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Law of the iterated logarithm for oscillating random walks conditioned to stay non-negative

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Abstract

We show that under a $3+\delta$ moment condition (where $\delta > 0$) there exists a 'Hartman–Winter' Law of the iterated logarithm for random walks conditioned to stay non-negative. We also show that under a second moment assumption the conditioned random walk eventually grows faster than $n^{1/2}(\log n)^{-(1+\varepsilon)}$ for any $\varepsilon > 0$ and yet slower than $n^{1/2}(\log n)^{-1}$. The results are proved using three key facts about conditioned random walks. The first is the relation of its step distribution to that of the original random walk given by Bertoin and Doney (Ann. Probab. 22 (1994) 2152). The second is the pathwise construction in terms of excursions in Tanaka (Tokyo J. Math. 12 (1989) 159) and the third is a new Skorohod-type embedding of the conditioned process in a Bessel-3 process.

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1. Introduction

Suppose that under the law $\mathbb{P}$, the process $S = \{S_n : n \geqslant 0\}$ is a random walk in $\mathbb{R}$ such that $S_0 = 0$ and the step distribution S_1 satisfies $\mathbb{E}(S_1) = 0$ and $\mathbb{E}(S_1^2) = \sigma^2 < \infty$.

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Suppose that H_k^- is the kth strict ascending ladder height of the reflected random walk $-S$ (with $H_0^- = 0$) and define the renewal function

$$v(x) = \sum_{k \geqslant 0} \mathbb{P}(H_k^- \leqslant x)$$

with the understanding that $v(x) = 0$ for $x < 0$. It is a non-decreasing right-continuous function. Also define $\tau^- = \inf\{n \geqslant 1 : S_n \in (-\infty, 0)\}$ the first time that S enters the negative portion of the real line and let $\mathscr{F}_k$ be the natural σ-algebra generated by the first k steps of S. Bertoin and Doney (1994) show that for each $A \in \mathscr{F}_k$ the limiting probabilities

$$\lim_{n\uparrow\infty} \mathbb{P}(A \mid \tau^- > n) \tag{1}$$

are well defined and that they induce a new measure $\mathbb{P}^\uparrow$ on the paths of S, the law of the random walk conditioned to stay non-negative. Further, if $\mathbb{P}_x$ is the translation of the measure $\mathbb{P}$ for which $S_0 = x \geqslant 0$ and $\mathbb{E}_x$ is the associated expectation operator then

$$\mathbb{P}_x^\uparrow(A) = \frac{1}{v(x)} \mathbb{E}_x(\mathbf{1}_{A \cap \{\tau^- > k\}} v(S_k)) \tag{2}$$

is the law of the random walk conditioned to stay non-negative but with initial value $x \geqslant 0$. In the sequel, $\mathbb{P}_0^\uparrow$ will sometimes be used for $\mathbb{P}^\uparrow$.

The results of Bertoin and Doney provide an analogue for random walks of the relationship between standard Brownian motion and Bessel-3 processes. It was shown by McKean (1963) that, in a similar sense to (1) and (2), a standard Brownian motion conditioned to stay non-negative has the same law as a Bessel-3 process with state space $[0, \infty)$ started from the origin.

Our aim is to investigate the asymptotics of the random walk conditioned to stay non-negative. Analogously to results for Bessel-3 processes obtained by Motoo (1959) we shall prove a 'Hartman–Winter' law of the iterated logarithm (LIL) as well as formulate integral tests that determine lower space–time envelopes. We use the terminology 'space–time envelope' to mean a function of time which eventually, with probability one, captures the path of the conditioned random walk completely above or below it. A lower space–time envelope thus contains the process from below. Although it is possible to provide some results concerning upper envelopes, as we are not able to obtain a precise integral test, we will not discuss them here.

Previous work on the conditioned random walk showed in a weak sense that the random walk conditioned to stay positive prefers paths that follow space–time trajectories 'in the neighbourhood' of $n^{1/2}$. For example, Iglehart (1974) showed that, under a third moment condition, rescaling the random walk $S_{[nt]}$ by $\sigma n^{1/2}$ where $t \in [0, 1]$ and then considering the law of this process in t conditioned to stay positive as n tends to infinity, one recovers essentially a rescaled Brownian meander. In Ritter (1981) it was shown that, under a second moment condition, the process S conditioned to stay positive grows no slower than $n^{1/2-\varepsilon}$ for any $\varepsilon > 0$ in a weak sense. (Note in the last two references the conditioning was on positivity rather than non-negativity.) In parallel with the writing of this paper, Biggins (2003) has also established results concerning the occupation of conditioned random walks below specified levels under second moment conditions. We now state our main theorem.

Theorem 1. *Let* $x \geqslant 0$.

(LIL) *Suppose that for some* $\delta > 0$, $\mathbb{E}|S_1|^{3+\delta} < \infty$, *then*

$$\limsup_{n\uparrow\infty} \frac{S_n}{\sqrt{2n\sigma^2 \log\log n}} = 1, \qquad \mathbb{P}_x^{\uparrow}\text{-}a.s.$$

(Lower space–time envelope) *Suppose that* $\mathbb{E}(S_1^2) < \infty$, $\psi(t) \downarrow 0$ *and* $\sqrt{t}\psi(t)\uparrow\infty$, *then*

$$\liminf_{n\uparrow\infty} \frac{S_n}{\sqrt{2n\sigma^2}\psi(n)} = \infty \text{ or } 0$$

accordingly as

$$\int^{\infty} \frac{\psi(t)}{t}\,\mathrm{d}t < \infty \text{ or} = \infty.$$

From the second part of this theorem it is also possible to deduce fine statements about a lower space–time envelope though there is no corresponding LIL. Our theorem shows that, under only a second moment, the functions

$$\psi_{k+1}(t) = \left[\prod_{i=0}^{k+1} \log_{(i)} t\right]^{-1}, \qquad \psi_{k+1}^{\varepsilon}(t) = \left[\left(\prod_{i=0}^{k} \log_{(i)} t\right)(\log_{(k+1)} t)^{1+\varepsilon}\right]^{-1},$$

where $\log_{(i)} t$ is the ith iterate of $\log t$ ($\log_{(0)} t = 1$) and $n \geqslant 1$, serve to produce

$$\liminf_{n\uparrow\infty} \frac{S_n}{\sqrt{2n\sigma^2}\psi_{k+1}(n)} = 0 \text{ and } \liminf_{n\uparrow\infty} \frac{S_n}{\sqrt{2n\sigma^2}\,\psi_{k+1}^{\varepsilon}(n)} = \infty.$$

The proof of Theorem 1 is essentially a consequence of being able to reconstruct $(S, \mathbb{P}^{\uparrow})$ in two different ways. The first comes from Tanaka (1989) and gives a pathwise construction of $(S, \mathbb{P}^{\uparrow})$ by systematically extracting excursions from the supremum of $(S, \mathbb{P})$, time reversing them and replacing them between the same end points. From Tanaka's construction, the lower space–time boundary results can easily be read off from existing theory for the fluctuation of random walks. The second reconstruction of $(S, \mathbb{P}^{\uparrow})$ is new and relies on piecing together appropriately stopped passages of the Bessel-3 process in the spirit of the Skorohod embedding problem. For this latter construction, knowing the distribution of S_1 under $\mathbb{P}_x^{\uparrow}$, as given by (2), will prove to be important in the calculations.

The paper is structured as follows. We begin with some preliminary results drawn from the literature on Bessel processes and random walks. Section 3 is devoted to Tanaka's decomposition and the lower space–time envelope results that follow.

In Section 4, we discuss a new Skorohod-type embedding which is the key tool for the upper envelope. Finally Section 5 deals with the proof of Theorem 1 (LIL).

On a final note, as we shall remark later, the need for a $3+\delta$ moment condition in the first part of Theorem 1 seems rather unnatural given that the behaviour of the conditioned process at large distances from the origin should be quite similar to the random walk itself, which in turn only requires second moments to satisfy the usual LIL. The stronger moment condition appears as a consequence of controlling the moments of the stopping times involved in our version of the Skorohod embedding; second moments suffice for the proofs of all the other results.

2. Space–time envelopes for Bessel-3 processes and random walks

Parts of the two theorems given in this section will be used to prove our main result given in the introduction.

A description of upper and lower space–time envelopes for Bessel-3 processes as follows is originally due to Motoo (1959).

Theorem 2. *Let $(X, \mathscr{P}_x)$ be a Bessel-3 process on $[0,\infty)$ started from $x \geqslant 0$.*

(i) *Suppose that $\phi \uparrow \infty$. Then*

$$\mathscr{P}_x(X_t > \sqrt{t}\phi(t) \text{ i.o. as } t \uparrow \infty) = 0 \text{ or } 1$$

according to the integral test

$$\int^{\infty} \frac{\phi(t)^3}{t} \mathrm{e}^{-1/2\phi(t)^2}\,\mathrm{d}t < \infty \text{ or } = \infty. \tag{3}$$

(ii) *Further suppose that $\psi \downarrow 0$, then*

$$\mathscr{P}_x(X_t < \sqrt{t}\psi(t) \text{ i.o. as } t \uparrow \infty) = 0 \text{ or } 1$$

according to the integral test

$$\int^{\infty} \frac{\psi(t)}{t}\,\mathrm{d}t < \infty \text{ or } = \infty. \tag{4}$$

Note that the original version of this theorem was stated for $x=0$. Since a Bessel-3 process started from the origin will hit any $x>0$ with probability one it is easy to see how to recover the results above via a simple space–time translation.

Despite the apparent gap in the literature for conditioned random walks, there are several theorems for random walks which offer integral tests similar to those of Motoo. In particular, we refer to the collective results of Khintchine (1924), Kolmogorov

(1929), Feller (1946), Hirsch (1965) and Csáki (1978) which we have summarized in the theorem below.

Theorem 3. *Consider the random walk* $(S, \mathbb{P})$.

(i) *Suppose that the step distribution is such that*

$$\limsup_{x\uparrow\infty} \log\log x \int_{|y|>x} y^2 \mathbb{P}(S_1 \in \mathrm{d}y) < \infty. \tag{5}$$

(*It would suffice then for example that* $\mathbb{E}(S_1^2 |\log\log|S_1||) < \infty$.) *Choose* $x \in \mathbb{R}$. *Let* $\phi \uparrow \infty$, *then*

$$\mathbb{P}_x(S_n > \sqrt{n\sigma^2}\phi(n) \text{ i.o. as } n \uparrow \infty) = 0 \text{ or } 1$$

accordingly as

$$\int^{\infty} \frac{\phi(t)}{t} \mathrm{e}^{-\phi(t)^2/2} \,\mathrm{d}t < \infty \text{ or } = \infty.$$

(ii) *Now suppose that* $\psi \downarrow 0$ *and that* $\mathbb{E}(S_1^2) < \infty$. *Then*

$$\mathbb{P}_x\left(\max_{k\leqslant n} S_k < \sqrt{n\sigma^2}\psi(n) \text{ i.o. as } n \uparrow \infty\right) = 0 \text{ or } 1$$

accordingly as

$$\int^{\infty} \frac{\psi(t)}{t} \,\mathrm{d}t < \infty \text{ or } = \infty. \tag{6}$$

Note that the moment condition in part (i) of the above theorem is heavier than necessary for the classical LIL where we take $\phi(t) = \sqrt{(2 \pm \varepsilon)\log\log t}$ for small $\varepsilon > 0$. For such cases it is a well established fact that under only second moments

$$\mathbb{P}_x\left(S_n > \sqrt{(2+\varepsilon)n\sigma^2 \log\log n} \text{ i.o. as } n \uparrow \infty\right) = 0 \text{ and}$$

$$\mathbb{P}_x\left(S_n > \sqrt{(2-\varepsilon)n\sigma^2 \log\log n} \text{ i.o. as } n \uparrow \infty\right) = 1.$$

3. Tanaka's decomposition and lower space–time envelopes

In exactly the same manner as Tanaka (1989), Afanasyev et al. (2003) give the following pathwise construction of a Markov chain from excursions of $(S, \mathbb{P})$ whose law is equal to that of $(S, \mathbb{P})$ conditioned to stay non-negative. Let $\{(H_k^+, \sigma_k^+)\}_{k\geqslant 0}$ be the sequence of weakly increasing ladder heights and times, respectively, of $(S, \mathbb{P})$ with $H_0^+ = \sigma_0^+ = 0$. Define $e_1, e_2, \ldots$, the sequence of excursions of $(S, \mathbb{P})$ from its supremum that have been time reversed;

$$e_n = (0, S_{\sigma_n^+} - S_{\sigma_n^+ - 1}, S_{\sigma_n^+} - S_{\sigma_n^+ - 2}, \ldots, S_{\sigma_n^+} - S_{\sigma_{n-1}^+ + 1}, S_{\sigma_n^+} - S_{\sigma_{n-1}^+})$$

for $n \geqslant 1$. Write for convenience $e_n = (e_n(0), e_n(1), \ldots, e_n(\sigma_n^+ - \sigma_{n-1}^+))$ as an alternative for the steps of each e_n. The process $S^\uparrow = \{S_n^\uparrow : n \geqslant 0\}$, constructed by gluing these time-reversed excursions end to end in the following way:

$$S_n^\uparrow = \begin{cases} e_1(n) & \text{for } 0 \leqslant n \leqslant \sigma_1^+, \\ H_1^+ + e_2(n - \sigma_1^+) & \text{for } \sigma_1^+ < n \leqslant \sigma_2^+, \\ \vdots & \vdots \\ H_{k-1}^+ + e_k(n - \sigma_{k-1}^+) & \text{for } \sigma_{k-1}^+ < n \leqslant \sigma_k^+ \\ \vdots & \vdots \end{cases}$$

forms a Markov chain whose finite-dimensional distributions under $\mathbb{P}$ are the same as those of the process S under law $\mathbb{P}^\uparrow$. In this sense we say that $(S, \mathbb{P}^\uparrow)$ and $(S^\uparrow, \mathbb{P})$ are the same process in law.

Remark 4. The Bessel-3 process has the special property that it is equal in law to the absolute value of a standard Brownian motion plus its local time at zero. Informally speaking we can think of the paths of Bessel-3 processes as the result of gluing end to end Brownian excursions away from the origin with a small 'nudge' upwards given by the increment in local time at the end of each excursion. To some extent, this construction is analogous to Tanaka's construction of the conditioned random walk. It is also analogous to another decomposition of conditioned random walks found in Bertoin (1993). That is the juxtaposition of successive excursions of S from $(-\infty, 0]$; where an excursion is considered to include the step out of $(-\infty, 0]$ but not the returning step to $(-\infty, 0]$.

Let $\{(M_k^+, \upsilon_k^+)\}_{k \geqslant 0}$ be the space–time points of increase of the 'future minimum' of $(S^\uparrow, \mathbb{P})$. That is, $M_0^+ = \upsilon_0^+ = 0$,

$$\upsilon_k^+ = \inf\left\{n > \upsilon_{k-1}^+ : \min_{r \geqslant n} S_r^\uparrow = S_n^\uparrow\right\} \text{ and } M_k^+ = S_{\upsilon_k^+}^\uparrow$$

for $k \geqslant 1$. From this construction, we can deduce that path for path, the sequence $\{(M_k^+, \upsilon_k^+)\}_{k \geqslant 0}$ corresponds precisely to $\{(H_k^+, \sigma_k^+)\}_{k \geqslant 0}$. (This can be best seen in a simple sketch where one can go from $(S, \mathbb{P})$ to $(S^\uparrow, \mathbb{P})$ and vice versa by systematically extracting the excursions, rotating them by 180° and then replacing them between the same end points.)

Let $L = \{L_n\}_{n \geqslant 0}$ be the local time at the maximum in $(S, \mathbb{P})$, that is

$$L_n = |\{k \leqslant n : \max_{i \leqslant k} S_i = S_k\}|.$$

Equivalently, L is the local time at the future minimum of $(S^{\uparrow}, \mathbb{P})$. The hierarchy

$$S_n \leqslant H^+_{L_n} = M^+_{L_n} \leqslant S^{\uparrow}_n \tag{7}$$

is now easy to see from the pathwise construction of $(S^{\uparrow}, \mathbb{P})$.

Proof of Theorem 1 (Lower space time envelope). Suppose that $\psi \downarrow 0$ satisfies the divergent integral test in part (ii) of Theorem 3. Since $H^+_{L_n} = \max_{i \leqslant n} S_i$, it follows

$$\mathbb{P}\left(H^+_{L_n} < \sqrt{\sigma^2 n}\psi(n) \text{ i.o. as } n \uparrow \infty\right) = 1,$$

which shows there exists a sequence of (random) times such that the last future minimum of $(S^{\uparrow}, \mathbb{P})$ is below the space–time curve $\sqrt{n\sigma^2}\psi(n)$ infinitely often. Let the increasing sequence of random times at which this occurs be denoted by $\{n_i\}_{i \geqslant 1}$. That is, with probability one, $H^+_{L_{n_i}} < \sqrt{n_i \sigma^2}\psi(n_i)$ for each i. Define $\triangle H^+_k = H^+_k - H^+_{k-1}$ and n'_i the next time index after n_i at which the process $H^+_{L_n}$ increases (that is $n'_i = \sigma^+_{L_{n_i}+1}$, the inverse local time of $L_{n_i} + 1$). Note that

$$\begin{aligned} S^{\uparrow}_{n'_i} &= H^+_{L_{n_i}} + \triangle H^+_{L_{n_i}+1} \\ &< \sqrt{n'_i \sigma^2}\psi(n'_i) + \triangle H^+_{L_{n_i}+1}, \end{aligned} \tag{8}$$

where we have used the assumption that $\sqrt{n}\psi(n)$ is an increasing function. Consider the sequence of positive iid random variables $\{\triangle H^+_k\}_{k \geqslant 1}$. Since $\triangle H^+_k$ has the same distribution as $S_{\sigma^+_1}$ under $\mathbb{P}$, it follows from Feller (1971) that $\mathbb{E}(\triangle H^+_k) = \sigma \mathrm{e}^{-\chi^+}/\sqrt{2}$, where

$$\chi^+ = \sum_{n \geqslant 1} \frac{1}{n}\left[\mathbb{P}(S_n > 0) - \frac{1}{2}\right]$$

and is finite. We thus have that $\limsup_{k \uparrow \infty} \triangle H^+_k / H^+_k = 0$. We now have for sufficiently large i and, say, any $\varepsilon \in (0, 1)$ that

$$\triangle H^+_{L_{n_i}+1} < \varepsilon(H^+_{L_{n_i}} + \triangle H^+_{L_{n_i}+1}) \text{ a.s.}$$

leading to

$$\triangle H^+_{L_{n_i}+1} < \frac{\varepsilon}{1-\varepsilon}\sqrt{n'_i \sigma^2}\psi(n'_i) \text{ a.s.}$$

for $\varepsilon \in (0, 1)$. Continuing from (8) we have in conclusion that there exist an increasing sequence of times such that with probability one,

$$S^{\uparrow}_{n'_i} < \frac{1}{1-\varepsilon}\sqrt{n'_i \sigma^2}\psi(n'_i)$$

for all sufficiently large i. Since the integral test which ψ satisfies is unaffected by multiplicative constants in front of ψ, we have shown that

$$\mathbb{P}^{\uparrow}(S_n < c\sqrt{\sigma^2 n}\psi(n) \text{ i.o. as } n \uparrow \infty) = 1$$

for any $c > 0$, and hence

$$\liminf_{n\uparrow\infty} \frac{S_n}{\sqrt{\sigma^2 n}\psi(n)} = 0,$$

$\mathbb{P}^{\uparrow}$-almost surely.

Consider now, under the same moment condition, the case that $\psi \downarrow 0$ satisfies the convergent integral test in part (ii) of Theorem 3. Using (7) we have

$$\begin{aligned} 1 &= \mathbb{P}\left(\max_{i\leqslant n} S_i \geqslant \sqrt{\sigma^2 n}\psi(n) \text{ ev. as } n\uparrow\infty\right) \\ &= \mathbb{P}\left(S_n^{\uparrow} \geqslant \sqrt{\sigma^2 n}\psi(n) \text{ ev. as } n\uparrow\infty\right) \\ &= \mathbb{P}^{\uparrow}\left(S_n \geqslant \sqrt{\sigma^2 n}\psi(n) \text{ ev. as } n\uparrow\infty\right), \end{aligned}$$

where 'ev.' means eventually. Again using the fact that we may replace ψ by $c\psi$ for any $c > 0$, and that the integral in part (ii) of Theorem 3 still converges, it follows easily that

$$\liminf_{n\uparrow\infty} \frac{S_n}{\sqrt{\sigma^2 n}\psi(n)} = \infty,$$

$\mathbb{P}^{\uparrow}$-almost surely.

On a final note, we remark that we have reached our conclusions under the measure $\mathbb{P}^{\uparrow}$. However, Theorem 3 in Biggins (2003) states that, when the conditioned random walk is started from $x > 0$, Tanaka's description applies to the development of the process from its all time minimum. Since the all time minimum occurs after an almost surely finite number of steps the results as stated in the second part of Theorem 1 follow. □

4. Skorohod embedding

In this section, we construct a Skorohod-type embedding for the conditioned random walk. We allow S_0 to have an arbitrary distribution μ with support in $[0,\infty)$, the corresponding measure is denoted by $\mathbb{P}^{\uparrow}_{\mu}$. Similarly $\mathscr{P}_\nu$ denotes the law of the Bessel-3 process X on $(0,\infty)$ such that X_0 has distribution ν, and $\mathscr{E}_\nu$ denotes the associated expectation operator. We assume now that the filtration $\{\mathscr{G}_t\}_{t\geqslant 0}$ to which X is adapted, is rich enough to make the following construction possible.

Theorem 5. *Let $\mu(\mathrm{d}x) = \mathbb{P}^{\uparrow}(S_0 \in \mathrm{d}x)$ and $\nu(\mathrm{d}x) = \mathbb{P}^{\uparrow}_{\mu}(v(S_0) \in \mathrm{d}x)$. Then there exists a stopping time $T < \infty$ a.s., such that X_T under $\mathscr{P}_\nu$ has the same distribution as $v(S_1)$ under $\mathbb{P}^{\uparrow}_{\mu}$.*

Proof. First we embed the random variable $\mathbb{E}^{\uparrow}(1/v(S_1)|S_0)$ into the Bessel-3 process. Let

$$\theta(x) = \mathbb{E}_x^{\uparrow}\left(\frac{1}{v(S_1)}\right) = \frac{\mathbb{P}_x(S_1 \geqslant 0)}{v(x)}, \tag{9}$$

then

$$\mathbb{E}^{\uparrow}(1/v(S_1)|S_0) = \theta(S_0) \quad \text{a.s.}$$

The pair $(v(S_0), \theta(S_0))$ have a joint distribution determined by μ, with the first coordinate having the marginal distribution v. If $\mathscr{G}_0$ is sufficiently rich, we may assume that X_0 has distribution v and that there is a $\mathscr{G}_0$-measurable random variable Θ such that $(v(S_0), \theta(S_0))$ and (X_0, Θ) have the same joint distribution. From the inequality $\theta(x) \leqslant 1/v(x)$ it follows that $v(S_0) \leqslant 1/\theta(S_0)$ and thus $X_0 \leqslant 1/\Theta$ a.s. Therefore, since the Bessel-3 process drifts to ∞ with probability 1,

$$\eta := \inf\{t \geqslant 0 : X_t = 1/\Theta\}$$

is an a.s. finite stopping time and $X_\eta = 1/\Theta$. In conclusion, $\mathbb{E}^{\uparrow}(1/v(S_1)|S_0)$ and $1/X_\eta$ coincide in distribution.

Next we show how to embed $v(S_1)$ into the Bessel-3 process $X'_t := X_{t+\eta}$, $t \geqslant 0$. Note that $X'_0 = 1/\Theta$. Thus, assuming that $X'_0 = 1/\theta$ for some real number $\theta > 0$ is equivalent to considering the conditional random walk, given the event $\mathbb{E}^{\uparrow}(1/v(S_1)|S_0) = \theta$, which we do in the sequel. Let $\pi_\theta(\mathrm{d}x)$ denote the corresponding conditional distribution of $v(S_1)$. Then

$$\int_1^{1/\theta} (x^{-1} - \theta)\, \pi_\theta(\mathrm{d}x) = \int_{1/\theta}^{\infty} (\theta - x^{-1})\, \pi_\theta(\mathrm{d}x) = \gamma(\theta) \quad \text{(say)}.$$

Note that $v(S_1)$ takes values only in $[1, \infty)$ and $\gamma(\theta)$ is strictly positive.

Now we follow a standard proof of the Skorohod embedding theorem for Brownian motion (see Kallenberg, 1997) with some adaptations. We define $F_\theta^+(\mathrm{d}b) = \pi_\theta(\mathrm{d}b)$ for $1/\theta \leqslant b < \infty$ and $F_\theta^-(\mathrm{d}a) = \pi_\theta(\mathrm{d}a)$ for $1 \leqslant a < 1/\theta$. If $\mathscr{G}_0$ is sufficiently rich, there exists a bivariate $\mathscr{G}_0$-measurable random variable (α, β) with values in $[1, 1/\theta) \times [1/\theta, \infty)$, independent of the Bessel-3 process and with distribution

$$\Pr((\alpha, \beta) \in \mathrm{d}a \times \mathrm{d}b) = \frac{1}{\gamma(\theta)}(a^{-1} - b^{-1}) F_\theta^-(\mathrm{d}a) F_\theta^+(\mathrm{d}b).$$

Then, setting

$$\eta' := \inf\{t \geqslant 0 : X'_t \notin (\alpha, \beta)\},$$

$X'_{\eta'}$ has distribution π_θ. To show this, we use the fact (Revuz and Yor, 1994) that the Bessel-3 process has scale function $s(x) = -1/x$, so that $\rho_t = 1/X'_t$ is in natural scale. Furthermore $\rho_0 = \theta$, thus

$$\mathscr{P}_{1/\theta}(X'_{\eta'} = \beta \mid \alpha, \beta) = \mathscr{P}_{1/\theta}(\rho_t \text{ hits } \beta^{-1} \text{ before } \alpha^{-1} \mid \alpha, \beta) = \frac{\alpha^{-1} - \theta}{\alpha^{-1} - \beta^{-1}}$$

and consequently for $1/\theta \leqslant b < \infty$,

$$\mathscr{P}_{1/\theta}(X'_{\eta'} \in \mathrm{d}b) = \int_1^{1/\theta} \frac{a^{-1}-\theta}{a^{-1}-b^{-1}} \frac{1}{\gamma(\theta)} (a^{-1}-b^{-1}) F_\theta^-(\mathrm{d}a) F_\theta^+(\mathrm{d}b) = \pi_\theta(\mathrm{d}b).$$

An identical calculation holds for $\mathscr{P}_{1/\theta}(X'_{\eta'} \in \mathrm{d}a)$ with $1 \leqslant a < 1/\theta$. In conclusion $X'_{\eta'}$ has distribution π_θ.

Now by defining the stopping time

$$T := \eta + \eta',$$

$X_T = X'_{\eta'}$ has the same distribution as $v(S_1)$. □

Iterating this construction the whole process $v(S_0), v(S_1), \ldots$, can be embedded into the Bessel-3 process X in much the same way provided some obvious changes are taken care of ($\theta(S_n)$ should be embedded, given the values of $v(S_0), \ldots, v(S_n)$).

Corollary 6 (Skorohod embedding). *There is a sequence of stopping times* $0 = T_0 \leqslant T_1 \leqslant T_2 \leqslant \cdots$ *such that the distribution of the process* $X_{T_0}, X_{T_1}, \ldots$, *under* $\mathscr{P}_v$ *is equal to the distribution of the process* $v(S_0), v(S_1), \ldots$, *under* $\mathbb{P}_\mu^\uparrow$.

In order to use the Skorohod embedding, we shall need to control the moments of the stopping times T_n. In particular we will need the following strong law of large numbers.

Let

$$\chi^- = \sum_{n \geqslant 1} \frac{1}{n} \left[\mathbb{P}(S_n < 0) - \frac{1}{2} \right]$$

and note χ^- is finite by Feller (1971). Also recall that $\mu(\mathrm{d}x) = \mathbb{P}^\uparrow(S_0 \in \mathrm{d}x)$ and $v(\mathrm{d}x) = \mathbb{P}_\mu^\uparrow(v(S_0) \in \mathrm{d}x)$.

Theorem 7 (Strong Law of Large Numbers). *Let* $\{T_n\}_{n \geqslant 0}$ *be as in Corollary* 6 *but with* $\mu = \delta_x$ *for* $x \geqslant 0$. *When* $\mathbb{E}|S_1|^{3+\delta}$ *for some* $\delta > 0$, *the sequence* $\{T_n\}_{n \geqslant 0}$ *obeys a Strong Law of Large Numbers,*

$$\lim_{n \uparrow \infty} \frac{T_n}{n} = 2\mathrm{e}^{2\chi^-}, \quad \mathscr{P}_{v(x)}\text{-almost surely.}$$

The intuition behind this is that both the conditioned random walk and the Bessel-3 process behave more and more like the random walk and Brownian motion, respectively, as they drift out to infinity. The Skorohod embedding described above should therefore resemble more and more the classical Skorohod embedding problem; that is that the increments $\triangle T_n$ should show similar behaviour to iid random variables in that they should obey a strong law of large numbers. Note that the Skorohod embedding above concerns the sequence $\{v(S_n)\}_{n \geqslant 0}$ rather than $\{S_n\}_{n \geqslant 0}$ and hence the limiting constant in our Strong Law of Large Numbers is not proportional to σ^2 as one might expect at first glance.

In order to prove Theorem 7 we require two propositions giving estimates for the expectation and higher moments of T.

Proposition 8. *Let T be as in Theorem 5 but with $\mu = \delta_x$ for $x \geqslant 0$. If $\mathbb{E}|S_1|^3 < \infty$, then T has finite expectation, and*

$$\mathscr{E}_{v(x)}(T) \to 2\mathrm{e}^{2\chi^-}$$

as $x \to \infty$.

Proof. Straightforward Itô calculus confirms that $U_t = X_t^2 - 3t$ is a local martingale. As T is the sum of hitting times of (albeit randomized) barriers, it can be shown in the usual way that $\mathscr{E}_{v(x)}(U_T) = \mathscr{E}_{v(x)}(U_0)$, thus taking advantage of Theorem 5 and the fact that $v(x) = 0$ for $x < 0$, we have

$$\begin{aligned} 3\mathscr{E}_{v(x)}(T) &= \mathscr{E}_{v(x)}(X_T^2) - v(x)^2 \\ &= \mathbb{E}_x^{\uparrow}(v(S_1)^2) - v(x)^2 \\ &= v(x)^{-1}(\mathbb{E}_x(v(S_1)^3) - v(x)^3) \\ &= v(x)^{-1}(\mathbb{E}_0((v(S_1+x) - v(x))^3) + 3v(x)\mathbb{E}_0((v(S_1+x) - v(x))^2) \\ &\quad + 3v(x)^2\mathbb{E}_x(v(S_1) - v(x))). \end{aligned}$$

As $\mathbb{E}_x(v(S_1)) = v(x)$, the last term vanishes.

Recall from Feller (1971) and Spitzer (1964) that the expectation $\mathbb{E}(H_1^-) = \sigma \mathrm{e}^{-\chi^-}/\sqrt{2}$, where χ^- was defined earlier in Theorem 7. The renewal theorem tells us that $v(x) \to \infty$ and

$$v(S_1 + x) - v(x) \to S_1 \frac{\sqrt{2}}{\sigma} \mathrm{e}^{\chi^-}$$

as $x \to \infty$. Also $v(s+x) - v(x) \leqslant v(s)$, since $\tilde{v}(s) = v(s+x) - v(x)$, $s \geqslant 0$, can be viewed as the renewal function of a delayed renewal process. It follows that $|v(S_1 + x) - v(x)| \leqslant v(|S_1|) \leqslant c(1 + |S_1|)$ for some $c > 0$. Therefore, by dominated convergence, it follows that the first term goes to 0 and the second term converges, hence

$$\mathscr{E}_{v(x)}(T) \to \mathbb{E}(S_1^2)\frac{2}{\sigma^2}\mathrm{e}^{2\chi^-},$$

which is the assertion. □

Proposition 9. *Let T be as in the proof of Theorem 5 but with $\mu = \delta_x$ for $x \geqslant 0$. If $\mathbb{E}|S_1|^{3+\delta} < \infty$ for some $\delta > 0$, then $\mathscr{E}_{v(x)}(T^p) = \mathrm{O}(1)$ for some $p > 1$, as $x \to \infty$.*

Proof. As $\mathscr{E}_{v(x)}(T^p) \leqslant 2^{p-1}(\mathscr{E}_{v(x)}(\eta^p) + \mathscr{E}_{v(x)}((\eta')^p))$, for $p > 1$, we deal with the stopping times η and η' separately. For η we use the fact that $X_t^2 - 3t$ and $X_t^4 - 10tX_t^2 + 15t^2$ are local martingales. (The second case is again easily deduced using

Itô’s formula.) By standard arguments we get

$$\mathscr{E}_{v(x)}X_\eta^2 - 3\mathscr{E}_{v(x)}\eta = v(x)^2, \quad \mathscr{E}_{v(x)}X_\eta^4 - 10\mathscr{E}_{v(x)}(\eta X_\eta^2) + 15\mathscr{E}_{v(x)}\eta^2 = v(x)^4.$$

Using $X_\eta = 1/\theta(x)$ and $v(x) \leqslant 1/\theta(x)$ it follows that

$$3\mathscr{E}_{v(x)}\eta = \theta(x)^{-2} - v(x)^2 \leqslant 2(1 - v(x)\theta(x))/\theta(x)^2,$$

$$15\mathscr{E}_{v(x)}\eta^2 = v(x)^4 - \theta(x)^{-4} + \frac{10(\theta(x)^{-2} - v(x)^2)}{3\theta(x)^2} \leqslant \frac{20(1 - v(x)\theta(x))}{3\theta(x)^4}$$

and by means of Cauchy–Schwartz

$$\mathscr{E}_{v(x)}\eta^{3/2} \leqslant (\mathscr{E}_{v(x)}\eta\mathscr{E}_{v(x)}\eta^2)^{1/2} \leqslant (1 - v(x)\theta(x))/\theta(x)^3.$$

Now $1 - v(x)\theta(x) = \mathbb{P}_x(S_1 < 0) \leqslant x^{-3}\mathbb{E}_x|S_1 - x|^3 = \mathrm{O}(x^{-3})$ by assumption, and $\theta(x)^{-1} \sim v(x) \sim x$ in view of the renewal theorem. Therefore,

$$\mathscr{E}_{v(x)}\eta^{3/2} = \mathrm{O}(1)$$

as $x \to \infty$.

In order to study η' we write θ for $\theta(x)$ and use the following formula for Bessel-3 processes (see Borodin and Salminen (1996), Eqs. 1.3.0.6a,b and 5.3.0.6a,b, p. 172 and 348)

$$\begin{aligned}&\mathscr{P}_{1/\theta}(\eta' \in \mathrm{d}t | \alpha = a, \beta = b)\\ &\quad = a\theta \Pr(\tau \in \mathrm{d}t, B_\tau = a) + b\theta \Pr(\tau \in \mathrm{d}t, B_\tau = b),\end{aligned}$$

where $\{B_t\}_{t \geqslant 0}$ is a standard Brownian motion starting at $B_0 = 1/\theta$, and the exit time $\tau = \tau_{a,b} := \inf\{t \geqslant 0 : B_t \notin (a,b)\}$. [Note that this equality follows simply from the fact that the law of a Bessel-3 process can be recovered from that of a Brownian motion killed on entering $(-\infty, 0]$ by applying a Doob h-transform with the function $h(x) = x$.] Thus,

$$\mathscr{E}_{1/\theta}((\eta')^p | \alpha = a, \beta = b) \leqslant (a+b)\theta E\tau^p.$$

Moreover, for $p > 1$ (cf. Hall and Heyde, 1980, p. 271)

$$E\tau^p \leqslant c(b-a)^{2p-2}(b - \theta^{-1})(\theta^{-1} - a)$$

for a suitable $c > 0$ (depending on p). Therefore, we arrive at the estimate

$$\begin{aligned}&\mathscr{E}_{1/\theta}(\eta')^p\\ &\quad \leqslant \frac{c\theta}{\gamma(\theta)} \int\int (a+b)(a^{-1} - b^{-1})(b-a)^{2p-2}(b - \theta^{-1})(\theta^{-1} - a) F_\theta^-(\mathrm{d}a) F_\theta^+(\mathrm{d}b)\\ &\quad = \frac{c\theta}{\gamma(\theta)} \int\int (a^{-1} + b^{-1})(b-a)^{2p-1}(b - \theta^{-1})(\theta^{-1} - a) F_\theta^-(\mathrm{d}a) F_\theta^+(\mathrm{d}b).\end{aligned}$$

Using the inequality

$$(b-a)^{2p-1} \leqslant 2^{2p-1}((b-\theta^{-1})^{2p-1} + (\theta^{-1}-a)^{2p-1})$$

and the definition of the measures F_θ^-, F_θ^+, we obtain by means of Fubini's theorem

$$\begin{aligned}\mathscr{E}_{1/\theta}(\eta')^p \leqslant \frac{2^{2p}c\theta}{\gamma(\theta)} \Bigg(& \mathbb{E}_x^\uparrow \left(\frac{|\theta^{-1}-v(S_1)|}{v(S_1)}\right) \mathbb{E}_x^\uparrow |v(S_1)-\theta^{-1}|^{2p} \\ & + \mathbb{E}_x^\uparrow|\theta^{-1}-v(S_1)| \mathbb{E}_x^\uparrow \left(\frac{|v(S_1)-\theta^{-1}|^{2p}}{v(S_1)}\right)\Bigg). \end{aligned} \tag{10}$$

Now $\theta(x)^{-1} - v(x) = \mathrm{O}(1)$, thus if $r \leqslant 3+\delta$

$$\begin{aligned}\mathbb{E}_x|v(S_1)-\theta^{-1}|^r &\leqslant \mathbb{E}_x(v(|S_1-x|) + |v(x)-\theta^{-1}|)^r \\ &\leqslant c\mathbb{E}_x(|S_1-x| + \mathrm{O}(1))^r \\ &= \mathrm{O}(1)\end{aligned}$$

by assumption. Therefore,

$$\mathbb{E}_x^\uparrow \left(\frac{|v(S_1)-\theta^{-1}|^r}{v(S_1)}\right) = \frac{1}{v(x)} \mathbb{E}_x|v(S_1)-\theta^{-1}|^r = \mathrm{O}(v(x)^{-1})$$

and if $r \leqslant 2+\delta$

$$\begin{aligned}\mathbb{E}_x^\uparrow|v(S_1)-\theta^{-1}|^r &= \frac{1}{v(x)} \mathbb{E}_x(v(S_1)|v(S_1)-\theta^{-1}|^r) \\ &\leqslant \frac{1}{v(x)} \mathbb{E}_x|v(S_1)-\theta^{-1}|^{r+1} + \frac{1}{v(x)\theta} \mathbb{E}_x|v(S_1)-\theta^{-1}|^r \\ &= \mathrm{O}(1).\end{aligned}$$

Returning to (10) it thus follows that

$$\mathscr{E}_{1/\theta}(\eta')^p = \mathrm{O}(\theta v(x)^{-1}\gamma(\theta)^{-1})$$

as $x \to \infty$, for $p > 1$ sufficiently close to 1. It remains to estimate $\gamma(\theta)$:

$$\begin{aligned}\gamma(\theta) &= \frac{1}{2}\int_1^\infty |\theta - y^{-1}|\pi_\theta(\mathrm{d}y) = \frac{1}{2}\mathbb{E}_x^\uparrow|\theta - v(S_1)^{-1}| \\ &= \frac{\theta}{2v(x)} \mathbb{E}_x|v(S_1)-\theta^{-1}| \sim c\theta v(x)^{-1}\end{aligned}$$

for some $c > 0$. Therefore $\mathscr{E}_{1/\theta}(\eta')^p = \mathrm{O}(1)$, as claimed. □

We are now ready to prove our Strong Law of Large Numbers.

Proof of Theorem 7. Let $\{T_n\}_{n\geqslant 0}$ be the sequence of stopping times given in Corollary 6 but with $\mu = \delta_x$ for some $x \geqslant 0$. For $n \geqslant 1$ let $\tau_n = T_n - T_{n-1}$ and $\mathscr{H}_{n-1} = \sigma(T_k, X_{T_k}; k \leqslant n-1)$. Now note that

$$\sum_{k=1}^{n} \frac{\tau_k - \mathscr{E}_{v(x)}(\tau_k|\mathscr{H}_{k-1})}{k} = \sum_{k=1}^{n} \frac{\tau_k - \mathscr{E}_{X_{T_{k-1}}}(\tau_1)}{k}$$

is a zero mean martingale. Theorem 2.18 of Hall and Heyde (1980) tells us that this martingale converges on the set

$$\left\{ \sum_{k=1}^{\infty} \frac{\mathscr{E}_{v(x)}([\tau_k - \mathscr{E}_{v(x)}(\tau_k|\mathscr{H}_{k-1})]^p|\mathscr{H}_{k-1})}{k^p} < \infty \right\}.$$

We would like to show that this event occurs with probability 1. From the Skorohod embedding together with the lower space–time envelopes proved in the previous section, we know that X_{T_k} tends to infinity almost surely. Using this together with Proposition 8 we note that to achieve convergence of the above sum almost surely, it suffices to establish the almost sure convergence of

$$\sum_{k=1}^{\infty} \frac{\mathscr{E}_{v(x)}(\tau_k^p|\mathscr{H}_{k-1})}{k^p} \text{ or equivalently } \sum_{k=1}^{\infty} \frac{\mathscr{E}_{v(S_{k-1})}(\tau_k^p)}{k^p};$$

but this follows automatically from Proposition 9 and the fact that S tends to infinity $\mathbb{P}_x^{\uparrow}$-almost surely (which itself follows from the lower space–time envelope).

Martingale convergence together with Kronecker's Lemma now implies that

$$\lim_{n\uparrow\infty} \frac{\sum_{k=1}^n \tau_k - \mathscr{E}_{X_{T_{k-1}}}(\tau_1)}{n} = 0$$

$\mathscr{P}_{v(x)}$-almost surely. From this we have

$$\lim_{n\uparrow\infty} \frac{T_n}{n} = \lim_{n\uparrow\infty} \frac{\sum_{k=1}^n \mathscr{E}_{X_{T_{k-1}}}(\tau_1)}{n} = 2\mathrm{e}^{2\chi^-}, \mathscr{P}_{v(x)}\text{-a.s.},$$

where the last equality is again a result of Proposition 8 and the fact that $\{X_{T_k}\}_{k\geqslant 0}$ drifts to infinity. The law of large numbers is thus established. □

Remark 10. The moment conditions that we have assumed in order to produce the asymptotics are essentially responsible for the moment condition in the first part of Theorem 1. It is difficult to see how this can be avoided using the current method; however, one might conjecture that an improvement is possible in view of the discussion following Theorem 7.

5. Proof of Theorem 1 (LIL)

We shall divide the proof into two parts by establishing first that

$$\limsup_{n\uparrow\infty} \frac{S_n}{\sqrt{\sigma^2 n \log\log n}} \leqslant 1 \text{ and then } \limsup_{n\uparrow\infty} \frac{S_n}{\sqrt{\sigma^2 n \log\log n}} \geqslant 1,$$

$\mathbb{P}_x^{\uparrow}$-almost surely for each $x \geqslant 0$.

For the upper bound, choose any $\varepsilon > 0$ and fix $\phi(t) = \sqrt{(2+\varepsilon)\log\log t}$. It is easy to check that this choice of ϕ produces a finite integral in (3) of Motoo's Theorem 2 and hence

$$\mathscr{P}_{v(x)}\left(X_t > \sqrt{t}\phi(t) \text{ i.o. as } t\uparrow\infty\right) = 0.$$

An immediate consequence of this is that

$$\mathscr{P}_{v(x)}\left(X_{T_n} \geqslant \sqrt{T_n}\phi(T_n) \text{ i.o. as } n\uparrow\infty\right) = 0$$

and hence

$$\mathscr{P}_{v(x)}\left(X_{T_n} < \sqrt{n}\phi(n)\sqrt{\frac{T_n}{n}}\,\frac{\phi(nT_n/n)}{\phi(n)} \text{ ev. as } n\uparrow\infty\right) = 1.$$

From the Strong Law of Large Numbers given in Theorem 7 and the uniform slow variation of ϕ on compacts it follows that

$$\mathscr{P}_{v(x)}\left(X_{T_n} < \sqrt{n}\phi(n)\sqrt{2}\mathrm{e}^{\chi^-} \text{ ev. as } n\uparrow\infty\right) = 1.$$

The Skorohod embedding result in Corollary 6 now tells us that

$$\mathbb{P}_x^{\uparrow}\left(v(S_n) < \sqrt{n}\phi(n)\sqrt{2}\mathrm{e}^{\chi^-} \text{ ev. as } n\uparrow\infty\right) = 1.$$

Note that when $\mu = \delta_x$, $v = \delta_{v(x)}$. Finally recalling that $\lim_{x\uparrow\infty} v(x)/x = \sigma^{-1}\sqrt{2}\mathrm{e}^{\chi^-}$ we have for any $\varepsilon > 0$ that $v(S_n) \geqslant (1-\varepsilon)\sigma^{-1}\sqrt{2}\mathrm{e}^{\chi^-} S_n$ eventually with $\mathbb{P}_x^{\uparrow}$-probability one. Since ε can be made arbitrarily small, it follows that

$$\limsup_{n\uparrow\infty} \frac{S_n}{\sqrt{\sigma^2 n \log\log n}} \leqslant 1,$$

$\mathbb{P}_x^{\uparrow}$-almost surely for any $x \geqslant 0$.

Now for the lower bound. We have from the classical LIL for zero mean random walks with finite variance that

$$\mathbb{P}\left(S_n > \sqrt{(2-\varepsilon)\sigma^2 n \log\log n} \text{ i.o. as } n\uparrow\infty\right) = 1.$$

Now recall from the pathwise construction of the random walk conditioned to stay non-negative that $S_n \leqslant S_n^{\uparrow}$. From this it follows immediately that

$$\mathbb{P}\left(S_n^{\uparrow} > \sqrt{(2-\varepsilon)\sigma^2 n \log\log n} \text{ i.o. as } n \uparrow \infty\right) = 1$$

and hence

$$\mathbb{P}^{\uparrow}\left(S_n > \sqrt{(2-\varepsilon)\sigma^2 n \log\log n} \text{ i.o. as } n \uparrow \infty\right) = 1.$$

Recalling Theorem 3 in Biggins (2002) we can again strengthen this statement to

$$\mathbb{P}_x^{\uparrow}\left(S_n > \sqrt{\sigma^2 n}\phi(n) \text{ i.o. as } n \uparrow \infty\right) = 1$$

for all $x \geqslant 0$. Since $\varepsilon > 0$ is arbitrarily small we have that

$$\limsup_{n\uparrow\infty} \frac{S_n}{\sqrt{\sigma^2\, n \log\log n}} \geqslant 1,$$

$\mathbb{P}_x^{\uparrow}$-almost surely. This concludes the proof of Theorem 1 (LIL) and the paper.

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