriately smooth functions.

- Associated to $(X, \mathbb{P}_x)$, $x \neq 0$ is the generator

$$\mathcal{L}^\circ f(x) = \lim_{t\downarrow 0} \frac{\mathbb{E}^\circ_x[f(X_t)] - f(x)}{t} = \lim_{t\downarrow 0} \frac{\mathbb{E}_x[|X_t|^{\alpha-d} f(X_t)] - |x|^{\alpha-d} f(x)}{|x|^{\alpha-d} t},$$

- That is to say

$$\mathcal{L}^\circ f(x) = \frac{1}{h(x)} \mathcal{L}(hf)(x),$$

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- Equivalently, the rate at which $(X, \mathbb{P}^\circ_x)$, $x \neq 0$ jumps given by

$$\Pi^\circ(x, B) := \frac{2^{\alpha-1}\Gamma((d+\alpha)/2)\Gamma(d/2)}{\pi^d |\Gamma(-\alpha/2)|} \int_{\mathbb{S}_{d-1}} \mathrm{d}\sigma_1(\phi) \int_{(0,\infty)} \mathbf{1}_B(r\phi) \frac{\mathrm{d}r}{r^{\alpha+1}} \frac{|x + r\phi|^{\alpha-d}}{|x|^{\alpha-d}},$$

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for $|x| > 0$ and $B \in \mathcal{B}(\mathbb{R}^d)$.

MAP OF $(X, \mathbb{P}^\circ_\cdot)$

Theorem

Suppose that f is a bounded measurable function on $[0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}_{d-1} \times \mathbb{S}_{d-1}$ such that $f(\cdot,\cdot,0,\cdot,\cdot) = 0$, then, for all $\theta \in \mathbb{S}_{d-1}$,

$$\mathbf{E}^\circ_{0,\theta}\left(\sum_{s>0} f(s, \xi_{s-}, \Delta\xi_s, \Theta_{s-}, \Theta_s)\right)$$
$$= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{S}_{d-1}} \int_{\mathbb{S}_{d-1}} \int_{\mathbb{R}} V^\circ_\theta(\mathrm{d}s, \mathrm{d}x, \mathrm{d}\vartheta)\sigma_1(\mathrm{d}\phi)\mathrm{d}y \frac{c(\alpha)\mathrm{e}^{yd}}{|\mathrm{e}^y\phi - \vartheta|^{\alpha+d}} f(s, x, -y, \vartheta, \phi),$$

where

$$V^\circ_\theta(\mathrm{d}s, dx, \mathrm{d}\vartheta) = \mathbf{P}^\circ_{0,\theta}(\xi_s \in \mathrm{d}x, \Theta_s \in \mathrm{d}\vartheta)\mathrm{d}s, \qquad x \in \mathbb{R}, \vartheta \in \mathbb{S}_{d-1}, s \geq 0,$$

is the space-time potential of (ξ, Θ) under $\mathbf{P}^\circ_{0,\theta}$.

Comparing the right-hand side above with that of the previous Theorem, it now becomes immediately clear that the the jump structure of (ξ, Θ) under $\mathbf{P}^\circ_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in \mathbb{S}_{d-1}$, is precisely that of $(-\xi, \Theta)$ under $\mathbf{P}_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in \mathbb{S}_{d-1}$.

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Theorem

Suppose that f is a bounded measurable function on $[0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}_{d-1} \times \mathbb{S}_{d-1}$ such that $f(\cdot,\cdot,0,\cdot,\cdot) = 0$, then, for all $\theta \in \mathbb{S}_{d-1}$,

$$
\begin{aligned}
&\mathbf{E}_{0,\theta}\left(\sum_{s>0} f(s, \xi_{s-}, \Delta\xi_s, \Theta_{s-}, \Theta_s)\right) \\
&= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{S}_{d-1}} \int_{\mathbb{S}_{d-1}} \int_{\mathbb{R}} V_\theta(\mathrm{d}s, \mathrm{d}x, \mathrm{d}\vartheta)\sigma_1(\mathrm{d}\phi)\mathrm{d}y \frac{c(\alpha)\mathrm{e}^{yd}}{|\mathrm{e}^y\phi - \vartheta|^{\alpha+d}} f(s, x, y, \vartheta, \phi),
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§9. Riesz–Bogdan–Żak transform

Riesz–Bogdan–Żak transform

- Define the transformation $K : \mathbb{R}^d \mapsto \mathbb{R}^d$, by

$$Kx = \frac{x}{|x|^2}, \qquad x \in \mathbb{R}^d \backslash \{0\}.$$

- This transformation inverts space through the unit sphere $\{x \in \mathbb{R}^d : |x| = 1\}$.
- Write $x \in \mathbb{R}^d$ in skew product form $x = (|x|, \mathrm{Arg}(x))$, and note that

$$Kx = (|x|^{-1}, \mathrm{Arg}(x)), \qquad x \in \mathbb{R}^d \backslash \{0\},$$

  showing that the K-transform 'radially inverts' elements of $\mathbb{R}^d$ through $\mathbb{S}_{d-1}$.
- In particular $K(Kx) = x$

Theorem (d-dimensional Riesz–Bogdan–Żak Transform, $d \geq 2$)

Suppose that X is a d-dimensional isotropic stable process with $d \geq 2$. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}, \qquad t \geq 0. \tag{3}$$

Then, for all $x \in \mathbb{R}^d \backslash \{0\}$, $(KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}^\circ_{Kx})$.

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Proof of Riesz–Bogdan–Żak transform

We give a proof, different to the original proof of Bogdan and Żak (2010).

- Recall that $X_t = e^{\xi_{\varphi(t)}} \Theta_{\varphi(t)}$, where

$$\int_0^{\varphi(t)} e^{\alpha \xi_u} \, du = t, \qquad t \geq 0.$$

- Note also that, as an inverse,

$$\int_0^{\eta(t)} |X_u|^{-2\alpha} du = t, \qquad t \geq 0.$$

- Differentiating,

$$\frac{d\varphi(t)}{dt} = e^{-\alpha \xi_{\varphi(t)}} \text{ and } \frac{d\eta(t)}{dt} = e^{2\alpha \xi_{\varphi \circ \eta(t)}}, \qquad \eta(t) < \tau^{\{0\}}.$$

and chain rule now tells us that

$$\frac{d(\varphi \circ \eta)(t)}{dt} = \left.\frac{d\varphi(s)}{ds}\right|_{s=\eta(t)} \frac{d\eta(t)}{dt} = e^{\alpha \xi_{\varphi \circ \eta(t)}}.$$

- Said another way,

$$\int_0^{\varphi \circ \eta(t)} e^{-\alpha \xi_u} du = t, \qquad t \geq 0,$$

or

$$\varphi \circ \eta(t) = \inf\{s > 0 : \int_0^s e^{-\alpha \xi_u} du > t\}$$

Proof of Riesz–Bogdan–Żak transform

We give a proof, different to the original proof of Bogdan and Żak (2010).

- Recall that $X_t = e^{\xi_{\varphi(t)}}\Theta_{\varphi(t)}$, where

$$\int_0^{\varphi(t)} e^{\alpha\xi_u}\, du = t, \qquad t \geq 0.$$

- Note also that, as an inverse,

$$\int_0^{\eta(t)} |X_u|^{-2\alpha} du = t, \qquad t \geq 0.$$

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$$KX_{\eta(t)} = \mathrm{e}^{-\xi_{\varphi\circ\eta(t)}}\Theta_{\varphi\circ\eta(t)}, \qquad t \geq 0,$$

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- It follows that $(KX_{\eta(t)}, t \geq 0)$ is a self-similar Markov process with underlying MAP $(-\xi, \Theta)$
- We have also seen that $(X, \mathbb{P}^\circ_x)$, $x \neq 0$, is also a self-similar Markov process with underlying MAP given by $(-\xi, \Theta)$.
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§10. Hitting spheres

PORT'S SPHERE HITTING PROBABILITY

- Recall that a stable process cannot hit points
- We are ultimately interested in the distribution of the position of X on first hitting of the sphere $\mathbb{S}_{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$.
- Define

$$\tau^{\odot} = \inf\{t > 0 : |X_t| = 1\}.$$

- We start with an easier result

Theorem (Port (1969))

If $\alpha \in (1,2)$, then

$$\mathbb{P}_x(\tau^{\odot} < \infty) = \frac{\Gamma\left(\frac{\alpha+d}{2} - 1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha - 1)} \begin{cases} {}_2F_1((d-\alpha)/2, 1-\alpha/2, d/2; |x|^2) & 1 > |x| \\ |x|^{\alpha-d} {}_2F_1((d-\alpha)/2, 1-\alpha/2, d/2; 1/|x|^2) & 1 \leq |x|. \end{cases}$$

Otherwise, if $\alpha \in (0,1]$, then $\mathbb{P}_x(\tau^{\odot} = \infty) = 1$ for all $x \in \mathbb{R}^d$.

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- If (ξ, Θ) is the underlying MAP then

$$\mathbb{P}_x(\tau^{\odot} < \infty) = \mathbf{P}_{\log|x|}(\tau^{\{0\}} < \infty) = \mathbf{P}_0(\tau^{\{\log(1/|x|)\}} < \infty),$$

where $\tau^{\{z\}} = \inf\{t > 0 : \xi_t = z\}$, $z \in \mathbb{R}$. (Note, the time change in the Lamperti–Kiu representation does not level out.)

- Using Sterling's formula, we have, $|\Gamma(x+iy)| = \sqrt{2\pi}e^{-\frac{\pi}{2}|y|}|y|^{x-\frac{1}{2}}(1+o(1))$, for $x, y \in \mathbb{R}$, as $y \to \infty$, uniformly in any finite interval $-\infty < a \le x \le b < \infty$. Hence,

$$\frac{1}{\Psi(z)} = \frac{\Gamma(-\frac{1}{2}iz)}{\Gamma(\frac{1}{2}(-iz+\alpha))}\frac{\Gamma(\frac{1}{2}(iz+d-\alpha))}{\Gamma(\frac{1}{2}(iz+d))} \sim |z|^{-\alpha}$$

uniformly on $\mathbb{R}$ as $|z| \to \infty$.

- From Kesten-Brestagnolle integral test we conclude that $(1+\Psi(z))^{-1}$ is integrable and each sphere $\mathbb{S}_{d-1}$ can be reached with positive probability from any x with $|x| \neq 1$ if and only if $\alpha \in (1,2)$.

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Proof of Port's hitting probability

- Note that

$$\frac{\Gamma(\frac{1}{2}(-\mathrm{i}z+\alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)}\,\frac{\Gamma(\frac{1}{2}(\mathrm{i}z+d))}{\Gamma(\frac{1}{2}(\mathrm{i}z+d-\alpha))}$$

so that $\Psi(-\mathrm{i}z)$, is well defined for $\mathrm{Re}(z)\in(-d,\alpha)$ with roots at 0 and $\alpha-d$.

- We can use the identity

$$\mathbb{P}_x(\tau^{\odot}<\infty)=\frac{u_\xi(\log(1/|x|))}{u_\xi(0)},$$

providing

$$u_\xi(x)=\frac{1}{2\pi\mathrm{i}}\int_{c+\mathrm{i}\mathbb{R}}\frac{\mathrm{e}^{-zx}}{\Psi(-\mathrm{i}z)}\mathrm{d}z,\qquad x\in\mathbb{R},$$

for $c\in(\alpha-d,0)$.

- Build the contour integral around simple poles at $\{-2n-(d-\alpha):n\geq 0\}$.

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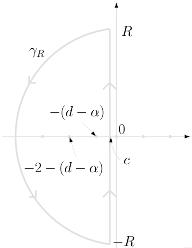

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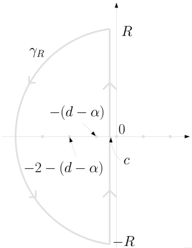

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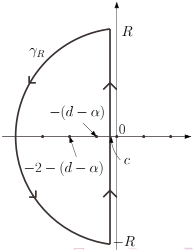

PROOF OF PORT'S HITTING PROBABILITY

- Now fix $x \leq 0$ and recall estimate $|1/\Psi(-\mathrm{i}z)| \lessapprox |z|^{-\alpha}$. The assumption $x \leq 0$ and the fact that the arc length of $\{c + Re^{\mathrm{i}\theta} : \theta \in (\pi/2, 3\pi/2)\}$ is πR, gives us

$$\left| \int_{c+Re^{\mathrm{i}\theta}:\theta\in(\pi/2,3\pi/2)} \frac{e^{-xz}}{\Psi(-\mathrm{i}z)} \mathrm{d}z \right| \leq CR^{-(\alpha-1)} \to 0$$

as $R \to \infty$ for some constant $C > 0$.

- Moreover,

$$u_\xi(x) = \sum_{n\geq 1} \mathrm{Res}\left(\frac{e^{-zx}}{\Psi(-\mathrm{i}z)}; z = -2n - (d-\alpha) \right)$$

$$= \sum_0^\infty (-1)^{n+1} \frac{\Gamma(n + (d-\alpha)/2)}{\Gamma(-n+\alpha/2)\Gamma(n+d/2)} \frac{e^{2nx}}{n!}$$

$$= e^{x(d-\alpha)} \frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)\Gamma(d/2)} {}_2F_1((d-\alpha)/2, 1-\alpha/2, d/2; e^{2x}),$$

Which also gives a value for $u_\xi(0)$.

- Hence, for $1 \leq |x|$,

$$\mathbb{P}_x(\tau^{\odot} < \infty) = \frac{u_\xi(\log(1/|x|))}{u_\xi(0)}$$

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Proof of Port's hitting probability

- Now fix $x \leq 0$ and recall estimate $|1/\Psi(-\mathrm{i}z)| \lesssim |z|^{-\alpha}$. The assumption $x \leq 0$ and the fact that the arc length of $\{c + R\mathrm{e}^{\mathrm{i}\theta} : \theta \in (\pi/2, 3\pi/2)\}$ is πR, gives us

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$$\begin{aligned} \mathbb{P}_x(\tau^\odot < \infty) &= \frac{u_\xi(\log(1/|x|))}{u_\xi(0)} \\ &= \frac{\Gamma\left(\frac{\alpha+d}{2} - 1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)} |x|^{\alpha-d} {}_2F_1((d-\alpha)/2, 1-\alpha/2, d/2; |x|^{-2}). \end{aligned}$$

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  as $R \to \infty$ for some constant $C > 0$.
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$$\begin{aligned} u_\xi(x) &= \sum_{n\geq 1} \mathrm{Res}\left(\frac{\mathrm{e}^{-zx}}{\Psi(-\mathrm{i}z)}; z = -2n - (d-\alpha) \right) \\ &= \sum_0^\infty (-1)^{n+1} \frac{\Gamma(n + (d-\alpha)/2)}{\Gamma(-n+\alpha/2)\Gamma(n+d/2)} \frac{\mathrm{e}^{2nx}}{n!} \\ &= \mathrm{e}^{x(d-\alpha)} \frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)\Gamma(d/2)} {}_2F_1((d-\alpha)/2, 1-\alpha/2, d/2; \mathrm{e}^{2x}), \end{aligned}$$

  Which also gives a value for $u_\xi(0)$.
- Hence, for $1 \leq |x|$,

$$\begin{aligned} \mathbb{P}_x(\tau^\odot < \infty) &= \frac{u_\xi(\log(1/|x|))}{u_\xi(0)} \\ &= \frac{\Gamma\left(\frac{\alpha+d}{2} - 1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)} |x|^{\alpha-d} {}_2F_1((d-\alpha)/2, 1-\alpha/2, d/2; |x|^{-2}). \end{aligned}$$

Proof of Port's hitting probability

- To deal with the case $|x| < 1$, we can appeal to the Riesz–Bogdan–Żak transform to help us.
- To this end we note that, for $|x| < 1$, $|Kx| > 1$

$$\mathbb{P}_{Kx}(\tau^{\odot} < \infty) = \mathbb{P}^{\circ}_{x}(\tau^{\odot} < \infty) = \mathbb{E}_x\left[\frac{|X_{\tau^{\odot}}|^{\alpha-d}}{|x|^{\alpha-d}}\mathbf{1}_{(\tau^{\odot}<\infty)}\right] = \frac{1}{|x|^{\alpha-d}}\mathbb{P}_x(\tau^{\odot} < \infty)$$

- Hence plugging in the expression for $|x| < 1$,

$$\mathbb{P}_x(\tau^{\odot} < \infty) = \frac{\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)}\,{}_2F_1((d-\alpha)/2, 1-\alpha/2, d/2; |x|^2),$$

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RIESZ REPRESENTATION OF PORT'S HITTING PROBABILITY

Theorem

Suppose $\alpha \in (1,2)$. For all $x \in \mathbb{R}^d$,

$$\mathbb{P}_x(\tau^{\odot} < \infty) = \frac{\Gamma\left(\frac{\alpha+d}{2} - 1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha - 1)} \int_{\mathbb{S}_{d-1}} |z - x|^{\alpha - d} \sigma_1(\mathrm{d}z),$$

where $\sigma_1(\mathrm{d}z)$ is the uniform measure on $\mathbb{S}_{d-1}$, normalised to have unit mass. In particular, for $y \in \mathbb{S}_{d-1}$,

$$\int_{\mathbb{S}_{d-1}} |z - y|^{\alpha - d} \sigma_1(\mathrm{d}z) = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha - 1)}{\Gamma\left(\frac{\alpha+d}{2} - 1\right)\Gamma\left(\frac{\alpha}{2}\right)}.$$

Proof of Riesz representation of Port's hitting probability

- We know that $|X_t - z|^{\alpha-d}$, $t \geq 0$ is a martingale.
- Hence we know that

$$M_t := \int_{\mathbb{S}_{d-1}} |z - X_{t\wedge\tau^{\odot}}|^{\alpha-d}\sigma_1(\mathrm{d}z), \qquad t \geq 0,$$

  is a martingale.
- Recall that $\lim_{t\to\infty}|X_t| = 0$ and $\alpha < d$ and hence

$$M_\infty := \lim_{t\to\infty} M_t = \int_{\mathbb{S}_{d-1}} |z - X_{\tau^{\odot}}|^{\alpha-d}\sigma_1(\mathrm{d}z)\mathbf{1}_{(\tau^{\odot}<\infty)} \stackrel{d}{=} C\mathbf{1}_{(\tau^{\odot}<\infty)}.$$

  where, despite the randomness in $X_{\tau^{\odot}}$, by rotational symmetry,

$$C = \int_{\mathbb{S}_{d-1}} |z - 1|^{\alpha-d}\sigma_1(\mathrm{d}z),$$

  and $1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$ is the 'North Pole' on $\mathbb{S}_{d-1}$.
- Since M is a UI martingale, taking expectations of M_∞

$$\int_{\mathbb{S}_{d-1}} |z - x|^{\alpha-d}\sigma_1(\mathrm{d}z) = \mathbb{E}_x[M_0] = \mathbb{E}_x[M_\infty] = C\mathbb{P}_x(\tau^{\odot} < \infty)$$

- Taking limits as $|x| \to 0$,
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Sphere inversions

SPHERE INVERSIONS

- Fix a point $b \in \mathbb{R}^d$ and a value $r > 0$.
- The spatial transformation $x^* : \mathbb{R}^d \backslash \{b\} \mapsto \mathbb{R}^d \backslash \{b\}$

$$x^* = b + \frac{r^2}{|x-b|^2}(x-b),$$

is called an *inversion through the sphere* $\mathbb{S}_{d-1}(b,r) := \{x \in \mathbb{R}^d : |x-b| = r\}$.

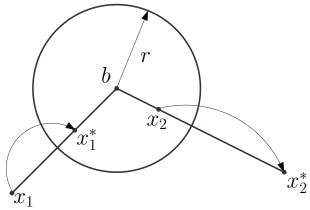


Figure: Inversion relative to the sphere $\mathbb{S}_{d-1}(b,r)$.

Inversion through $\mathbb{S}_{d-1}(b, r)$: key properties

Inversion through $\mathbb{S}_{d-1}(b, r)$

$$x^* = b + \frac{r^2}{|x-b|^2}(x-b),$$

The following can be deduced by straightforward algebra

- Self inverse

$$x = b + r^2 \frac{(x^*-b)}{|x^*-b|^2}$$

- Symmetry

$$r^2 = |x^*-b||x-b|$$

- Difference

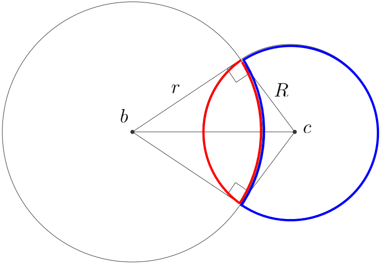
$$|x^*-y^*| = \frac{r^2|x-y|}{|x-b||y-b|}$$

- Differential

$$\mathrm{d}x^* = \frac{r^{2d}}{|x-b|^{2d}}\mathrm{d}x$$

INVERSION THROUGH $\mathbb{S}_{d-1}(b,r)$: KEY PROPERTIES

- The sphere $\mathbb{S}_{d-1}(c,R)$ maps to itself under inversion through $\mathbb{S}_{d-1}(b,r)$ provided the former is orthogonal to the latter, which is equivalent to $r^2 + R^2 = |c-b|^2$.



- In particular, the area contained in the blue segment is mapped to the area in the red segment and vice versa.

SPHERE INVERSION WITH REFLECTION

A variant of the sphere inversion transform takes the form

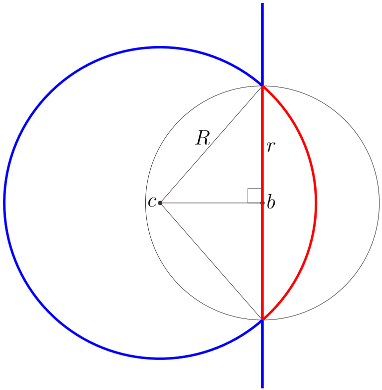
$$x^{\diamond} = b - \frac{r^2}{|x-b|^2}(x-b),$$

and has properties

- Self inverse
$$x = b - \frac{r^2}{|x^{\diamond}-b|^2}(x^{\diamond}-b),$$
- Symmetry
$$r^2 = |x^{\diamond}-b||x-b|,$$
- Difference
$$|x^{\diamond}-y^{\diamond}| = \frac{r^2|x-y|}{|x-b||y-b|}.$$
- Differential
$$\mathrm{d}x^{\diamond} = \frac{r^{2d}}{|x-b|^{2d}}\mathrm{d}x$$

SPHERE INVERSION WITH REFLECTION

- Fix $b \in \mathbb{R}^d$ and $r > 0$. The sphere $\mathbb{S}_{d-1}(c, R)$ maps to itself through $\mathbb{S}_{d-1}(b, r)$ providing $|c - b|^2 + r^2 = R^2$.



- However, this time, the exterior of the sphere $\mathbb{S}_{d-1}(c, R)$ maps to the interior of the sphere $\mathbb{S}_{d-1}(c, R)$ and vice versa. For example, the region in the exterior of $\mathbb{S}_{d-1}(c, R)$ contained by blue boundary maps to the portion of the interior of $\mathbb{S}_{d-1}(c, R)$ contained by the red boundary.

§11. Spherical hitting distribution

PORT'S SPHERE HITTING DISTRIBUTION

A richer version of the previous theorem:

Theorem (Port (1969))

Define the function

$$h^{\odot}(x,y) = \frac{\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)}\frac{||x|^2-1|^{\alpha-1}}{|x-y|^{\alpha+d-2}}$$

for $|x| \neq 1$, $|y| = 1$. *Then, if* $\alpha \in (1,2)$,

$$\mathbb{P}_x(X_{\tau^{\odot}} \in \mathrm{d}y) = h^{\odot}(x,y)\sigma_1(\mathrm{d}y)\mathbf{1}_{(|x|\neq 1)} + \delta_x(\mathrm{d}y)\mathbf{1}_{(|x|=1)}, \qquad |y| = 1,$$

where $\sigma_1(\mathrm{d}y)$ *is the surface measure on* $\mathbb{S}_{d-1}$, *normalised to have unit total mass.*

Otherwise, if $\alpha \in (0,1]$, $\mathbb{P}_x(\tau^{\odot} = \infty) = 1$, *for all* $|x| \neq 1$.

PROOF OF PORT'S SPHERE HITTING DISTRIBUTION

- Write $\mu_x^{\odot}(\mathrm{d}z) = \mathbb{P}_x(X_{\tau^{\odot}} \in \mathrm{d}z)$ on $\mathbb{S}_{d-1}$ where $x \in \mathbb{R}^d \backslash \mathbb{S}_{d-1}$.
- Recall the expression for the resolvent of the stable process in Theorem 17 which states that, due to transience,

$$\int_0^\infty \mathbb{P}_x(X_t \in \mathrm{d}y)\mathrm{d}t = C(\alpha)|x-y|^{\alpha-d}\mathrm{d}y, \qquad x, y \in \mathbb{R}^d,$$

where $C(\alpha)$ is an unimportant constant in the following discussion.

- The measure $\mu_x^{\odot}$ is the solution to the 'functional fixed point equation'

$$|x-y|^{\alpha-d} = \int_{\mathbb{S}_{d-1}} |z-y|^{\alpha-d}\mu(\mathrm{d}z), \qquad y \in \mathbb{S}_{d-1}.$$

Note that $y \in \mathbb{S}_{d-1}$, so the occupation of y from x, will at least see the the process pass through the sphere $\mathbb{S}_{d-1}$ somewhere first (if not y).

- With a little work, we can show it is the unique solution in the class of probability measures.

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Proof of Port's sphere hitting distribution

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Proof of Port's sphere hitting distribution

Recall, for $y^* \in \mathbb{S}_{d-1}$, from the Riesz representation of the sphere hitting probability,

$$\frac{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)}{\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)} = \int_{\mathbb{S}_{d-1}} |z^* - y^*|^{\alpha-d}\sigma_1(\mathrm{d}z^*).$$

we are going to manipulate this identity using sphere inversion to solve the fixed point equation **first assuming that** $|x| > 1$

- Apply the sphere inversion with respect to the sphere $\mathbb{S}_{d-1}(x, (|x|^2-1)^{1/2})$ remembering that this transformation maps $\mathbb{S}_{d-1}$ to itself and using

$$\frac{1}{|z^* - x|^{d-1}}\sigma_1(\mathrm{d}z^*) = \frac{1}{|z - x|^{d-1}}\sigma_1(\mathrm{d}z)$$

$$(|x|^2 - 1) = |z^* - x||z - x| \quad \text{and} \quad |z^* - y^*| = \frac{(|x|^2-1)|z-y|}{|z-x||y-x|}$$

- We have

$$\frac{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)}{\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)} = \int_{\mathbb{S}_{d-1}} |z^* - x|^{d-1}|z^* - y^*|^{\alpha-d}\frac{\sigma_1(\mathrm{d}z^*)}{|z^* - x|^{d-1}}$$

$$= \frac{(|x|^2-1)^{\alpha-1}}{|y-x|^{\alpha-d}}\int_{\mathbb{S}_{d-1}} \frac{|z-y|^{\alpha-d}}{|z-x|^{\alpha+d-2}}\sigma_1(\mathrm{d}z).$$

- For the case $|x| < 1$, use Riesz–Bogdan–Żak theorem again!

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- We have

$$\frac{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)}{\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)} = \int_{\mathbb{S}_{d-1}} |z^*-x|^{d-1}|z^*-y^*|^{\alpha-d}\frac{\sigma_1(\mathrm{d}z^*)}{|z^*-x|^{d-1}}$$
$$= \frac{(|x|^2-1)^{\alpha-1}}{|y-x|^{\alpha-d}}\int_{\mathbb{S}_{d-1}} \frac{|z-y|^{\alpha-d}}{|z-x|^{\alpha+d-2}}\sigma_1(\mathrm{d}z).$$

- For the case $|x|<1$, use Riesz–Bogdan–Żak theorem again!

§12. Spherical entrance/exit distribution

BLUMENTHAL–GETOOR–RAY EXIT / ENTRANCE DISTRIBUTION

Theorem

Define the function

$$g(x,y) = \pi^{-(d/2+1)}\,\Gamma(d/2)\,\sin(\pi\alpha/2)\frac{\left|1-|x|^2\right|^{\alpha/2}}{\left|1-|y|^2\right|^{\alpha/2}}|x-y|^{-d}$$

for $x, y \in \mathbb{R}^d \backslash \mathbb{S}_{d-1}$. *Let*

$$\tau^{\oplus} := \inf\{t>0 : |X_t|<1\} \text{ and } \tau_a^{\ominus} := \inf\{t>0 : |X_t|>1\}.$$

(i) *Suppose that* $|x|<1$, *then*

$$\mathbb{P}_x(X_{\tau^{\ominus}} \in \mathrm{d}y) = g(x,y)\mathrm{d}y, \qquad |y| \geq 1.$$

(ii) *Suppose that* $|x|>1$, *then*

$$\mathbb{P}_x(X_{\tau^{\oplus}} \in \mathrm{d}y,\ \tau^{\oplus}<\infty) = g(x,y)\mathrm{d}y, \qquad |y| \leq 1.$$

Proof of B–G–R entrance/exit distribution (i)

- Appealing again to the potential density and the strong Markov property, it suffices to find a solution to

$$|x-y|^{\alpha-d} = \int_{|z|\geq 1} |z-y|^{\alpha-d}\mu(\mathrm{d}z), \qquad |y|>1,$$

  with a straightforward argument providing uniqueness.
- The proof is complete as soon as we can verify that

$$|x-y|^{\alpha-d} = c_{\alpha,d}\int_{|z|\geq 1} |z-y|^{\alpha-d}\frac{|1-|x|^2|^{\alpha/2}}{|1-|z|^2|^{\alpha/2}}|x-z|^{-d}\mathrm{d}z$$

  for $|y|>1>|x|$, where

$$c_{\alpha,d} = \pi^{-(1+d/2)}\,\Gamma(d/2)\,\sin(\pi\alpha/2).$$

Proof of B–G–R entrance/exit distribution (I)

- Appealing again to the potential density and the strong Markov property, it suffices to find a solution to

$$|x-y|^{\alpha-d} = \int_{|z|\geq 1} |z-y|^{\alpha-d} \mu(\mathrm{d}z), \qquad |y|>1,$$

with a straightforward argument providing uniqueness.

- The proof is complete as soon as we can verify that

$$|x-y|^{\alpha-d} = c_{\alpha,d} \int_{|z|\geq 1} |z-y|^{\alpha-d} \frac{|1-|x|^2|^{\alpha/2}}{|1-|z|^2|^{\alpha/2}} |x-z|^{-d} \mathrm{d}z$$

for $|y|>1>|x|$, where

$$c_{\alpha,d} = \pi^{-(1+d/2)}\, \Gamma(d/2)\, \sin(\pi\alpha/2).$$

Proof of B–G–R entrance/exit distribution (I)

- Transform $z \mapsto z^\diamond$ (sphere inversion with reflection) through the sphere $\mathbb{S}_{d-1}(x, (1-|x|^2)^{1/2})$, noting in particular that

$$|z^\diamond - y^\diamond| = (1-|x|^2)\frac{|z-y|}{|z-x||y-x|} \quad \text{and} \quad |z|^2 - 1 = \frac{|z-x|^2}{1-|x|^2}(1-|z^\diamond|^2)$$

and

$$\mathrm{d}z^\diamond = (1-|x|^2)^d |z-x|^{-2d}\mathrm{d}z, \qquad z \in \mathbb{R}^d.$$

- For $|x| < 1 < |y|$,

$$\int_{|z|\geq 1} |z-y|^{\alpha-d}\frac{|1-|x|^2|^{\alpha/2}}{|1-|z|^2|^{\alpha/2}}|x-z|^{-d}\mathrm{d}z = |y-x|^{\alpha-d}\int_{|z^\diamond|\leq 1}\frac{|z^\diamond - y^\diamond|^{\alpha-d}}{|1-|z^\diamond|^2|^{\alpha/2}}\mathrm{d}z^\diamond.$$

- Now perform similar transformation $z^\diamond \mapsto w$ (inversion with reflection), albeit through the sphere $\mathbb{S}_{d-1}(y^\diamond, (1-|y^\diamond|^2)^{1/2})$.

$$|y-x|^{\alpha-d}\int_{|z^\diamond|\leq 1}\frac{|z^\diamond - y^\diamond|^{\alpha-d}}{|1-|z^\diamond|^2|^{\alpha/2}}\mathrm{d}z^\diamond = |y-x|^{\alpha-d}\int_{|w|\geq 1}\frac{|1-|y^\diamond|^2|^{\alpha/2}}{|1-|w|^2|^{\alpha/2}}|w-y^\diamond|^{-d}\mathrm{d}w.$$

PROOF OF B–G–R ENTRANCE/EXIT DISTRIBUTION (I)

- Transform $z \mapsto z^\diamond$ (sphere inversion with reflection) through the sphere $\mathbb{S}_{d-1}(x, (1-|x|^2)^{1/2})$, noting in particular that

$$|z^\diamond - y^\diamond| = (1-|x|^2)\frac{|z-y|}{|z-x||y-x|} \quad \text{and} \quad |z|^2 - 1 = \frac{|z-x|^2}{1-|x|^2}(1-|z^\diamond|^2)$$

and

$$\mathrm{d}z^\diamond = (1-|x|^2)^d |z-x|^{-2d}\mathrm{d}z, \qquad z \in \mathbb{R}^d.$$

- For $|x| < 1 < |y|$,

$$\int_{|z|\geq 1} |z-y|^{\alpha-d}\frac{|1-|x|^2|^{\alpha/2}}{|1-|z|^2|^{\alpha/2}}|x-z|^{-d}\mathrm{d}z = |y-x|^{\alpha-d}\int_{|z^\diamond|\leq 1}\frac{|z^\diamond - y^\diamond|^{\alpha-d}}{|1-|z^\diamond|^2|^{\alpha/2}}\mathrm{d}z^\diamond.$$

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Proof of B–G–R entrance/exit distribution (I)

- Transform $z \mapsto z^{\diamond}$ (sphere inversion with reflection) through the sphere $\mathbb{S}_{d-1}(x,(1-|x|^2)^{1/2})$, noting in particular that

$$|z^{\diamond}-y^{\diamond}| = (1-|x|^2)\frac{|z-y|}{|z-x||y-x|} \quad \text{and} \quad |z|^2-1 = \frac{|z-x|^2}{1-|x|^2}(1-|z^{\diamond}|^2)$$

and

$$\mathrm{d}z^{\diamond} = (1-|x|^2)^d|z-x|^{-2d}\mathrm{d}z, \qquad z \in \mathbb{R}^d.$$

- For $|x| < 1 < |y|$,

$$\int_{|z|\geq 1}|z-y|^{\alpha-d}\frac{|1-|x|^2|^{\alpha/2}}{|1-|z|^2|^{\alpha/2}}|x-z|^{-d}\mathrm{d}z = |y-x|^{\alpha-d}\int_{|z^{\diamond}|\leq 1}\frac{|z^{\diamond}-y^{\diamond}|^{\alpha-d}}{|1-|z^{\diamond}|^2|^{\alpha/2}}\mathrm{d}z^{\diamond}.$$

- Now perform similar transformation $z^{\diamond} \mapsto w$ (inversion with reflection), albeit through the sphere $\mathbb{S}_{d-1}(y^{\diamond},(1-|y^{\diamond}|^2)^{1/2})$.

$$|y-x|^{\alpha-d}\int_{|z^{\diamond}|\leq 1}\frac{|z^{\diamond}-y^{\diamond}|^{\alpha-d}}{|1-|z^{\diamond}|^2|^{\alpha/2}}\mathrm{d}z^{\diamond} = |y-x|^{\alpha-d}\int_{|w|\geq 1}\frac{|1-|y^{\diamond}|^2|^{\alpha/2}}{|1-|w|^2|^{\alpha/2}}|w-y^{\diamond}|^{-d}\mathrm{d}w.$$

Proof of B–G–R entrance/exit distribution (I)

Thus far:

$$\int_{|z|\geq 1} |z-y|^{\alpha-d} \frac{|1-|x|^2|^{\alpha/2}}{|1-|z|^2|^{\alpha/2}} |x-z|^{-d} \mathrm{d}z = |y-x|^{\alpha-d} \int_{|w|\geq 1} \frac{|1-|y^\diamond|^2|^{\alpha/2}}{|1-|w|^2|^{\alpha/2}} |w-y^\diamond|^{-d} \mathrm{d}w.$$

- Taking the integral in red and decomposition into generalised spherical polar coordinates

$$\int_{|v|\geq 1} \frac{1}{|1-|w|^2|^{\alpha/2}} |w-y^\diamond|^{-d} \mathrm{d}w = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_1^\infty \frac{r^{d-1}\mathrm{d}r}{|1-r^2|^{\alpha/2}} \int_{\mathbb{S}_{d-1}(0,r)} |z-y^\diamond|^{-d} \sigma_r(\mathrm{d}z)$$

- Poisson's formula (the probability that a Brownian motion hits a sphere of radius $r > 0$) states that

$$\int_{\mathbb{S}_{d-1}(0,r)} \frac{r^{d-2}(r^2-|y^\diamond|^2)}{|z-y^\diamond|^d} \sigma_r(\mathrm{d}z) = 1, \qquad |y^\diamond| < 1 < r.$$

gives us

$$\int_{|v|\geq 1} \frac{1}{|1-|w|^2|^{\alpha/2}} |w-y^\diamond|^{-d} \mathrm{d}w = \frac{\pi^{d/2}}{\Gamma(d/2)} \int_1^\infty \frac{2r}{(r^2-1)^{\alpha/2}(r^2-|y^\diamond|^2)} \mathrm{d}r$$

$$= \frac{\pi}{\sin(\alpha\pi/2)} \frac{1}{(1-|y^\diamond|^2)^{\alpha/2}}$$

- Plugging everything back in gives the result for $|x| < 1$.

Proof of B–G–R entrance/exit distribution (I)

Thus far:

$$\int_{|z|\geq 1}|z-y|^{\alpha-d}\frac{|1-|x|^2|^{\alpha/2}}{|1-|z|^2|^{\alpha/2}}|x-z|^{-d}\mathrm{d}z = |y-x|^{\alpha-d}\int_{|w|\geq 1}\frac{|1-|y^\diamond|^2|^{\alpha/2}}{|1-|w|^2|^{\alpha/2}}|w-y^\diamond|^{-d}\mathrm{d}w.$$

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- Poisson's formula (the probability that a Brownian motion hits a sphere of radius $r>0$) states that

$$\int_{\mathbb{S}_{d-1}(0,r)}\frac{r^{d-2}(r^2-|y^\diamond|^2)}{|z-y^\diamond|^d}\sigma_r(\mathrm{d}z) = 1, \qquad |y^\diamond|<1<r.$$

gives us

$$\int_{|v|\geq 1}\frac{1}{|1-|w|^2|^{\alpha/2}}|w-y^\diamond|^{-d}\mathrm{d}w = \frac{\pi^{d/2}}{\Gamma(d/2)}\int_1^\infty\frac{2r}{(r^2-1)^{\alpha/2}(r^2-|y^\diamond|^2)}\mathrm{d}r$$
$$= \frac{\pi}{\sin(\alpha\pi/2)}\frac{1}{(1-|y^\diamond|^2)^{\alpha/2}}$$

- Plugging everything back in gives the result for $|x|<1$.

Proof of B–G–R entrance/exit distribution (II)

The interesting part of the proof is the derivation of the the identity in (ii) (i.e. $|x| > 1$) from the identity in (i) (i.e. $|x| < 1$).

- Start by noting from the Riesz–Bogdan–Żak transform that, for $|x| > 1$,

$$\mathbb{P}_x(X_{\tau^{\oplus}} \in D) = \mathbb{P}^{\circ}_{Kx}(KX_{\tau^{\ominus}} \in D),$$

  where $Kx = x/|x|^2$, $|Kx - Kz| = |x - z|/|x||z|$ and $KD = \{Kx : x \in D\}$.

- Noting that $\mathrm{d}(Kz) = |z|^{-2d}\mathrm{d}z$, we have

$$\begin{aligned}
&\mathbb{P}_x(X_{\tau^{\oplus}} \in D)\\
&= \int_{KD} \frac{|y|^{\alpha-d}}{|Kx|^{\alpha-d}} g(Kx, y)\mathrm{d}y\\
&= c_{\alpha,d} \int_{KD} |z|^{d-\alpha}|Kx|^{d-\alpha} \frac{|1 - |Kx|^2|^{\alpha/2}}{|1 - |y|^2|^{\alpha/2}} |Kx - y|^{-d}\mathrm{d}y\\
&= c_{\alpha,d} \int_{D} |z|^{2d} \frac{|1 - |x|^2|^{\alpha/2}}{|1 - |z|^2|^{\alpha/2}} |x - z|^{-d}\mathrm{d}(Kz)\\
&= c_{\alpha,d} \int_{D} \frac{|1 - |x|^2|^{\alpha/2}}{|1 - |z|^2|^{\alpha/2}} |x - z|^{-d}\mathrm{d}z
\end{aligned}$$

Proof of B–G–R entrance/exit distribution (ii)

The interesting part of the proof is the derivation of the the identity in (ii) (i.e. $|x| > 1$) from the identity in (i) (i.e. $|x| < 1$).

- Start by noting from the Riesz–Bogdan–Żak transform that, for $|x| > 1$,

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- Noting that $\mathrm{d}(Kz) = |z|^{-2d}\mathrm{d}z$, we have

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&\mathbb{P}_x(X_{\tau^\oplus} \in D) \\
&= \int_{KD} \frac{|y|^{\alpha-d}}{|Kx|^{\alpha-d}} g(Kx, y)\mathrm{d}y \\
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&= c_{\alpha,d} \int_{D} \frac{|1-|x|^2|^{\alpha/2}}{|1-|z|^2|^{\alpha/2}} |x - z|^{-d}\mathrm{d}z
\end{aligned}$$

§13. Radial excursion theory

Excursions from the radial minimum

Recall that we can represent an isotropic Lévy process through the Lamperti transform

$$X_t := \mathrm{e}^{\xi_{\varphi(t)}}\Theta_{\varphi(t)} \qquad t \geq 0,$$

where

$$\varphi(t) = \inf\left\{ s > 0 : \int_0^s \mathrm{e}^{\alpha \xi_u}\,\mathrm{d}u > t \right\}$$

and (ξ, Θ) with probabilities $\mathbf{P}_{x,\theta}$, $x \neq 0$, $\theta \in \mathbb{S}_{d-1}$, is a MAP. Recall also that, although corollated to Θ, ξ alone is a Lévy process.

- Let $\ell = (\ell_t, t \geq 0)$, the local time at 0 of the reflected Lévy process $\xi_t - \underline{\xi}_t$, $t \geq 0$, where $\underline{\xi}_t := \inf_{s \leq t} \xi_s$, $t \geq 0$.
- The process ℓ serves as an adequate choice for the local time of the Markov process $(\xi - \underline{\xi}, \Theta)$ on the set $\{0\} \times \mathbb{S}_{d-1}$.
- Define

$$\mathtt{g}_t = \sup\{s < t : \xi_s = \underline{\xi}_s\} \text{ and } \mathtt{d}_t = \inf\{s > t : \xi_s = \underline{\xi}_s\}.$$

- For all $t > 0$ such that $\mathtt{d}_t > \mathtt{g}_t$ the process

$$(\epsilon_{\mathtt{g}_t}(s), \Theta^{\epsilon}_{\mathtt{g}_t}(s)) := (\xi_{\mathtt{g}_t + s} - \xi_{\mathtt{g}_t}, \Theta_{\mathtt{g}_t + s}), \qquad s \leq \zeta_{\mathtt{g}_t} := \mathtt{d}_t - \mathtt{g}_t,$$

  codes the excursions of $(\xi - \underline{\xi}, \Theta)$ from the set $(0, \mathbb{S}_{d-1})$ or equivalently, excursions of $(X_t / \inf_{s \leq t}|X_s|, t \geq 0)$, from $\mathbb{S}_{d-1}$, or equivalently an excursion of X from its running radial infimum.
- Moreover, we see that, for all $t > 0$ such that $\mathtt{d}_t > \mathtt{g}_t$,

$$X_{\mathtt{g}_t + s} = \mathrm{e}^{\xi_{\mathtt{g}_t}}\mathrm{e}^{\epsilon_{\mathtt{g}_t}(s)}\Theta^{\epsilon}_{\mathtt{g}_t}(s) = |X_{\mathtt{g}_t}|\mathrm{e}^{\epsilon_{\mathtt{g}_t}(s)}\Theta^{\epsilon}_{\mathtt{g}_t}(s), \qquad s \leq \zeta_{\mathtt{g}_t}.$$

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$$(\epsilon_{\mathtt{g}_t}(s), \Theta^{\epsilon}_{\mathtt{g}_t}(s)) := (\xi_{\mathtt{g}_t+s} - \xi_{\mathtt{g}_t}, \Theta_{\mathtt{g}_t+s}), \qquad s \leq \zeta_{\mathtt{g}_t} := \mathtt{d}_t - \mathtt{g}_t,$$

codes the excursions of $(\xi - \underline{\xi}, \Theta)$ from the set $(0, \mathbb{S}_{d-1})$ or equivalently, excursions of $(X_t/\inf_{s\leq t}|X_s|, t \geq 0)$, from $\mathbb{S}_{d-1}$, or equivalently an excursion of X from its running radial infimum.
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$$X_{\mathtt{g}_t+s} = \mathrm{e}^{\xi_{\mathtt{g}_t}}\mathrm{e}^{\epsilon_{\mathtt{g}_t}(s)}\Theta^{\epsilon}_{\mathtt{g}_t}(s) = |X_{\mathtt{g}_t}|\mathrm{e}^{\epsilon_{\mathtt{g}_t}(s)}\Theta^{\epsilon}_{\mathtt{g}_t}(s), \qquad s \leq \zeta_{\mathtt{g}_t}.$$

Excursions from the radial minimum

Recall that we can represent an isotropic Lévy process through the Lamperti transform

$$X_t := \mathrm{e}^{\xi_{\varphi(t)}}\Theta_{\varphi(t)} \qquad t \geq 0,$$

where

$$\varphi(t) = \inf\left\{s > 0 : \int_0^s \mathrm{e}^{\alpha\xi_u}\,\mathrm{d}u > t\right\}$$

and (ξ, Θ) with probabilities $\mathbf{P}_{x,\theta}$, $x \neq 0$, $\theta \in \mathbb{S}_{d-1}$, is a MAP. Recall also that, although corollated to Θ, ξ alone is a Lévy process.

- Let $\ell = (\ell_t, t \geq 0)$, the local time at 0 of the reflected Lévy process $\xi_t - \underline{\xi}_t$, $t \geq 0$, where $\underline{\xi}_t := \inf_{s \leq t} \xi_s$, $t \geq 0$.
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Recall that we can represent an isotropic Lévy process through the Lamperti transform

$$X_t := e^{\xi_{\varphi(t)}} \Theta_{\varphi(t)} \qquad t \geq 0,$$

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Recall that we can represent an isotropic Lévy process through the Lamperti transform

$$X_t := \mathrm{e}^{\xi_{\varphi(t)}}\Theta_{\varphi(t)} \qquad t \geq 0,$$

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EXCURSIONS FROM THE RADIAL MINIMUM

- The classical theory of exit systems in Maisonneuve (1975) now implies that there exists a family of *excursion measures*, $\mathbb{N}_\theta$, $\theta \in \mathbb{S}_{d-1}$, such that:
- the map $\theta \mapsto \mathbb{N}_\theta$ is a kernel from $\mathbb{S}_{d-1}$ to $\mathbb{R} \times \mathbb{S}_{d-1}$, such that $\mathbb{N}_\theta(1 - e^{-\zeta}) < \infty$ and $\mathbb{N}_\theta$ is carried by the set $\{(\epsilon(0), \Theta^\epsilon(0) = (0, \theta)\}$ and $\{\zeta > 0\}$;
- we have the *exit formula*

$$\mathbf{E}_{x,\theta}\left[\sum_{\mathtt{g}\in G} F((\xi_s, \Theta_s) : s < \mathtt{g}) H((\epsilon_{\mathtt{g}}, \Theta^\epsilon_{\mathtt{g}}))\right]$$
$$= \mathbf{E}_{x,\theta}\left[\int_0^\infty F((\xi_s, \Theta_s) : s < t)\mathbb{N}_{\Theta_t}(H(\epsilon, \Theta^\epsilon))\mathrm{d}\ell_t\right],$$

  for $x \neq 0$, where F and H are continuous on the space of càdlàg paths on $\mathbb{R} \times \mathbb{S}_{d-1}$) and $G = \{\mathtt{g}_s : s \geq 0\}$
- under any measure $\mathbb{N}_\theta$ the process $(\epsilon, \Theta^\epsilon)$ is Markovian with the same *transition* semigroup as (ξ, Θ) stopped at its first hitting time of $(-\infty, 0] \times \mathbb{S}_{d-1}$.
- The couple $(\ell, \mathbb{N}_\cdot)$ is called an exit system. The pair ℓ and the kernels $\mathbb{N}_\theta$, $\theta \in \mathbb{S}_{d-1}$, are not unique, but once ℓ is chosen the measures $\mathbb{N}_\theta$ are determined but for a ℓ-neglectable set.

Excursions from the radial minimum

- The classical theory of exit systems in Maisonneuve (1975) now implies that there exists a family of *excursion measures*, $\mathbb{N}_\theta$, $\theta \in \mathbb{S}_{d-1}$, such that:
- the map $\theta \mapsto \mathbb{N}_\theta$ is a kernel from $\mathbb{S}_{d-1}$ to $\mathbb{R} \times \mathbb{S}_{d-1}$, such that $\mathbb{N}_\theta(1 - e^{-\zeta}) < \infty$ and $\mathbb{N}_\theta$ is carried by the set $\{(\epsilon(0), \Theta^\epsilon(0) = (0, \theta)\}$ and $\{\zeta > 0\}$;
- we have the *exit formula*

$$\mathbf{E}_{x,\theta}\left[\sum_{\mathfrak{g}\in G} F((\xi_s, \Theta_s) : s < \mathfrak{g}) H((\epsilon_{\mathfrak{g}}, \Theta^\epsilon_{\mathfrak{g}}))\right]$$
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- The couple $(\ell, \mathbb{N}_\cdot)$ is called an exit system. The pair ℓ and the kernels $\mathbb{N}_\theta$, $\theta \in \mathbb{S}_{d-1}$, are not unique, but once ℓ is chosen the measures $\mathbb{N}_\theta$ are determined but for a ℓ-neglectable set.

RADIAL LADDER MAP

- For bounded measurable f on $\mathbb{R}^d$ and $G(\infty) := \sup\{s \geq 0 : |X_s| = \inf_{u \leq s} |X_u|\}$,

$$\mathbb{E}_x[f(X_{G(\infty)})] = \mathbf{E}_{\log|x|,\arg(x)}\left[\sum_{t \in G} f(\mathrm{e}^{\xi_t}\Theta_t)\mathbf{1}(\zeta_t = \infty)\right]$$

$$= \mathbf{E}_{\log|x|,\arg(x)}\left[\int_0^\infty f(\mathrm{e}^{\xi_t}\Theta_t)\mathbb{N}_{\Theta_t}(\zeta = \infty)\mathrm{d}\ell_t\right]$$

$$= \mathbf{E}_{\log|x|,\arg(x)}\left[\int_0^{\ell_\infty} f(\mathrm{e}^{-H_t^-}\Theta_t^-)\mathbb{N}_{\Theta_t^-}(\zeta = \infty)\mathrm{d}t\right]$$

where $(H_t^-, \Theta_t^-) = (-\xi_{\ell_t^{-1}}, \Theta_{\ell_t^{-1}})$, $t < \ell_\infty$.

- Define the potential

$$U_x^-(\mathrm{d}z) := \int_0^\infty \mathbf{P}_{\log|x|,\arg(x)}(\mathrm{e}^{-H_t^-}\Theta_t^- \in \mathrm{d}z,\ t < \ell_\infty)\mathrm{d}t, \qquad |z| \leq |x|.$$

- As X is transient, (H^-, Θ^-) experiences killing at Θ^--dependent rate $\mathbb{N}_\theta(\zeta = \infty)$, $\theta \in \mathbb{S}_{d-1}$. Isotropy implies $\mathbb{N}_\theta(\zeta = \infty)$ independent of θ. Scaling of local time ℓ chosen so that $\mathbb{N}_\theta(\zeta = \infty) = 1$.
- In conclusion, we reach the identity

$$\mathbb{E}_x[f(X_{G(\infty)})] = \int_{|z|<|x|} f(z)U_x^-(\mathrm{d}z)$$

Radial ladder MAP

- For bounded measurable f on $\mathbb{R}^d$ and $G(\infty) := \sup\{s \geq 0 : |X_s| = \inf_{u \leq s} |X_u|\}$,

$$
\begin{aligned}
\mathbb{E}_x[f(X_{G(\infty)})] &= \mathbf{E}_{\log|x|,\arg(x)}\left[\sum_{t \in G} f(\mathrm{e}^{\xi_t}\Theta_t)\mathbf{1}(\zeta_t = \infty)\right] \\
&= \mathbf{E}_{\log|x|,\arg(x)}\left[\int_0^\infty f(\mathrm{e}^{\xi_t}\Theta_t)\mathbb{N}_{\Theta_t}(\zeta = \infty)\mathrm{d}\ell_t\right] \\
&= \mathbf{E}_{\log|x|,\arg(x)}\left[\int_0^{\ell_\infty} f(\mathrm{e}^{-H_t^-}\Theta_t^-)\mathbb{N}_{\Theta_t^-}(\zeta = \infty)\mathrm{d}t\right]
\end{aligned}
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where $(H_t^-, \Theta_t^-) = (-\xi_{\ell_t^{-1}}, \Theta_{\ell_t^{-1}})$, $t < \ell_\infty$.

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U_x^-(\mathrm{d}z) := \int_0^\infty \mathbf{P}_{\log|x|,\arg(x)}(\mathrm{e}^{-H_t^-}\Theta_t^- \in \mathrm{d}z,\ t < \ell_\infty)\mathrm{d}t, \qquad |z| \leq |x|.
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- As X is transient, (H^-, Θ^-) experiences killing at Θ^--dependent rate $\mathbb{N}_\theta(\zeta = \infty)$, $\theta \in \mathbb{S}_{d-1}$. Isotropy implies $\mathbb{N}_\theta(\zeta = \infty)$ independent of θ. Scaling of local time ℓ chosen so that $\mathbb{N}_\theta(\zeta = \infty) = 1$.
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Radial ladder MAP

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$$\begin{aligned}\mathbb{E}_x[f(X_{G(\infty)})] &= \mathbf{E}_{\log|x|,\arg(x)}\left[\sum_{t \in G} f(\mathrm{e}^{\xi_t}\Theta_t)\mathbf{1}(\zeta_t = \infty)\right] \\ &= \mathbf{E}_{\log|x|,\arg(x)}\left[\int_0^\infty f(\mathrm{e}^{\xi_t}\Theta_t)\mathbb{N}_{\Theta_t}(\zeta = \infty)\mathrm{d}\ell_t\right] \\ &= \mathbf{E}_{\log|x|,\arg(x)}\left[\int_0^{\ell_\infty} f(\mathrm{e}^{-H_t^-}\Theta_t^-)\mathbb{N}_{\Theta_t^-}(\zeta = \infty)\mathrm{d}t\right]\end{aligned}$$

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$$\begin{aligned}
\mathbb{E}_x[f(X_{G(\infty)})] &= \mathbf{E}_{\log|x|,\arg(x)}\left[\sum_{t \in G} f(e^{\xi_t}\Theta_t)\mathbf{1}(\zeta_t = \infty)\right] \\
&= \mathbf{E}_{\log|x|,\arg(x)}\left[\int_0^\infty f(e^{\xi_t}\Theta_t)\mathbb{N}_{\Theta_t}(\zeta = \infty)\mathrm{d}\ell_t\right] \\
&= \mathbf{E}_{\log|x|,\arg(x)}\left[\int_0^{\ell_\infty} f(e^{-H_t^-}\Theta_t^-)\mathbb{N}_{\Theta_t^-}(\zeta = \infty)\mathrm{d}t\right]
\end{aligned}$$

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Point of closest reach

Theorem (Point of Closest Reach to the origin)

The law of the point of closest reach to the origin is given by

$$\mathbb{P}_x(X_{G(\infty)} \in \mathrm{d}y) = \pi^{-d/2} \frac{\Gamma(d/2)^2}{\Gamma((d-\alpha)/2)\,\Gamma(\alpha/2)} \frac{(|x|^2 - |y|^2)^{\alpha/2}}{|x-y|^d |y|^\alpha} \mathrm{d}y, \qquad 0 < |y| < |x|.$$

Point of closest reach: sketch proof

- First define, for $x \neq 0$, $|x| > r$, $\delta > 0$ and continuous, positive and bounded f on $\mathbb{R}^d$,

$$\Delta_r^\delta f(x) := \frac{1}{\delta}\mathbb{E}_x\left[f(\arg(X_{G_\infty})), |X_{G_\infty}| \in [r-\delta, r]\right].$$

- Then, with the help of Blumenthal–Getoor–Ray first entry distribution,

$$\begin{aligned}
&\Delta_r^\delta f(x)\\
&= \frac{1}{\delta}\int_{|y|\in[r-\delta,r]} \mathbb{P}_x(X_{\tau_r^\oplus} \in \mathrm{d}y;\ \tau_r^\oplus < \infty)\mathbb{E}_y\left[f(\arg(X_{G_\infty}));\ |X_{G_\infty}| \in (r-\delta, |y|]\right]\\
&= \frac{1}{\delta}C_{\alpha,d}\int_{|y|\in[r-\delta,r]} \mathrm{d}y\left|\frac{r^2-|x|^2}{r^2-|y|^2}\right|^{\alpha/2}|y-x|^{-d}\mathbb{E}_y\left[f(\arg(X_{G_\infty}));\ |X_{G_\infty}| \in (r-\delta, |y|]\right]\\
&= \frac{1}{\delta}C_{\alpha,d}|r^2-|x|^2|^{\alpha/2}\int_{|y|\in(r-\delta,r]} \mathrm{d}y\frac{|y-x|^{-d}}{|r^2-|y|^2|^{\alpha/2}}\int_{r-\delta\leq|z|\leq|y|} U_y^-(\mathrm{d}z)f(\arg(z)),
\end{aligned}$$

Lemma

Suppose that f is a bounded continuous function on $\mathbb{R}^d$. Then

$$\lim_{\delta\to 0}\sup_{|y|\in(r-\delta,r]}\left|\frac{\int_{r-\delta\leq|z|\leq|y|} U_y^-(\mathrm{d}z)f(z)}{\int_{r-\delta\leq|z|\leq|y|} U_y^-(\mathrm{d}z)} - f(y)\right| = 0.$$

POINT OF CLOSEST REACH: SKETCH PROOF

- First define, for $x \neq 0$, $|x| > r$, $\delta > 0$ and continuous, positive and bounded f on $\mathbb{R}^d$,

$$\Delta_r^\delta f(x) := \frac{1}{\delta}\mathbb{E}_x\left[f(\arg(X_{\mathtt{G}_\infty})), |X_{\mathtt{G}_\infty}| \in [r-\delta, r]\right].$$

- Then, with the help of Blumenthal–Getoor–Ray first entry distribution,

$$\begin{aligned}
&\Delta_r^\delta f(x)\\
&= \frac{1}{\delta}\int_{|y|\in[r-\delta,r]} \mathbb{P}_x(X_{\tau_r^\oplus} \in \mathrm{d}y;\, \tau_r^\oplus < \infty)\mathbb{E}_y\left[f(\arg(X_{\mathtt{G}_\infty}));\, |X_{\mathtt{G}_\infty}| \in (r-\delta, |y|]\right]\\
&= \frac{1}{\delta}C_{\alpha,d}\int_{|y|\in[r-\delta,r]} \mathrm{d}y\left|\frac{r^2-|x|^2}{r^2-|y|^2}\right|^{\alpha/2} |y-x|^{-d}\mathbb{E}_y\left[f(\arg(X_{\mathtt{G}_\infty}));\, |X_{\mathtt{G}_\infty}| \in (r-\delta, |y|]\right]\\
&= \frac{1}{\delta}C_{\alpha,d}|r^2-|x|^2|^{\alpha/2}\int_{|y|\in(r-\delta,r]} \mathrm{d}y\frac{|y-x|^{-d}}{|r^2-|y|^2|^{\alpha/2}}\int_{r-\delta\le|z|\le|y|} U_y^-(\mathrm{d}z)f(\arg(z)),
\end{aligned}$$

Lemma

Suppose that f is a bounded continuous function on $\mathbb{R}^d$. Then

$$\lim_{\delta\to 0}\sup_{|y|\in(r-\delta,r]}\left|\frac{\int_{r-\delta\le|z|\le|y|} U_y^-(\mathrm{d}z)f(z)}{\int_{r-\delta\le|z|\le|y|} U_y^-(\mathrm{d}z)} - f(y)\right| = 0.$$

Point of closest reach: sketch proof

- First define, for $x \neq 0$, $|x| > r$, $\delta > 0$ and continuous, positive and bounded f on $\mathbb{R}^d$,

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&= \frac{1}{\delta}C_{\alpha,d}|r^2-|x|^2|^{\alpha/2}\int_{|y|\in(r-\delta,r]} \mathrm{d}y \frac{|y-x|^{-d}}{|r^2-|y|^2|^{\alpha/2}}\int_{r-\delta\le|z|\le|y|} U_y^-(\mathrm{d}z)f(\arg(z)),
\end{aligned}$$

Lemma

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Point of closest reach: sketch proof

- Hence

$$\Delta_r^\delta f(x) \overset{\delta\downarrow 0}{\sim} \frac{1}{\delta} C_{\alpha,d} |r^2 - |x|^2|^{\alpha/2} \int_{|y|\in(r-\delta,r]} \mathrm{d}y \frac{|y-x|^{-d}}{|r^2-|y|^2|^{\alpha/2}} f(\arg(y)) \int_{r-\delta\le|z|\le|y|} U_y^-(\mathrm{d}z)$$

and for $|y| \in (r-\delta, r]$,

$$\int_{r-\delta\le|z|\le|y|} U_y^-(\mathrm{d}z) = \mathbb{P}_y(\tau^\oplus_{r-\delta} = \infty) = \mathbf{P}(\underline{\xi}_\infty \ge \log((r-\delta)/y))$$

- The right hand side above can be determined explicitly thanks to the known Wiener–Hopf factorisation of ξ
- Note also

$$\Delta_r^\delta f(x) \overset{\delta\downarrow 0}{\sim} C_{\alpha,d}|r^2-|x|^2|^{\alpha/2}\frac{1}{\delta}\int_{r-\delta}^r \rho^{d-1}\mathrm{d}\rho \frac{\mathbf{P}(\underline{\xi}_\infty \ge \log((r-\delta)/y))}{|r^2-\rho^2|^{\alpha/2}} \int_{\rho\mathbb{S}_{d-1}} \sigma_\rho(\mathrm{d}\theta)|\rho\theta - x|^{-d} f(\theta$$

Lemma

Let $D_{\alpha,d} = \Gamma(d/2)/\Gamma((d-\alpha)/2)\Gamma(\alpha/2)$. Then

$$\lim_{\delta\to 0} \sup_{|y|\in[r-\delta,r]} \left| (\rho^2 - (r-\delta)^2)^{-\alpha/2} r^\alpha \mathbf{P}(\underline{\xi}_\infty \ge \log((r-\delta)/y)) - \frac{2D_{\alpha,d}}{\alpha} \right| = 0$$

Point of closest reach: sketch proof

- Hence

$$\Delta_r^\delta f(x) \overset{\delta\downarrow 0}{\sim} \frac{1}{\delta} C_{\alpha,d} |r^2 - |x|^2|^{\alpha/2} \int_{|y|\in(r-\delta,r]} \mathrm{d}y \frac{|y-x|^{-d}}{|r^2-|y|^2|^{\alpha/2}} f(\arg(y)) \int_{r-\delta\le|z|\le|y|} U_y^-(\mathrm{d}z)$$

and for $|y| \in (r-\delta, r]$,

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Point of closest reach: sketch proof

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Lemma

Let $D_{\alpha,d} = \Gamma(d/2)/\Gamma((d-\alpha)/2)\Gamma(\alpha/2)$. Then

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More excursion theory-based results

Theorem (Triple law at first entrance/exit of a ball)

Fix $r > 0$ and define, for $x, z, y, v \in \mathbb{R}^d \backslash \{0\}$,

$$\chi_x(z, y, v) := \pi^{-3d/2} \frac{\Gamma((d+\alpha)/2)}{|\Gamma(-\alpha/2)|} \frac{\Gamma(d/2)^2}{\Gamma(\alpha/2)^2} \frac{||z|^2 - |x|^2|^{\alpha/2} ||y|^2 - |z|^2|^{\alpha/2}}{|z|^\alpha |z-x|^d |z-y|^d |v-y|^{\alpha+d}}.$$

(i) *Write*

$$G(\tau_r^\oplus) = \sup\{s < \tau_r^\oplus : |X_s| = \inf_{u \leq s} |X_u|\}$$

for the instant of closest reach of the origin before first entry into $r\mathbb{S}_{d-1}$. For $|x| > |z| > r$, $|y| > |z|$ and $|v| < r$,

$$\mathbb{P}_x(X_{G(\tau_r^\oplus)} \in \mathrm{d}z,\ X_{\tau_r^\oplus -} \in \mathrm{d}y,\ X_{\tau_r^\oplus} \in \mathrm{d}v;\ \tau_r^\oplus < \infty) = \chi_x(z, y, v)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}v.$$

(ii) *Define $\mathcal{G}(t) = \sup\{s < t : |X_s| = \sup_{u \leq s} |X_u|\}$, $t \geq 0$, and write*

$$\mathcal{G}(\tau_r^\ominus) = \sup\{s < \tau_r^\ominus : |X_s| = \sup_{u \leq s} |X_u|\}.$$

for the instant of furtherest reach from the origin immediately before first exit from $r\mathbb{S}_{d-1}$. For $|x| < |z| < r$, $|y| < |z|$ and $|v| > r$,

$$\mathbb{P}_x(X_{\mathcal{G}(\tau_r^\ominus)} \in \mathrm{d}z,\ X_{\tau_r^\ominus -} \in \mathrm{d}y,\ X_{\tau_r^\ominus} \in \mathrm{d}v) = \chi_x(z, y, v)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}v.$$

More excursion theory-based results

Theorem

Write $M_t = \sup_{s\leq t}|X_t|$, $t\geq 0$. *For all bounded measurable* $f:\mathbb{B}_d\mapsto\mathbb{R}$ *and* $x\in\mathbb{R}\backslash\{0\}$

$$\lim_{t\to\infty}\mathbb{E}_x[f(X_t/M_t)] = \pi^{-d/2}\frac{\Gamma((d+\alpha)/2)}{\Gamma(\alpha/2)}\int_{\mathbb{S}_{d-1}}\sigma_1(\mathrm{d}\phi)\int_{|w|<1}f(w)\frac{|1-|w|^2|^{\alpha/2}}{|\phi-w|^d}\mathrm{d}w,$$

where $\sigma_1(\mathrm{d}y)$ *is the surface measure on* $\mathbb{S}_{d-1}$*, normalised to have unit mass.*

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