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Part 8. Markov processes and renewal theory

THE EXTENDED HYPERGEOMETRIC CLASS OF LÉVY PROCESSES

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By A. E. KYPRIANOU, J. C. PARDO and A. R. WATSON

Abstract

We review and extend the class of hypergeometric Lévy processes explored in Kuznetsov and Pardo (2013) with a view to computing fluctuation identities related to stable processes. We give the Wiener–Hopf factorisation of a process in the extended class, characterise its exponential functional, and give three concrete examples arising from transformations of stable processes.

Keywords: Lévy process; hypergeometric Lévy process; extended hypergeometric Lévy process; Wiener–Hopf factorisation; exponential functional; stable process; path-censored stable process; conditioned stable process; hitting distribution; hitting probability

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1. Introduction

The simple definition of a Lévy process—a stochastic process with stationary independent increments—has been sufficient to fuel a vast field of study for many decades, and Lévy processes have been employed in many successful applied models. However, historically, there have been few classes of processes for which many functionals could be computed explicitly. In recent years, the field has seen a proliferation of examples which have proved to be more analytically tractable; in particular, we single out spectrally negative Lévy processes [19], Lamperti-stable processes [5, 6, 9], β- and θ-processes [15, 16], and finally the inspiration for this work, hypergeometric Lévy processes [6, 17, 20, 25]. It is also worth mentioning that the close relationship which appears to hold between hypergeometric Lévy processes and stable processes has also allowed the computation of several identities for the latter; see [17, 25, 26].

In this work, we review the hypergeometric class of Lévy processes introduced by Kuznetsov and Pardo [17], and introduce a new class of extended hypergeometric processes which have many similar properties. In particular, for an extended hypergeometric process ξ, we compute the Wiener–Hopf factors and find that its ladder height processes are related to Lamperti-stable subordinators; we characterise explicitly the distribution of the exponential functional of ξ/δ for any $\delta > 0$. We also give three examples of processes connected via the Lamperti representation to α-stable processes; these fall into the hypergeometric class when $\alpha \leq 1$, and into the extended hypergeometric class when $\alpha > 1$. Finally, we give some new identities for the stable process when $\alpha > 1$.

First we discuss the results of Kuznetsov and Pardo [17]. For a choice of parameters $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$ from the set $\mathcal{A}_{\mathrm{HG}} = \{\beta \leq 1, \ \gamma \in (0, 1), \ \hat{\beta} \geq 0, \ \hat{\gamma} \in (0, 1)\}$, we define

$$\psi(z) = -\frac{\Gamma(1 - \beta + \gamma - z)}{\Gamma(1 - \beta - z)} \frac{\Gamma(\hat{\beta} + \hat{\gamma} + z)}{\Gamma(\hat{\beta} + z)},$$

which we view as a meromorphic function on $\mathbb{C}$. Let ξ be a Lévy process started from 0, with law $\mathbb{P}$ and expectation operator $\mathbb{E}$. We say that ξ is a member of the *hypergeometric class* of Lévy processes if it has Laplace exponent ψ, in the sense that

$$\mathbb{E}[e^{z\xi_1}] = e^{\psi(z)}, \qquad z \in i\mathbb{R}. \tag{1}$$

Note that, in general, when the Laplace exponent ψ of a Lévy process ξ is a meromorphic function, relation (1) actually holds on any neighbourhood of $0 \in \mathbb{C}$ which does not contain a pole of ψ; thus, in this article we generally do not specify the domain of Laplace exponents which may arise.

In [17], it was shown that, for any choice of parameters in $\mathcal{A}_{\mathrm{HG}}$, there is a Lévy process with Laplace exponent ψ, and its Wiener–Hopf factorisation, in the following sense, is found. The *(spatial) Wiener–Hopf factorisation* of a Lévy process ξ with Laplace exponent ψ consists of the equation

$$\psi(z) = -\kappa(-z)\hat{\kappa}(z), \qquad z \in i\mathbb{R},$$

where κ and $\hat{\kappa}$ are the Laplace exponents of subordinators H and $\widehat{H}$, respectively, this time in the sense that $\mathbb{E}[e^{-\lambda H_1}] = e^{-\kappa(\lambda)}$ for $\mathrm{Re}\,\lambda \geq 0$. The subordinators H and $\widehat{H}$ are known as the ascending and descending ladder heights, and are related via a time change to the running maximum and running minimum of the process ξ. For more details, see [22, Chapter 6]. The insight into the structure of ξ given by the Wiener–Hopf factorisation allows one to simplify first passage problems for ξ; see [22, Chapter 7] for a collection of results.

Kuznetsov and Pardo [17] computed

$$\kappa(z) = \frac{\Gamma(1 - \beta + \gamma + z)}{\Gamma(1 - \beta + z)}, \qquad \hat{\kappa}(z) = \frac{\Gamma(\hat{\beta} + \hat{\gamma} + z)}{\Gamma(\hat{\beta} + z)},$$

thus demonstrating that the ascending and descending ladder height processes are Lamperti-stable subordinators (see [6]). They also considered the exponential functional of a hypergeometric Lévy process ξ. For each $\delta > 0$, the random variable

$$I(\xi/\delta) = \int_0^\infty e^{-\xi_t/\delta}\,\mathrm{d}t$$

is almost surely (a.s.) finite provided that ξ drifts to $+\infty$. This random variable is known as the *exponential functional* of the Lévy process ξ, and it has been studied extensively in general; the paper of Bertoin and Yor [3] gives a survey of the literature, and mentions, among other aspects, applications to diffusions in random environments, mathematical finance, and fragmentation theory. In the context of self-similar Markov processes (ssMps), the exponential functional appears in the entrance law of a positive ssMp (pssMp) started at 0 (see, e.g. [2]), and Pardo [31] related the exponential functional of a Lévy process to envelopes of its associated pssMp; furthermore, it is related to the hitting time of points for pssMps, and we shall make use of it in this capacity in our example of Subsection 4.2.

For the purpose of characterising the distribution of $I(\xi/\delta)$, its Mellin transform

$$\mathcal{M}(s) = \mathbb{E}[I(\xi/\delta)^{s-1}]$$

is useful. For ξ in the hypergeometric class, $\mathcal{M}$ was calculated by Kuznetsov and Pardo [17] in terms of gamma and double gamma functions; we recall and extend this in Section 3.

We now give a brief outline of the main body of the paper. In Section 2 we demonstrate that the parameter set $\mathcal{A}_{HG}$ may be extended by changing the domains of the two parameters β and $\hat{\beta}$, and find the Wiener–Hopf factorisation of a process ξ in this new class, identifying explicitly the ladder height processes. In Section 3 we find an expression for the Mellin transform $\mathcal{M}$ in this new case, making use of an auxiliary hypergeometric Lévy process. In Section 4 we give three examples where the extended hypergeometric class is of use, on the way extending the result in [7] on the Wiener–Hopf factorisation of the Lamperti representation associated with the radial part of a stable process.

2. The extended hypergeometric class

We begin by defining the set of admissible parameters

$$\mathcal{A}_{EHG} = \{\beta \in [1, 2], \ \gamma, \hat{\gamma} \in (0, 1), \ \hat{\beta} \in [-1, 0]; \ 1 - \beta + \hat{\beta} + \gamma \geq 0, \ 1 - \beta + \hat{\beta} + \hat{\gamma} \geq 0\}.$$

We are interested in proving the existence and investigating the properties of a Lévy process ξ whose Laplace exponent is given by the meromorphic function

$$\psi(z) = -\frac{\Gamma(1 - \beta + \gamma - z)}{\Gamma(1 - \beta - z)} \frac{\Gamma(\hat{\beta} + \hat{\gamma} + z)}{\Gamma(\hat{\beta} + z)}, \qquad z \in \mathbb{C},$$

when $(\beta, \gamma, \hat{\beta}, \hat{\gamma}) \in \mathcal{A}_{EHG}$. To facilitate more concise expressions below, we also define

$$\eta = 1 - \beta + \gamma + \hat{\beta} + \hat{\gamma}.$$

We now present our main result on the existence and properties of ξ.

Proposition 1. *There exists a Lévy process ξ such that $\mathbb{E}[e^{z\xi_1}] = e^{\psi(z)}$. Its Wiener–Hopf factorisation is expressible as*

$$\psi(z) = -(-\hat{\beta} - z) \frac{\Gamma(1 - \beta + \gamma - z)}{\Gamma(2 - \beta - z)} (\beta - 1 + z) \frac{\Gamma(\hat{\beta} + \hat{\gamma} + z)}{\Gamma(1 + \hat{\beta} + z)}.$$

Its Lévy measure possesses the density

$$\pi(x) = \begin{cases} -\dfrac{\Gamma(\eta)}{\Gamma(\eta - \hat{\gamma})\,\Gamma(-\gamma)} e^{-(1 - \beta + \gamma)x} \, {}_2F_1(1 + \gamma, \eta; \eta - \hat{\gamma}; e^{-x}), & x > 0, \\[2ex] -\dfrac{\Gamma(\eta)}{\Gamma(\eta - \gamma)\,\Gamma(-\hat{\gamma})} e^{(\hat{\beta} + \hat{\gamma})x} \, {}_2F_1(1 + \hat{\gamma}, \eta; \eta - \gamma; e^{x}), & x < 0, \end{cases} \tag{2}$$

where ${}_2F_1$ is the Gauss hypergeometric function.
If $\beta \in (1, 2)$ and $\hat{\beta} \in (-1, 0)$, the process ξ is killed at rate

$$q = \frac{\Gamma(1 - \beta + \gamma)}{\Gamma(1 - \beta)} \frac{\Gamma(\hat{\beta} + \hat{\gamma})}{\Gamma(\hat{\beta})}.$$

Otherwise, the process has infinite lifetime and

(i) *ξ drifts to $+\infty$ if $\beta > 1$ and $\hat{\beta} = 0$,*

(ii) *ξ drifts to $-\infty$ if $\beta = 1$ and $\hat{\beta} < 0$,*

(iii) *ξ oscillates if $\beta = 1$ and $\hat{\beta} = 0$; in this case, ξ is a hypergeometric Lévy process.*

Furthermore, the process ξ has no Gaussian component, and is of bounded variation with zero drift when $\gamma + \hat{\gamma} < 1$ and of unbounded variation when $\gamma + \hat{\gamma} \geq 1$.

Proof. We remark that there is nothing to do in case (iii) since such processes are analysed in [17]; however, the proof we give below also carries through in this case.

First we identify the proposed ascending and descending ladder processes. Then, when we have shown that ψ really is the Laplace exponent of a Lévy process, this is proof of the Wiener–Hopf factorisation.

Before we begin, we must review the definitions of special subordinators and the $\mathcal{T}$-transformations of subordinators. Suppose that υ is the Laplace exponent of a subordinator H, in the sense that $\mathbb{E}[e^{-zH_1}] = e^{-\upsilon(z)}$. We say that H is a *special subordinator* and that υ is a *special Bernstein function* if the function

$$\upsilon^*(z) = \frac{z}{\upsilon(z)}, \qquad z \geq 0,$$

is also the Laplace exponent of a subordinator. The function υ^* is said to be *conjugate* to υ. Special Bernstein functions play an important role in potential theory; see, for example, [34] for more details.

Again taking υ to be the Laplace exponent of a subordinator, not necessarily special, we define, for $c \geq 0$, the transformation

$$\mathcal{T}_c \upsilon(z) = \frac{z}{z+c} \upsilon(z+c), \qquad z \geq 0.$$

It is then known (see [13, 23]) that $\mathcal{T}_c \upsilon$ is the Laplace exponent of a subordinator. Furthermore, if υ is in fact a special Bernstein function then $\mathcal{T}_c \upsilon$ is also a special Bernstein function.

We are now in a position to identify the ladder height processes in the Wiener–Hopf factorisation of ξ. Let the (proposed) ascending factor be given for $z \geq 0$ by

$$\kappa(z) = (-\hat{\beta} + z)\frac{\Gamma(1 - \beta + \gamma + z)}{\Gamma(2 - \beta + z)}.$$

Then some simple algebraic manipulation shows that $\kappa(z) = (\mathcal{T}_{-\hat{\beta}}\upsilon)^*(z)$, where

$$\upsilon(z) = \frac{\Gamma(2 - \beta + \hat{\beta} + z)}{\Gamma(1 - \beta + \hat{\beta} + \gamma + z)},$$

provided that υ is a special Bernstein function. This follows immediately from Example 2 of Kyprianou and Rivero [24], under the constraint $1 - \beta + \hat{\beta} + \gamma \geq 0$ which is included in the parameter set $\mathcal{A}_{\mathrm{EHG}}$. Note that υ is in fact the Laplace exponent of a Lamperti-stable subordinator (see [6]), although we do not use this fact.

Proceeding similarly for the descending factor, we obtain

$$\hat{\kappa}(z) = (\beta - 1 + z)\frac{\Gamma(\hat{\beta} + \hat{\gamma} + z)}{\Gamma(1 + \hat{\beta} + z)} = (\mathcal{T}_{\beta - 1}\hat{\upsilon})^*(z), \qquad z \geq 0,$$

where

$$\hat{\upsilon}(z) = \frac{\Gamma(2 - \beta + \hat{\beta} + z)}{\Gamma(1 - \beta + \hat{\beta} + \hat{\gamma} + z)}, \qquad z \geq 0,$$

and again the function $\hat{\upsilon}$ is a special Bernstein function provided that $1 - \beta + \hat{\beta} + \hat{\gamma} \geq 0$. As before, $\hat{\upsilon}$ is the Laplace exponent of a Lamperti-stable subordinator.

We have now shown that both κ and $\hat{\kappa}$ are Laplace exponents of subordinators; we wish to show that the function

$$\psi(z) = -\kappa(-z)\hat{\kappa}(z)$$

is the Laplace exponent of a Lévy process. For this purpose, we apply the theory of *philanthropy*

developed by Vigon [35, Chapter 7]. This states, in part, that it is sufficient for both of the subordinators corresponding to κ and $\hat{\kappa}$ to be 'philanthropists', which means that their Lévy measures possess decreasing densities.

We recall our discussion of $\mathcal{T}$-transforms and special Bernstein functions. We have already stated that when υ is a special Bernstein function, then so is $\mathcal{T}_c\upsilon$; furthermore, we may show that its conjugate satisfies

$$(\mathcal{T}_c\upsilon)^*(z) = \mathcal{E}_c\upsilon^*(z) + \upsilon^*(c), \qquad z \geq 0,$$

where $\mathcal{E}_c$ is the *Esscher transform*, given by

$$\mathcal{E}_c\upsilon^*(z) = \upsilon^*(z + c) - \upsilon^*(c), \qquad z \geq 0.$$

The Esscher transform of the Laplace exponent of any subordinator is again the Laplace exponent of a subordinator; and if the subordinator corresponding to υ^* possesses a Lévy density π_{υ^*} then the Lévy density of $\mathcal{E}_c\upsilon^*$ is given by $x \mapsto e^{-cx}\pi_{\upsilon^*}(x)$ for $x > 0$.

Returning to our Wiener–Hopf factors, we have

$$\kappa(z) = (\mathcal{T}_{-\hat{\beta}}\upsilon)^*(z) = \mathcal{E}_{-\hat{\beta}}\upsilon^*(z) + \upsilon^*(-\hat{\beta}), \qquad z \geq 0,$$

where υ^* is the Laplace exponent conjugate to υ. Now, υ is precisely the type of special Bernstein function considered in [24, Example 2]. In that work, the authors even established that the subordinator corresponding to υ^* has a decreasing Lévy density π_{υ^*}. Finally, the Lévy density of the subordinator corresponding to κ is $x \mapsto e^{\hat{\beta}x}\pi_{\upsilon^*}(x)$, and this is then clearly also decreasing.

We have thus shown that the subordinator whose Laplace exponent is κ is a philanthropist. By a very similar argument, the subordinator corresponding to $\hat{\kappa}$ is also a philanthropist. As we have stated, the theory developed by Vigon now shows that the function ψ really is the Laplace exponent of a Lévy process ξ, with the Wiener–Hopf factorisation claimed.

We now proceed to calculate the Lévy measure of ξ. A fairly simple way to do this is to make use of the theory of 'meromorphic Lévy processes', as developed in [18]. We first show that ξ is in the meromorphic class. Initially, suppose that

$$1 - \beta + \hat{\beta} + \gamma > 0, \qquad 1 - \beta + \hat{\beta} + \hat{\gamma} > 0; \tag{3}$$

we relax this assumption later. Looking at the expression for ψ, we see that it has zeros $(\zeta_n)_{n\geq 1}$ and $(-\hat{\zeta}_n)_{n\geq 1}$, and (simple) poles $(\rho_n)_{n\geq 1}$ and $(-\hat{\rho}_n)_{n\geq 1}$ given by

$$\begin{aligned} \zeta_1 &= -\hat{\beta}, & \zeta_n &= n - \beta, & n &\geq 2, \\ & & \rho_n &= n - \beta + \gamma, & n &\geq 1, \\ \hat{\zeta}_1 &= \beta - 1, & \hat{\zeta}_n &= \hat{\beta} + n - 1, & n &\geq 2, \\ & & \hat{\rho}_n &= \hat{\beta} + \hat{\gamma} + n - 1, & n &\geq 1, \end{aligned}$$

which satisfy the interlacing condition

$$\cdots < -\hat{\rho}_2 < -\hat{\zeta}_2 < -\hat{\rho}_1 < -\hat{\zeta}_1 < 0 < \zeta_1 < \rho_1 < \zeta_2 < \rho_2 < \cdots.$$

To show that ξ belongs to the meromorphic class, apply [18, Theorem 1(v)] when ξ is killed, and [18, Corollary 2] in the unkilled case. The proof is a routine calculation using the Weierstrass representation [14, Formula 8.322] to expand κ and $\hat{\kappa}$ as infinite products; we omit it for the sake of brevity.

We now calculate the Lévy density. For a process in the meromorphic class, it is known that the Lévy measure has a density of the form

$$\pi(x) = \mathbf{1}_{\{x>0\}} \sum_{n\geq 1} a_n \rho_n e^{-\rho_n x} + \mathbf{1}_{\{x<0\}} \sum_{n\geq 1} \hat{a}_n \hat{\rho}_n e^{\hat{\rho}_n x} \tag{4}$$

for some coefficients $(a_n)_{n\geq 1}$ and $(\hat{a}_n)_{n\geq 1}$, where ρ_n and $\hat{\rho}_n$ are as above. Furthermore, from [18, Equation (8)] we see that

$$a_n \rho_n = -\operatorname{Res}(\psi(z) \colon z = \rho_n),$$

and correspondingly for $\hat{a}_n \hat{\rho}_n$. (This remark is made in [18, p. 1111].) From here, it is simple to compute

$$a_n \rho_n = -\frac{(-1)^{n-1}}{(n-1)!} \frac{1}{\Gamma(1-\gamma-n)} \frac{\Gamma(\eta+n-1)}{\Gamma(\eta-\hat{\gamma}+n-1)}, \qquad n \geq 1,$$

and similarly for $\hat{a}_n \hat{\rho}_n$. Expression (2) follows by substituting in (4) and using the series definition of the hypergeometric function.

Thus far we have been working under the assumption that (3) holds. Suppose now that this fails and we have, say, $1 - \beta + \hat{\beta} + \hat{\gamma} = 0$. Then $\zeta_1 = \rho_1$, which is to say that the first zero-pole pair to the right of the origin is removed. It is clear that ξ still falls into the meromorphic class, and indeed, our expression for π remains valid: although the initial pole ρ_1 no longer exists, the corresponding coefficient $a_1 \rho_1$ vanishes as well. Similarly, we may allow $1 - \beta + \hat{\beta} + \gamma = 0$, in which case the zero-pole pair to the left of the origin is removed; or we may allow both expressions to be zero, in which case both pairs are removed. The proof carries through in all cases.

The claim about the large-time behaviour of ξ follows from the Wiener–Hopf factorisation: $\kappa(0) = 0$ if and only if the range of ξ is a.s. unbounded above, and $\hat{\kappa}(0) = 0$ if and only if the range of ξ is a.s. unbounded below, so we need to examine the values of $\kappa(0)$ and $\hat{\kappa}(0)$ only in each of the four parameter regimes.

Finally, we prove the claims about the Gaussian component and variation of ξ. This proof proceeds along the same lines as that in [17]. Firstly, we observe using [14, Formula 8.328.1] that

$$\psi(i\theta) = O(|\theta|^{\gamma+\hat{\gamma}}) \quad \text{as } |\theta| \to \infty. \tag{5}$$

Applying [1, Proposition I.2(i)] shows that ξ has no Gaussian component. Then, using [14, Formulae 9.131.1 and 9.122.2], we see that

$$\pi(x) = O(|x|^{-(1+\gamma+\hat{\gamma})}) \quad \text{as } x \to 0,$$

and together with the necessary and sufficient condition $\int_{\mathbb{R}} (1 \wedge |x|) \pi(x) \, \mathrm{d}x < \infty$ for bounded variation, this proves the claim about the variation. In the bounded variation case, applying [1, Proposition I.2(ii)] with (5) shows that ξ has zero drift. This completes the proof.

Henceforth we use the phrase *extended hypergeometric class* of Lévy processes to describe any process ξ satisfying Proposition 1.

Remark 1. If ξ is a process in the extended hypergeometric class, with parameters $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$, then the dual process $-\xi$ also lies in this class, and has parameters $(1 - \hat{\beta}, \hat{\gamma}, 1 - \beta, \gamma)$.

Remark 2. Note that we may instead extend the parameter range $\mathcal{A}_{\mathrm{HG}}$ by moving only β, or only $\hat{\beta}$. To be precise, both

$$\mathcal{A}_{\mathrm{EHG}}^{\beta} = \{\beta \in [1, 2],\ \gamma, \hat{\gamma} \in (0, 1),\ \hat{\beta} \geq 0;\ 1 - \beta + \hat{\beta} + \gamma \leq 0,\ 1 - \beta + \hat{\beta} + \hat{\gamma} \geq 0\}$$

and

$$\mathcal{A}_{\mathrm{EHG}}^{\hat{\beta}} = \{\beta \leq 1,\ \gamma, \hat{\gamma} \in (0, 1),\ \hat{\beta} \in [-1, 0];\ 1 - \beta + \hat{\beta} + \gamma \geq 0,\ 1 - \beta + \hat{\beta} + \hat{\gamma} \leq 0\}$$

are suitable parameter regimes, and we can develop a similar theory for such processes; for instance, for parameters in $\mathcal{A}_{\mathrm{EHG}}^{\beta}$, we have the Wiener–Hopf factors

$$\kappa(z) = \frac{\Gamma(2 - \beta + \gamma + z)}{\Gamma(2 - \beta + z)}, \qquad \hat{\kappa}(z) = \frac{\beta - 1 + z}{\beta - 1 - \gamma + z} \frac{\Gamma(\hat{\beta} + \hat{\gamma} + z)}{\Gamma(\hat{\beta} + z)}.$$

However, we are not aware of any examples of processes in these classes.

3. The exponential functional

Suppose that ξ is a Lévy process in the extended hypergeometric class with $\beta > 1$, which is to say that either ξ is killed or it drifts to $+\infty$.

We are then interested in the *exponential functional* of the process, defined for any $\delta > 0$ by

$$I(\xi/\delta) = \int_0^\infty \mathrm{e}^{-\xi_t/\delta}\, \mathrm{d}t.$$

(Since ξ/δ is not in the extended hypergeometric class, we are in fact studying exponential functionals of a slightly larger collection of processes.) This is an a.s. finite random variable under the conditions we have just outlined.

It will emerge that the best way to characterise the distribution of $I(\xi/\delta)$ is via its *Mellin transform*, $\mathcal{M}(s) = \mathbb{E}[I(\xi/\delta)^{s-1}]$, whose domain of definition in the complex plane is a vertical strip to be determined.

In the case of a hypergeometric Lévy process with $\hat{\beta} > 0$, it was shown in [17] that the Mellin transform of the exponential functional is given for $\operatorname{Re} s \in (0, 1 + \hat{\beta}\delta)$ by

$$\mathcal{M}_{\mathrm{HG}}(s) = C\, \Gamma(s) \frac{G((1 - \beta)\delta + s; \delta)}{G((1 - \beta + \gamma)\delta + s; \delta)} \frac{G((\hat{\beta} + \hat{\gamma})\delta + 1 - s; \delta)}{G(\hat{\beta}\delta + 1 - s; \delta)},$$

where C is a normalising constant such that $\mathcal{M}_{\mathrm{HG}}(1) = 1$ and G is the double gamma function; see [17] for a definition of this special function.

Our goal in this section is the following result; it characterises the law of the exponential functional for the extended hypergeometric class.

Proposition 2. *Let ξ be a Lévy process in the extended hypergeometric class with $\beta > 1$, and set $\theta = \delta(\beta - 1)$. Then the Mellin transform $\mathcal{M}$ of $I(\xi/\delta)$ is given by*

$$\mathcal{M}(s) = c\widetilde{\mathcal{M}}(s) \frac{\Gamma(\delta(1 - \beta + \gamma) + s)}{\Gamma(-\delta\hat{\beta} + s)} \frac{\Gamma(\delta(\beta - 1) + 1 - s)}{\Gamma(\delta(\hat{\beta} + \hat{\gamma}) + 1 - s)}, \qquad \operatorname{Re} s \in (0, 1 + \theta), \quad (6)$$

where $\widetilde{\mathcal{M}}$ is the Mellin transform of $I(\zeta/\delta)$, and ζ is an auxiliary Lévy process in the hypergeometric class, with parameters $(\beta - 1, \gamma, \hat{\beta} + 1, \hat{\gamma})$. The constant c is such that $\mathcal{M}(1) = 1$.

Proof. The process ξ/δ has Laplace exponent ψ_δ given by $\psi_\delta(z) = \psi(z/\delta)$. The relationship with ζ arises from the calculation

$$\psi_\delta(z) = \frac{-\hat\beta - z/\delta}{1 - \beta + \gamma - z/\delta}\frac{\beta - 1 + z/\delta}{\hat\beta + \hat\gamma + z/\delta}\frac{\Gamma(2 - \beta + \gamma - z/\delta)}{\Gamma(2 - \beta - z/\delta)}\frac{\Gamma(1 + \hat\beta + \hat\gamma + z/\delta)}{\Gamma(1 + \hat\beta + z/\delta)}$$

$$= \frac{-\hat\beta - z/\delta}{1 - \beta + \gamma - z/\delta}\frac{\beta - 1 + z/\delta}{\hat\beta + \hat\gamma + z/\delta}\tilde\psi_\delta(z), \tag{7}$$

where $\tilde\psi_\delta$ is the Laplace exponent of a Lévy process ζ/δ, with ζ as in the statement of the theorem.

Let $f(s)$ denote the right-hand side of (6). The proof now proceeds via the 'verification result' [17, Proposition 2].

Recall that a Lévy process with Laplace exponent ϕ is said to satisfy the *Cramér condition with Cramér number θ* if there exist $z_0 < 0$ and $\theta \in (0, -z_0)$ such that $\phi(z)$ is defined for all $z \in (z_0, 0)$ and $\phi(-\theta) = 0$. Inspecting the Laplace exponent ψ_δ shows that ξ/δ satisfies the Cramér condition with Cramér number $\theta = \delta(\beta - 1)$.

Furthermore, ζ/δ satisfies the Cramér condition with Cramér number $\tilde\theta = \delta(\hat\beta + 1)$. It follows from [32, Lemma 2] that $\widetilde{\mathcal{M}}(s)$ is finite in the strip $\operatorname{Re} s \in (0, 1 + \tilde\theta)$; and by the properties of Mellin transforms of positive random variables, it is analytic and zero-free in its domain of definition. The constraints in the parameter set $\mathcal{A}_{\mathrm{EHG}}$ ensure that $\tilde\theta \geq \theta$; this, together with inspecting the right-hand side of (6) and comparing again with the conditions in $\mathcal{A}_{\mathrm{EHG}}$, demonstrates that $\mathcal{M}(s)$ is analytic and zero-free in the strip $\operatorname{Re} s \in (0, 1 + \theta)$.

We must then check the functional equation $f(s + 1) = -sf(s)/\psi_\delta(-s)$ for $s \in (0, \theta)$. Apply (7) to write

$$-\frac{s}{\psi_\delta(-s)} = -\frac{s}{\tilde\psi_\delta(-s)}\frac{1 - \beta + \gamma + s/\delta}{-\hat\beta + s/\delta}\frac{\hat\beta + \hat\gamma - s/\delta}{\beta - 1 - s/\delta}$$

$$= \frac{\widetilde{\mathcal{M}}(s + 1)}{\widetilde{\mathcal{M}}(s)}\frac{\delta(1 - \beta + \gamma) + s}{-\delta\hat\beta + s}\frac{\delta(\hat\beta + \hat\gamma) - s}{\delta(\beta - 1) - s}$$

$$= \frac{\widetilde{\mathcal{M}}(s + 1)}{\widetilde{\mathcal{M}}(s)}\frac{\Gamma(-\delta\hat\beta + s)}{\Gamma(-\delta\hat\beta + s + 1)}\frac{\Gamma(\delta(1 - \beta + \gamma) + s + 1)}{\Gamma(\delta(1 - \beta + \gamma) + s)}$$

$$\times \frac{\Gamma(\delta(\hat\beta + \hat\gamma) + 1 - s)}{\Gamma(\delta(\hat\beta + \hat\gamma) - s)}\frac{\Gamma(\delta(\beta - 1) - s)}{\Gamma(\delta(\beta - 1) + 1 - s)},$$

making use of the same functional equation for the Mellin transform $\widetilde{\mathcal{M}}$. It is then clear that the right-hand side is equal to $f(s + 1)/f(s)$.

Finally, it remains to check that $|f(s)|^{-1} = o(\exp(2\pi|\operatorname{Im}(s)|))$ as $|\operatorname{Im} s| \to \infty$, uniformly in $\operatorname{Re} s \in (0, 1 + \theta)$. The following asymptotic relation may be derived from Stirling's asymptotic formula for the gamma function:

$$\log \Gamma(z) = z \log z - z + O(\log z). \tag{8}$$

Since Stirling's asymptotic formula is uniform in $|\arg(z)| < \pi - \omega$ for any choice of $\omega > 0$, it follows that (8) holds uniformly in the strip $\operatorname{Re} s \in (0, 1 + \theta)$; see [29, Chapter 8, Section 4]. We thus obtain

$$\log\left|\frac{\Gamma(\delta(1 - \beta + \gamma) + s)}{\Gamma(-\delta\hat\beta + s)}\frac{\Gamma(\delta(\beta - 1) + 1 - s)}{\Gamma(\delta(\hat\beta + \hat\gamma) + 1 - s)}\right|^{-1} = O(\log s) = o(|\operatorname{Im} s|),$$

and comparing this with the proof of [17, Theorem 2], where the asymptotic behaviour of $\widetilde{\mathcal{M}}(s)$ is given, we see that this is sufficient for our purposes. Hence, $\mathcal{M}(s) = f(s)$ when $\operatorname{Re} s \in (0, 1 + \theta)$. This completes the proof.

This Mellin transform may be inverted to give an expression for the density of $I(\xi/\delta)$ in terms of a series whose terms are defined iteratively, but we do not pursue this here. For details of this approach, see [17, Section 4].

4. Three examples

It is well known that hypergeometric Lévy processes appear as the Lamperti transforms of stable processes killed passing below 0, conditioned to stay positive and conditioned to hit 0 continuously; see [17, Theorem 1]. In this section we briefly present three additional examples in which the extended hypergeometric class comes into play. The examples may all be obtained in the same way: begin with a stable process, modify its path in some way to obtain a pssMp, and then apply the Lamperti transform to obtain a new Lévy process. We therefore start with a short description of these concepts.

We work with the (strictly) stable process X with scaling parameter α and positivity parameter ρ, and which is defined as follows. For (α, ρ) in the set

$$
\begin{aligned}
\mathcal{A}_{\mathrm{st}} = {} & \{(\alpha, \rho) \colon \alpha \in (0, 1),\ \rho \in (0, 1)\} \cup \left\{(\alpha, \rho) = \left(1, \tfrac{1}{2}\right)\right\} \\
& \cup \left\{(\alpha, \rho) \colon \alpha \in (1, 2),\ \rho \in \left(1 - \frac{1}{\alpha}, \frac{1}{\alpha}\right)\right\},
\end{aligned}
$$

let X, with probability laws $(\mathrm{P}_x)_{x\in\mathbb{R}}$, be the Lévy process with characteristic exponent

$$
\Psi(\theta) = \begin{cases} c|\theta|^{\alpha}\big(1 - \mathrm{i}\beta \tan \tfrac{1}{2}\pi\alpha\ \mathrm{sgn}\,\theta\big), & \alpha \in (0, 2) \setminus \{1\}, \\ c|\theta|, & \alpha = 1, \end{cases} \qquad \theta \in \mathbb{R},
$$

where $c = \cos(\pi\alpha(\rho - \tfrac{1}{2}))$ and $\beta = \tan(\pi\alpha(\rho - \tfrac{1}{2}))/\tan(\tfrac{1}{2}\pi\alpha)$; by this we mean that $\mathrm{E}_0[\mathrm{e}^{\mathrm{i}\theta X_1}] = \mathrm{e}^{-\Psi(\theta)}$. This Lévy process has absolutely continuous Lévy measure with density

$$
c_+ x^{-(\alpha+1)}\mathbf{1}_{\{x>0\}} + c_- |x|^{-(\alpha+1)}\mathbf{1}_{\{x<0\}}, \qquad x \in \mathbb{R},
$$

where

$$
c_+ = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha\rho)\,\Gamma(1 - \alpha\rho)}, \qquad c_- = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha\hat{\rho})\,\Gamma(1 - \alpha\hat{\rho})},
$$

and $\hat{\rho} = 1 - \rho$.

The parameter set $\mathcal{A}_{\mathrm{st}}$ and the characteristic exponent Ψ represent, up to a multiplicative constant in Ψ, all (strictly) stable processes which jump in both directions, except for Brownian motion and symmetric Cauchy processes with nonzero drift.

The choice of α and ρ as parameters is explained as follows. The process X satisfies the *α-scaling property*, namely,

$$
\text{under } \mathrm{P}_x, \text{ the law of } (cX_{tc^{-\alpha}})_{t\ge 0} \text{ is } \mathrm{P}_{cx} \text{ for all } x \in \mathbb{R} \text{ and } c > 0. \tag{9}
$$

The second parameter satisfies $\rho = \mathrm{P}_0\{X_t > 0\}$.

A *pssMp* with *self-similarity index* $\alpha > 0$ is a standard Markov process $Y = (Y_t)_{t\ge 0}$ with filtration $(\mathcal{G}_t)_{t\ge 0}$ and probability laws $(\mathrm{P}_x)_{x>0}$, on $[0, \infty)$, which has 0 as an absorbing state

and which satisfies the scaling property (9) (with Y in place of X). Here, we mean 'standard' in the sense of [4], which is to say, $(\mathcal{G}_t)_{t \geq 0}$ is a complete, right-continuous filtration, and Y has càdlàg paths and is strong Markov and quasi-left-continuous.

In the seminal paper [27], Lamperti described a one-to-one correspondence between pssMps and Lévy processes, as we now outline (but, it should be noted that our definition of a pssMp differs slightly from Lamperti's; for the connection, see [36, Section 0]).

Let $S(t) = \int_0^t (Y_u)^{-\alpha} \, du$. This process is continuous and strictly increasing until Y reaches 0. Let $(T(s))_{s \geq 0}$ be its inverse, and define

$$\xi_s = \log Y_{T(s)}, \qquad s \geq 0.$$

Then $\xi := (\xi_s)_{s \geq 0}$ is a Lévy process started at $\log x$, possibly killed at an independent exponential time; the law of the Lévy process and the rate of killing do not depend on the value of x. The real-valued process ξ with probability laws $(\mathbb{P}_y)_{y \in \mathbb{R}}$ is called the *Lévy process associated to Y*, or the *Lamperti transform of Y*. (Our choice here of the symbol $\mathbb{P}_y$ is an indication that in the coming examples, the Lamperti transforms that arise will in fact be in the extended hypergeometric class.)

Equivalent definitions of S and T, in terms of ξ instead of Y, are given by taking $T(s) = \int_0^s \exp(\alpha \xi_u) \, du$ and S as its inverse. Then

$$Y_t = \exp(\xi_{S(t)}) \quad \text{for all } t \geq 0,$$

and this shows that the Lamperti transform is a bijection.

4.1. The path-censored stable process

Let X be the stable process defined in Section 4. In [26], the present authors considered a 'path-censored' version of the stable process, formed by erasing the time spent in the negative half-line. To be precise, define

$$A_t = \int_0^t \mathbf{1}_{\{X_s > 0\}} \, ds, \qquad t \geq 0,$$

and let $\gamma(t) = \inf\{s \geq 0 \colon A_s > t\}$ be its right-continuous inverse. Also, define

$$T_0 = \inf\{t \geq 0 \colon X_{\gamma(t)} = 0\},$$

which is finite or infinite a.s. according to whether $\alpha > 1$ or $\alpha \leq 1$. Then the process

$$Y_t = X_{\gamma(t)} \mathbf{1}_{\{t < T_0\}}, \qquad t \geq 0,$$

is a pssMp, called the *path-censored stable process*.

In Theorems 5.3 and 5.5 of [26], it was shown that the Laplace exponent ψ^Y of the Lamperti transform ξ^Y associated with Y is given by

$$\psi^Y(z) = \frac{\Gamma(\alpha\rho - z)}{\Gamma(-z)} \frac{\Gamma(1 - \alpha\rho + z)}{\Gamma(1 - \alpha + z)},$$

and there it was remarked that, when $\alpha \leq 1$, this process is in the hypergeometric class with parameters

$$(\beta, \gamma, \hat{\beta}, \hat{\gamma}) = (1, \alpha\rho, 1 - \alpha, \alpha\hat{\rho}).$$

It is readily seen from our definition that, when $\alpha > 1$, the process ξ^Y is in the extended hypergeometric class, with the same set of parameters.

From the Lamperti transform we know that

$$I(-\alpha\xi^Y) = \inf\{u \geq 0 \colon Y_u = 0\} = \int_0^{T_0} \mathbf{1}_{\{X_t > 0\}} \, \mathrm{d}t,$$

where the latter is the occupation time of $(0, \infty)$ up to first hitting 0 for the stable process. This motivates the following proposition, whose proof is a direct application of Proposition 2.

Proposition 3. *The Mellin transform of the random variable $I(-\alpha\xi^Y)$ is given by*

$$\mathcal{M}(s) = c \frac{G(2/\alpha - 1 + s; 1/\alpha)}{G(2/\alpha - \rho + s; 1/\alpha)} \frac{G(1/\alpha + \rho + 1 - s; 1/\alpha)}{G(1/\alpha + 1 - s; 1/\alpha)} \frac{\Gamma(1/\alpha - \rho + s)}{\Gamma(\rho + 1 - s)} \Gamma\left(2 - \frac{1}{\alpha} - s\right)$$

for $\operatorname{Re} s \in (\rho - 1/\alpha, 2 - 1/\alpha)$, where c is a normalising constant such that $\mathcal{M}(1) = 1$.

Remark 3. When X is in the class $\mathcal{C}_{k,l}$ introduced by Doney [12], which is to say that

$$\rho + k = \frac{l}{\alpha}$$

for $k, l \in \mathbb{Z}$, equivalent expressions in terms of gamma and trigonometric functions may be found via repeated application of certain identities of the double-gamma function; see, for example, [17, Equations (19) and (20)].

For example, when $k, l \geq 0$, we have

$$\mathcal{M}(s) = c(-1)^l (2\pi)^{l(1/\alpha - 1)} \left(\frac{1}{\alpha}\right)^{l(1 - 2/\alpha)} \frac{\Gamma((1 - l)/\alpha + k + s)}{\Gamma(l/\alpha + 1 - k - s)} \frac{\Gamma(2 - 1/\alpha - s)}{\Gamma(2 - l - \alpha - \alpha s)}$$

$$\times \prod_{j=1}^l \Gamma\left(\frac{j}{\alpha} + 1 - s\right) \Gamma\left(\frac{2}{\alpha} - \left(\frac{j}{\alpha} + 1 - s\right)\right) \prod_{i=0}^{k-1} \frac{\sin(\pi\alpha(s + i))}{\pi},$$

and, when $k < 0$ and $l \geq 0$,

$$\mathcal{M}(s) = c(-1)^l (2\pi)^{l(1/\alpha - 1)} \left(\frac{1}{\alpha}\right)^{l(1 - 2/\alpha)}$$

$$\times \frac{\Gamma((1 - l)/\alpha + k + s)\Gamma(2 - 1/\alpha - s)\Gamma(l + 1 + \alpha - \alpha s)\Gamma(2 - l + \alpha k + \alpha s)}{\Gamma(l/\alpha + 1 - k - s)}$$

$$\times \prod_{j=1}^l \Gamma\left(\frac{j}{\alpha} + 1 - s\right) \Gamma\left(\frac{2}{\alpha} - \left(\frac{j}{\alpha} + 1 - s\right)\right) \prod_{i=2}^{-k-1} \frac{\pi}{\sin(\pi\alpha(s - i))}.$$

Similar expressions may be obtained when $k \geq 0$, $l < 0$, and $k, l < 0$.

4.2. The radial part of the symmetric stable process

If X is a symmetric stable process—that is, $\rho = \frac{1}{2}$—then the process

$$R_t = |X_t|, \qquad t \geq 0,$$

is a pssMp, which we call the *radial part* of X. The Lamperti transform, ξ^R, of this process was studied by Caballero *et al.* [7] in dimension d; these authors computed the Wiener–Hopf factorisation of ξ^R under the assumption that $\alpha < d$, finding that the process is a hypergeometric Lévy process. Using the extended hypergeometric class, we extend this result, in one dimension, by finding the Wiener–Hopf factorisation when $\alpha > 1$.

In Kuznetsov *et al.* [21], the following theorem was proved using the work of Caballero *et al.* [7].

Theorem 1. (Laplace exponent.) *The Laplace exponent of the Lévy process* $2\xi^R$ *is given by*

$$\psi^R(2z) = -2^\alpha \frac{\Gamma(\alpha/2 - z)}{\Gamma(-z)} \frac{\Gamma(1/2 + z)}{\Gamma((1-\alpha)/2 + z)}. \tag{10}$$

We now identify the Wiener–Hopf factorisation of ξ^R; it depends on the value of α. However, note the factor 2^α in (10). In the context of the Wiener–Hopf factorisation, we could ignore this factor by picking an appropriate normalisation of local time; however, another approach is as follows.

Write $R' = \frac{1}{2}R$, and denote by $\xi^{R'}$ the Lamperti transform of R'. Then the scaling of space on the level of the self-similar process is converted by the Lamperti transform into a scaling of time, so that $\xi_s^R = \log 2 + \xi_{s2^\alpha}^{R'}$. In particular, if we write ψ' for the characteristic exponent of $\xi^{R'}$, it follows that $\psi' = 2^{-\alpha}\psi^R$. This allows us to disregard the inconvenient constant factor in (10), if we work with $\xi^{R'}$ instead of ξ^R.

The following corollary is now simple when we bear in mind the hypergeometric class of Lévy processes introduced in Section 2. We emphasise that this Wiener–Hopf factorisation was derived by different methods in [7, Theorem 7] for $\alpha < 1$, though not for $\alpha = 1$.

Corollary 1. (Wiener–Hopf factorisation, $\alpha \in (0, 1]$.) *The Wiener–Hopf factorisation of* $2\xi^{R'}$ *when* $\alpha \in (0, 1]$ *is given by*

$$\psi'(2z) = -\frac{\Gamma(\alpha/2 - z)}{\Gamma(-z)} \frac{\Gamma(1/2 + z)}{\Gamma((1-\alpha)/2 + z)}$$

and $2\xi^{R'}$ *is a Lévy process of the hypergeometric class with parameters*

$$(\beta, \gamma, \hat{\beta}, \hat{\gamma}) = \left(1, \tfrac{1}{2}\alpha, \tfrac{1}{2}(1-\alpha), \tfrac{1}{2}\alpha\right).$$

Proof. It suffices to compare the characteristic exponent with that of a hypergeometric Lévy process.

When $\alpha > 1$, the process $\xi^{R'}$ is not a hypergeometric Lévy process, but it is in the extended hypergeometric class, and we have the following result, which is new.

Theorem 2. (Wiener–Hopf factorisation, $\alpha \in (1, 2)$.) *The Wiener–Hopf factorisation of* $2\xi^{R'}$ *when* $\alpha \in (1, 2)$ *is given by*

$$\psi'(2z) = -\left(\tfrac{1}{2}(\alpha - 1) - z\right)\frac{\Gamma(\alpha/2 - z)}{\Gamma(1-z)} z \frac{\Gamma(1/2 + z)}{\Gamma((3-\alpha)/2 + z)} \tag{11}$$

and $2\xi^{R'}$ *is a Lévy process in the extended hypergeometric class, with parameters*

$$(\beta, \gamma, \hat{\beta}, \hat{\gamma}) = \left(1, \tfrac{1}{2}\alpha, \tfrac{1}{2}(1-\alpha), \tfrac{1}{2}\alpha\right).$$

Proof. Simply use Theorem 1; using the formula $x\Gamma(x) = \Gamma(x + 1)$ yields (11). That this is indeed the Wiener–Hopf factorisation follows once we recognise $2\xi^{R'}$ as a process in the extended hypergeometric class, and apply Proposition 1.

As an illustration of the utility of the extended hypergeometric class, we now derive an expression for the Mellin transform of the exponential functional for the dual process $-\xi^{R'}$. This quantity is linked by the Lamperti representation to the hitting time of 0 for X; see Section 4. In particular, defining $T_0 = \inf\{t \geq 0 : X_t = 0\}$, we have

$$T_0 = \int_0^\infty e^{\alpha\xi_t^R} \, dt = \int_0^\infty e^{\alpha\xi_{2^\alpha t}^{R'}} \, dt = 2^{-\alpha} \int_0^\infty e^{\alpha\xi_s^{R'}} \, ds = 2^{-\alpha} I(-\alpha\xi^{R'}).$$

Since $-2\xi^{R'}$ is an extended hypergeometric Lévy process which drifts to $+\infty$ and has parameters $(\frac{1}{2}(\alpha + 1), \frac{1}{2}\alpha, 0, \frac{1}{2}\alpha)$, we can apply the theory just developed to compute the Mellin transform of $I(-\alpha\xi^{R'})$. Denote this by $\mathcal{M}$, that is,

$$\mathcal{M}(s) = \mathbb{E}[I(-\alpha\xi^{R'})^{s-1}]$$

for some range of $s \in \mathbb{C}$ to be determined.

Proposition 4. *For* $\mathrm{Re}\, s \in (-1/\alpha, 2 - 1/\alpha)$,

$$
\begin{aligned}
\mathrm{E}_1[T_0^{s-1}] &= 2^{-\alpha(s-1)}\mathcal{M}(s) \\
&= 2^{-\alpha(s-1)}\frac{\sqrt{\pi}}{\Gamma(1/\alpha)\Gamma(1 - 1/\alpha)}\frac{\Gamma(1 + \alpha/2 - \alpha s/2)}{\Gamma((1-\alpha)/2 + \alpha s/2)}\Gamma\left(\frac{1}{\alpha} - 1 + s\right) \\
&\quad \times \frac{\Gamma(2 - 1/\alpha - s)}{\Gamma(2 - s)}.
\end{aligned}
\tag{12}
$$

Proof. Let ζ be a hypergeometric Lévy process with parameters $(\frac{1}{2}(\alpha - 1), \frac{1}{2}\alpha, 1, \frac{1}{2}\alpha)$, and denote by $\widetilde{\mathcal{M}}$ the Mellin transform of the exponential functional $I(\frac{1}{2}\alpha\zeta)$, so we know $\widetilde{\mathcal{M}}$ to be finite for $\mathrm{Re}\, s \in (0, 1 + 2/\alpha)$ from the argument in the proof of Proposition 2.

We can now use Proposition 2 to carry out the calculation below, provided that $\mathrm{Re}\, s \in (0, 2 - 1/\alpha)$. In it, G is the double-gamma function, as defined in [17, Section 3], and we use [17, Equation (25)] in the third line and the identity $x\Gamma(x) = \Gamma(x + 1)$ in the final line. For normalisation constants C (and C') to be determined, we have

$$
\begin{aligned}
\mathcal{M}(s) &= C\widetilde{\mathcal{M}}(s)\frac{\Gamma(1/\alpha + s)}{\Gamma(s)}\frac{\Gamma(2 - 1/\alpha - s)}{\Gamma(2 - s)} \\
&= C\frac{G(3/\alpha - 1 + s; 2/\alpha)}{G(3/\alpha + s; 2/\alpha)}\frac{G(2/\alpha + 2 - s; 2/\alpha)}{G(2/\alpha + 1 - s; 2/\alpha)}\Gamma(s)\frac{\Gamma(1/\alpha + s)}{\Gamma(s)}\frac{\Gamma(2 - 1/\alpha - s)}{\Gamma(2 - s)} \\
&= C\frac{\Gamma(1 + \alpha/2 - \alpha s/2)}{\Gamma((3 - \alpha)/2 + \alpha/2s)}\Gamma\left(\frac{1}{\alpha} + s\right)\frac{\Gamma(2 - 1/\alpha - s)}{\Gamma(2 - s)} \\
&= C'\frac{\Gamma(1 + \alpha/2 - \alpha s/2)}{\Gamma((1 - \alpha)/2 + \alpha s/2)}\Gamma\left(\frac{1}{\alpha} - 1 + s\right)\frac{\Gamma(2 - 1/\alpha - s)}{\Gamma(2 - s)}.
\end{aligned}
$$

The condition $\mathcal{M}(1) = 1$ means that we can calculate

$$C' = \frac{\sqrt{\pi}}{\Gamma(1/\alpha)\Gamma(1 - 1/\alpha)},$$

and this gives the Mellin transform explicitly for $\mathrm{Re}\, s \in (0, 2 - 1/\alpha)$.

We now expand the domain of $\mathcal{M}$. Note that, in contrast to the general case of Proposition 2, the right-hand side of (12) is well defined when $\mathrm{Re}\, s \in (-1/\alpha, 2 - 1/\alpha)$, and is indeed analytic in this region. (The reason for this difference is the cancellation of a simple pole and zero at the point 0.) Theorem 2 of [28] shows that, if the Mellin transform of a probability measure is analytic in a neighbourhood of the point $1 \in \mathbb{C}$, then it is analytic in a strip $\mathrm{Re}\, s \in (a, b)$, where $-\infty \leq a < 1 < b \leq \infty$; furthermore, the function has singularities at a and b, if they are finite. It then follows that the right-hand side of (12) must actually be equal to $\mathcal{M}$ in all of $\mathrm{Re}\, s \in (-1/\alpha, 2 - 1/\alpha)$, and this completes the proof.

We remark that the distribution of T_0 has been characterised previously by Yano *et al.* [37] and Cordero [11], using rather different methods; and the Mellin transform above was also obtained, again via the Lamperti transform but without the extended hypergeometric class, in Kuznetsov *et al.* [21].

It is also fairly straightforward to produce the following hitting distribution. Define

$$\sigma_{-1}^1 = \inf\{t \ge 0 \colon X_t \notin [-1, 1]\},$$

the *first exit time of* $[-1, 1]$ for X. We give the distribution of the position of the symmetric stable process X at time σ_{-1}^1, provided this occurs before X hits 0. Note that, when $\alpha \in (0, 1]$, the process does not hit 0, so the distribution is simply that found by Rogozin [33].

Proposition 5. *Let X be the symmetric stable process with $\alpha \in (1, 2)$. Then, for $|x| < 1$ and $y > 1$,*

$$\frac{\mathrm{P}_x\{|X_{\sigma_{-1}^1}| \in \mathrm{d}y; \sigma_{-1}^1 < T_0\}}{\mathrm{d}y}$$

$$= \frac{\sin(\pi\alpha/2)}{\pi}|x|(1 - |x|)^{\alpha/2}y^{-1}(y - 1)^{-\alpha/2}(y - |x|)^{-1}$$

$$+ \frac{1}{2}\frac{\sin(\pi\alpha/2)}{\pi}y^{-1}(y - 1)^{-\alpha/2}|x|^{(\alpha-1)/2}\int_0^{1-|x|} t^{\alpha/2-1}(1 - t)^{-(\alpha-1)/2}\,\mathrm{d}t.$$

Proof. The starting point of the proof is the 'second factorisation identity' [22, Exercise 6.7],

$$\int_0^\infty \mathrm{e}^{-qz}\mathbb{E}[\mathrm{e}^{-\beta(\xi_{S_z^+}-z)}; S_z^+ < \infty]\,\mathrm{d}x = \frac{\kappa(q) - \kappa(\beta)}{(q - \beta)\kappa(q)}, \qquad q, \beta > 0,$$

where $S_z^+ = \inf\{t \ge 0 \colon \xi_t > z\}$. We now invert in q and z, in that order; this is a lengthy but routine calculation, and we omit it. We then apply the Lamperti transform: if $g(z, \cdot)$ is the density of the measure $\mathbb{P}\{\xi_{S_z^+} - z \in \cdot; S_z^+ < \infty\}$ then

$$\mathrm{P}_x\{|X_{\sigma_{-1}^1}| \in \mathrm{d}y; \sigma_{-1}^1 < T_0\} = y^{-1}g(\log|x|^{-1}, \log y),$$

and this completes the proof.

The following hitting probability emerges after integrating in the above proposition.

Corollary 2. *For $|x| < 1$,*

$$\mathrm{P}_x\{T_0 < \sigma_{-1}^1\} = (1 - |x|)^{\alpha/2} - \frac{1}{2}|x|^{(\alpha-1)/2}\int_0^{1-|x|} t^{\alpha/2-1}(1 - t)^{-(\alpha-1)/2}\,\mathrm{d}t.$$

Finally, it is not difficult to produce the following slightly more general result. Applying the Markov property at time T_0 gives

$$\mathrm{P}_x\{X_{\sigma_{-1}^1} \in \mathrm{d}y; \sigma_{-1}^1 < T_0\} = \mathrm{P}_x\{X_{\sigma_{-1}^1} \in \mathrm{d}y\} - \mathrm{P}_x\{X_{\sigma_{-1}^1} \in \mathrm{d}y; T_0 < \sigma_{-1}^1\}$$

$$= \mathrm{P}_x\{X_{\sigma_{-1}^1} \in \mathrm{d}y\} - \mathrm{P}_x\{T_0 < \sigma_{-1}^1\}\mathrm{P}_0\{X_{\sigma_{-1}^1} \in \mathrm{d}y\}.$$

The hitting distributions on the right-hand side were found by Rogozin [33], and substituting yields the following corollary.

Corollary 3. *For* $|x| < 1$ *and* $|y| > 1$,

$$\frac{\mathrm{P}_x\{X_{\sigma^1_{-1}} \in \mathrm{d}y; \sigma^1_{-1} < T_0\}}{\mathrm{d}y}$$

$$= \frac{\sin(\pi\alpha/2)}{\pi}(1-x)^{\alpha/2}(1+x)^{\alpha/2}(y-1)^{-\alpha/2}(y+1)^{-\alpha/2}(y-x)^{-1}$$

$$- \left[(1-|x|)^{\alpha/2} - \frac{1}{2}|x|^{(\alpha-1)/2}\int_0^{1-|x|} t^{\alpha/2-1}(1-t)^{-(\alpha-1)/2}\,\mathrm{d}t\right]$$

$$\times \frac{\sin(\pi\alpha/2)}{\pi}(y-1)^{-\alpha/2}(y+1)^{-\alpha/2}y^{-1}.$$

4.3. The radial part of the symmetric stable process conditioned to avoid 0

We have just computed the Lamperti transform $\xi^{R'}$ of the pssMp $R' = \frac{1}{2}|X|$, where X is a symmetric stable process. In this subsection we consider instead the symmetric stable process conditioned to avoid 0, and obtain its Lamperti transform.

In [30], Pantí showed (among many other results) that the function

$$h(x) = \begin{cases} -\Gamma(1-\alpha)\dfrac{\sin(\pi\alpha\hat{\rho})}{\pi}x^{\alpha-1}, & x > 0, \\[2mm] -\Gamma(1-\alpha)\dfrac{\sin(\pi\alpha\rho)}{\pi}x^{\alpha-1}, & x < 0, \end{cases}$$

is invariant for the stable process killed upon hitting 0, and defines the family of measures $(\mathrm{P}_x^{\updownarrow})_{x\neq 0}$ given via the Doob h-transform:

$$\mathrm{P}_x^{\updownarrow}(\Lambda) = \frac{1}{h(x)}\mathrm{E}_x[h(X_t)\mathbf{1}_\Lambda; t < T_0], \qquad x \neq 0,$$

for $\Lambda \in \mathcal{F}_t = \sigma(X_s,\, s \leq t)$. In [30] it was also shown that the laws $\mathrm{P}_x^{\updownarrow}$ arise as limits of the stable process conditioned not to have hit 0 up to an exponential time of rate q, as $q \downarrow 0$. The canonical process associated with the laws $(\mathrm{P}_x^{\updownarrow})_{x\neq 0}$ is therefore called the *stable process conditioned to avoid 0*, and we shall denote it by $X^{\updownarrow}$.

Consider now the process $R^{\updownarrow} = \frac{1}{2}|X^{\updownarrow}|$. This is a pssMp, and we may consider its Lamperti transform, which we will denote by $\xi^{\updownarrow}$. The characteristics of the generalised Lamperti representation of $X^{\updownarrow}$ have been computed explicitly in [10], and the Laplace exponent, $\psi^{\updownarrow}$, of $\xi^{\updownarrow}$ could be computed from this information; however, the harmonic transform gives us the following straightforward relationship between Laplace exponents:

$$\psi^{\updownarrow}(z) = \psi'(z + \alpha - 1).$$

This allows us to calculate

$$\psi^{\updownarrow}(2z) = -\frac{\Gamma(1/2-z)}{\Gamma((1-\alpha)/2-z)}\frac{\Gamma(\alpha/2+z)}{\Gamma(z)},$$

which demonstrates that $2\xi^{\updownarrow}$ is a process in the extended hypergeometric class with parameters

$$(\beta, \gamma, \hat{\beta}, \hat{\gamma}) = \left(\tfrac{1}{2}(\alpha+1), \tfrac{1}{2}\alpha, 0, \tfrac{1}{2}\alpha\right).$$

The present authors and A. Kuznetsov previously computed $\psi^{\updownarrow}$ in [21], where we also observed that the process $\xi^{\updownarrow}$ is the dual Lévy process to $\xi^{R'}$, and remarked that this implies a certain time-reversal relation between R and $R^{\updownarrow}$; see [8, Section 2].

5. Concluding remarks

In this section we offer some comments on how our approach may be adapted in order to offer new insight on an existing class of processes, the Lamperti-stable processes. These were defined in general in the work of Caballero *et al.* [6]; the one-dimensional Lamperti-stable processes are defined as follows. We say that a Lévy process ξ is in the *Lamperti-stable class* if it has no Gaussian component and its Lévy measure has density

$$\pi(x) = \begin{cases} c_+ e^{\beta x} (e^x - 1)^{-(\alpha+1)}\, dx, & x > 0, \\ c_- e^{-\delta x} (e^{-x} - 1)^{-(\alpha+1)}\, dx, & x < 0, \end{cases}$$

for some choice of parameters $(\alpha, \beta, \delta, c_+, c_-)$ such that $\alpha \in (0, 2)$ and $\beta, \delta, c_+, c_- \geq 0$.

The one-dimensional Lamperti-stable processes form a proper subclass of the β-class of Lévy processes of Kuznetsov [15]. It was observed in [17] that there is an intersection between the hypergeometric class and the Lamperti-stable class. In particular, the Lamperti representations of killed and conditioned stable processes (see [5]) fall within the hypergeometric class; and generally speaking, setting $\beta = \hat{\beta}$ in the hypergeometric class and choosing γ and $\hat{\gamma}$ as desired, we obtain a Lamperti-stable process.

Not all Lamperti-stable processes, however, may be obtained in this way, and we now outline how the ideas developed in this work can be used to characterise another subset of the Lamperti-stable processes.

Define the set of parameters

$$\mathcal{A}_{EHL} = \{\beta \in [1, 2],\ \gamma \in (1, 2),\ \hat{\gamma} \in (-1, 0)\},$$

and, for $(\beta, \gamma, \hat{\gamma}) \in \mathcal{A}_{EHL}$, let

$$\psi(z) = \frac{\Gamma(1 - \beta + \gamma - z)}{\Gamma(2 - \beta - z)} \frac{\Gamma(\beta + \hat{\gamma} + z)}{\Gamma(\beta + z)}.$$

Note that this is the negative of the usual hypergeometric Laplace exponent, with $\beta = \hat{\beta}$. We claim that the following proposition holds.

Proposition 6. *There exists a Lévy process ξ with Laplace exponent ψ. Its Wiener–Hopf factorisation $\psi(z) = -\kappa(-z)\hat{\kappa}(z)$ is given by the components*

$$\kappa(z) = \frac{\Gamma(1 - \beta + \gamma + z)}{\Gamma(2 - \beta + z)}, \qquad \hat{\kappa}(z) = (\beta - 1 + z)\frac{\Gamma(\beta + \hat{\gamma} + z)}{\Gamma(\beta + z)}.$$

The ascending ladder height process is a Lamperti-stable subordinator, and the descending factor satisfies

$$\hat{\kappa}(z) = (\mathcal{T}_{\beta-1}\upsilon)^*(z), \qquad \upsilon(z) = \frac{\Gamma(1 + z)}{\Gamma(1 + \hat{\gamma} + z)}.$$

Here υ is the Laplace exponent of a Lamperti-stable subordinator.

The process ξ has no Gaussian component and has a Lévy density given by

$$\pi(x) = \begin{cases} \dfrac{\Gamma(\gamma + \hat{\gamma} + 1)}{\Gamma(1 + \gamma)\Gamma(-\gamma)} e^{(\beta+\hat{\gamma})x}(e^x - 1)^{-(\gamma+\hat{\gamma}+1)}, & x > 0, \\[2ex] \dfrac{\Gamma(\gamma + \hat{\gamma} + 1)}{\Gamma(1 + \hat{\gamma})\Gamma(-\hat{\gamma})} e^{-(1-\beta+\gamma)x}(e^{-x} - 1)^{-(\gamma+\hat{\gamma}+1)}, & x < 0. \end{cases}$$

Thus, ξ falls within the Lamperti-stable class, and

$$(\alpha, \beta, \delta) = (\gamma + \hat{\gamma}, \ \beta + \hat{\gamma}, \ 1 - \beta + \gamma).$$

The proposition may be proved in much the same way as Proposition 1, first using the theory of philanthropy to prove existence, and then the theory of meromorphic Lévy processes to deduce the Lévy measure.

We have thus provided an explicit spatial Wiener–Hopf factorisation of a subclass of Lamperti-stable processes disjoint from that given by the hypergeometric processes.

Acknowledgements

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