New families of subordinators with explicit transition probability semigroup

James Burridge[1] Alexey Kuznetsov[2] Mateusz Kwásnicki[3]
Andreas E. Kyprianou[4]

[1]Portsmouth University
[2]York University, Toronto, Canada.
[3]Wroclaw University of Technology
[4]Unversity of Bath

Continuous state, discrete time branching processes

James Burridge, a physicist, approached me and told me about some of his remarkable results:

- Consider the following continuous-state, discrete-time branching process:

$$Z_0 = x > 0 \text{ and for } n \geq 1, \;\; Z_n \sim \Gamma(kZ_{n-1}, \theta),$$

  where $\Gamma(k, \theta)$ is a Gamma distribution with density $x^{k-1}\mathrm{e}^{-x/\theta}/\Gamma(k)\theta^k$. [Note $\Gamma(nk, \theta) =^d \Gamma(k, \theta)^{*n}$.]
- Define the total progeny $Z^* = \sum_{i=0}^{\infty} Z_i$. Remarkably, in a rather clever way, James had computed by hand the probability density of Z:

$$\mathbb{P}(Z^* \in \mathrm{d}z) = \frac{x\theta^{-kz}(z-x)^{kz-1}\mathrm{e}^{(x-z)/\theta}}{z\Gamma(kz)}\mathrm{d}z, \qquad z \geq 0.$$

- Classical theory predicts that $\mathbb{E}[\mathrm{e}^{-\lambda Z^*}]$ is a reasonably explicit identity, but the inversion to obtain the distribution of Z^* was simply an unknown computation.

Hint of a new family of subordinators

- James' result is strongly suggestive of the existence of a new class of subordinator with explicit semi-group - which is not that common.
- Suppose that $\{Z_t : t \geq 0\}$ is a continuous-state continuous-time branching processes. Then it is well known that $L_t = Z_{\theta(t)}$, $t < \zeta$, is a spectrally positive Lévy process

$$\zeta = \inf\{t > 0 : Z_t = 0\} \text{ and } \int_0^{\theta(t)} Z_u \mathrm{d}u = t.$$

- Inspection of this representation tells us that

$$\inf\{t > 0 : L_t < 0\} = \int_0^{\zeta} Z_t \mathrm{d}t$$

- The law of τ_0^- when $Z_0 = x$ is also that of

$$\tau_x^+ := \inf\{t > 0 : -L_t > x\},$$

and the process $\{\tau_x^+ : x \geq 0\}$ is a subordinator.

Infact.....

Example

There exists a subordinator Y with Laplace exponent is given by

$$\Phi_Y(z) = -cW_{-1}\left(-\frac{1}{\theta c}\exp\left(-\frac{1+\theta z}{\theta c}\right)\right) + cW_{-1}\left(-\frac{1}{\theta c}\exp\left(-\frac{1}{\theta c}\right)\right) - z, \qquad z \geq 0,$$

with no drift and no killing, where W_{-1} is the branch of the Lambert W-function (a solution to $we^w = z$) that is a decreasing and maps $[-1/e, 0)$ onto $(-\infty, -1]$ and $c, \theta > 0$ are parametric constants.
The transition probability density of Y is

$$p_Y(t, y) = \frac{c\theta^{-1}t}{\Gamma(1 + c(t + y))}\left(\frac{y}{\theta}\right)^{c(t+y)-1} \mathrm{e}^{-\frac{y}{\theta} + at}, \qquad y, t > 0,$$

where $a := 0$ if $\theta c \leq 1$ and $a := -1/\theta - cW_{-1}\left(-\frac{1}{\theta c}\mathrm{e}^{-\frac{1}{\theta c}}\right)$ if $\theta c > 1$. The density of the Lévy measure is given by

$$\pi_Y(y) = \frac{c\theta^{-1}}{\Gamma(1 + cy)}\left(\frac{y}{\theta}\right)^{cy-1} \mathrm{e}^{-\frac{y}{\theta}}, \qquad y > 0.$$

A note on Lambert functions

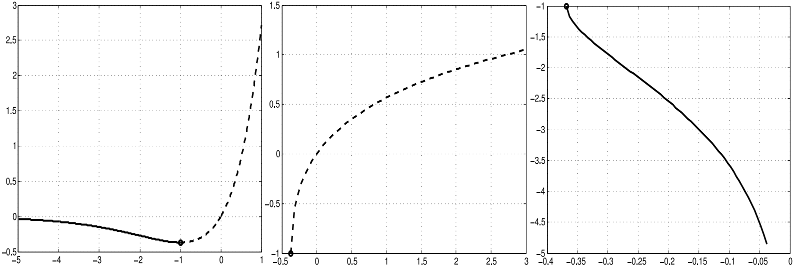

Figure : The two real branches of the Lambert W-function: $W_0(z)$ is an increasing function which maps $[-1/e, \infty)$ onto $[-1, \infty)$, and $W_{-1}(z)$ is a decreasing function which maps $[-1/e, 0)$ onto $(-\infty, -1]$.

In the end it's trivially Kendall's identity!

- Suppose that ξ is a spectrally negative Lévy process and suppose further that it has a transition density: $\mathbb{P}(\xi_t \in \mathrm{d}x) = p_\xi(x,t)\mathrm{d}x$.
- Then Kendall's identity states that

$$\mathbb{P}(\tau_x^+ \in \mathrm{d}t) = \frac{x}{t} p_\xi(t,x)\mathrm{d}t, \qquad x,t>0,$$

How to produce this example (and others)

- Start with a subordinator X with no drift and no killing and Laplace exponent

$$\Phi_X(q) = -\log \mathbb{E}[e^{-qX_1}] = \int_{(0,\infty)} (1 - e^{-qx})\Pi_X(dx), \qquad q \geq 0.$$

- Define $\xi_t = t - X_t$, $t \geq 0$ and let $Y_x = \inf\{t > 0 : \xi_t > x\}$.
- It is well know that $\tau_x^+ = \inf\{t > 0 : \xi_t > x\}$ has Laplace exponent, say $\phi(q)$, which is the root of the equation

$$\psi_\xi(z) = z - \Phi_X(z) = q.$$

and which necessarily takes the form

$$\phi(q) = \kappa + q + \int_{(0,\infty)} (1 - e^{-qx})\Pi_Y(dx), \qquad q \geq 0.$$

- Define the subordinator Y to have Laplace exponent $\Phi_Y(q) := \phi(q) - q - \kappa$ (strip off killing and drift).
- Use Kendall's identity to tell you more about ϕ.

How to produce this example (and others)

Lemma

If the transition semi-group of X is absolutely continuous with respect to Lebesgue measure (written $p_X(t,x)$), then the transition semi-group of Y is given by

$$p_Y(t,y) = \frac{t}{t+y} e^{\kappa t} p_X\left(t+y, y\right), \quad y > 0.$$

and the Levy measure of Y is given by

$$\Pi_Y(\mathrm{d}y) = \frac{1}{y} p_X(y,y)\mathrm{d}y, \quad y > 0.$$

The first part is from Kendall's identity, the second part is a consequence of the fact that (weakly)

$$\Pi_Y(\mathrm{d}x) = \lim_{t \downarrow 0} \frac{1}{t} \mathbb{P}(Y_t \in \mathrm{d}x).$$

The previous example is the result of taking $\xi_t = t - G_t$, where G is a Gamma subordinator with exponent $\Phi_X(q) = c \ln(1 + \theta q)$, $q \geq 0$.

Example 2

Example (X is a Poisson process)

For $c > 0$ there exists a subordinator Y with Laplace exponent

$$\Phi_Y(z) = W_0\left(-ce^{-c-z}\right) - W_0\left(-ce^{-c}\right), \qquad z \geq 0,$$

where W_0 is the principle branch of the Lambert-W function. The process Y is a compound Poisson process. The distribution of Y_t is supported on $\{0, 1, 2, \cdots\}$ and is given by

$$\mathbb{P}(Y_t = n) = ct\frac{(c(n+t))^{n-1}}{n!}\mathrm{e}^{-c(n+t)+at}, \quad n \geq 0,$$

where $a := 0$ if $c \leq 1$ and $a := c + W_0\left(-ce^{-c}\right)$ if $c > 1$. The Lévy measure is given by

$$\Pi_Y(\{n\}) = \frac{n^{n-1}}{n!}c^n\mathrm{e}^{-cn}, \quad n \geq 1.$$

This is a subordinator whose distribution at time t is that of a generalised Poisson distribution (a.k.a. Borel distribution). This example is generated by taking $\xi_t = t - N_{ct}$, where N is a Poisson process.

Example 3

Example (X is a stable subordinator)

Assume that $\alpha \in (0,1)$ and $c > 0$. For $q \geq 0$ define $\phi(q)$, $q \geq 0$, as the unique positive solution to the equation $z - cz^\alpha = q$. Then the function

$$\Phi_Y(z) = \phi(z) - c^{\frac{1}{1-\alpha}} - z$$

is the Laplace exponent of a subordinator. The transition probability density of the subordinator Y is given by

$$p_Y(t,y) = t \exp\left(c^{\frac{1}{1-\alpha}} t\right) \frac{(c(t+y))^{-\frac{1}{\alpha}}}{t+y} g\left(y(c(t+y))^{-\frac{1}{\alpha}}; \alpha\right) \qquad x, t > 0,$$

where $g(x;\alpha)$ is the density of an $\mathbb{R}_+$-valued stable random variable with index α. The density of the Lévy measure is given by

$$\pi_Y(y) = c^{-\frac{1}{\alpha}} y^{-\frac{1}{\alpha}-1} g\left(c^{-\frac{1}{\alpha}} y^{1-\frac{1}{\alpha}}; \alpha\right), \qquad y > 0.$$

Another example is possible here by taking

$$\psi_\xi(z) = z + z^\alpha, \qquad z \geq 0$$

for $\alpha \in (1,2)$.

Example 4

Example (X is a Bessel subordinator)

For $q > 0$ define $\phi(q)$ as the unique solution to the equation

$$z - c \ln \left(1 + \theta z + \sqrt{(1+\theta z)^2 - 1} \right) = q.$$

Then the function $\Phi_Y(z) = \phi(z) - \phi(0) - z$ is the Laplace exponent of a finite mean subordinator. The transition probability density of the subordinator Y is given by

$$p_Y(t, y) = cty^{-1} e^{\phi(0)t - \frac{y}{\theta}} I_{c(t+y)} \left(\frac{y}{\theta} \right),$$

where $I_\nu(x)$ denotes the modified Bessel function of the first kind. The density of the Lévy measure is given by

$$\pi_Y(y) = cy^{-1} e^{-\frac{y}{\theta}} I_{cy} \left(\frac{y}{\theta} \right).$$

Example 5

Example (X is a Geometric stable subordinator)

Assume that $c > 0$, $\theta > 0$ and $\alpha \in (0,1)$. For $q > 0$ define $\phi(q)$ as the unique solution to the equation

$$z - c \ln\left(1 + (\theta z)^{\alpha}\right) = q.$$

Then the function $\Phi_Y(z) = \phi(z) - \phi(0) - z$ is the Laplace exponent of a finite mean subordinator. The transition probability density of the subordinator Y is given by

$$p_Y(t,y) = e^{\phi(0)t} \frac{\alpha c t}{y} \sum_{k \geq 0} \frac{(-1)^k (1 + c(t+y))_k}{\Gamma(1 + \alpha(c(t+y)+k)) k!} \left(\frac{y}{\theta}\right)^{\alpha(c(t+y)+k)}, \qquad y, t > 0,$$

where $(a)_k := a(a+1)\cdots(a+k-1)$ denotes the Pocchammer symbol. The density of the Lévy measure is given by

$$\pi_Y(y) = \frac{\alpha c}{y} \sum_{k \geq 0} \frac{(-1)^k (1 + cy)_k}{\Gamma(1 + \alpha(cy+k)) k!} \left(\frac{y}{\theta}\right)^{\alpha(cy+k)}, \qquad y > 0.$$

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Quiz time

- Q: What happens if we feed in $\xi_t = t - (\text{inverse Gaussian})_t$?
- A: An inverse Gaussian comes out for Y.
- Q: Apart from the inverse Guassian example above, what happens if you build subordinator Y_t from $\xi_t := t - X_t$, then build Y^*_t from $\xi^*_t := t - Y_t$ etc etc?
- A: Nothing. Y^* will essentially be a version Y generated by taking $c_1 \xi_{c_2 t}$ for some constants $c_1, c_2 > 0$.

Complete monotonicity I

A result that seems not to be known:

Theorem

If ξ has jump measure which has completely monotone density, then $\{\tau_x^+ : x \geq 0\}$ (and hence Y) has jump measure which has completely monotone density.

In general the Wiener-Hopf factorisation gives

$$\psi_\xi(z) = (z - c)\phi_H(z), \qquad z \geq 0,$$

where $\phi_H(z)$ is the Laplace exponent of the subordinator characterising the descending ladder height processes.
The assumption on $\pi(-x)$ implies that ϕ_H has a Lévy density which is completely monotone.
Esscher transform away the killing in the ascending ladder process:

$$\psi_\xi(z + c) = z\phi_H(z + c).$$

Note that $\phi_H(z + c)$ still has a completely monotone Lévy density. Look in operator theory literature and discover that the inverse of $z\times$(complete Bernstein) is again a complete Bernsein. Relate back to $\phi = \psi_\xi^{-1}$.

Complete monotonicity II

Another result that seems not to be known:

Theorem

If $\xi_t = t - X_t$ and X belongs to the Thorin class of subordinators ($x\pi_X(x)$ is completely monotone) then so does $\{\tau_x^+ : x \geq 0\}$ (and hence so does Y).

A variant of the previous proof but making more use of the representation of complete Bernstein functions through Pick functions.

Strange analytical conclusions

Challenge: These facts imply that the following functions are completely monotone

$$
\begin{aligned}
f_1(y) &= \frac{y^{cy}\mathrm{e}^{-y}}{\Gamma(1+cy)}, \quad c>0,\ y>0, \\
f_2(y) &= y^{-\frac{1}{\alpha}} g(y^{1-\frac{1}{\alpha}};\alpha), \quad \alpha \in (0,1),\ y>0, \\
f_3(y) &= \mathrm{e}^{-y} I_{cy}(y), \quad c>0,\ y>0, \\
f_4(y) &= \sum_{k\geq 0} \frac{(-1)^k (1+cy)_k}{\Gamma(1+\alpha(cy+k))k!} y^{\alpha(cy+k)}, \quad c>0,\ \alpha\in(0,1),\ y>0,
\end{aligned}
$$

Can you prove it directly??

Kendall's identity is doing something analytical

The Laplace transform of the found transition densities is gives the Laplace exponent. This can also be reinterpreted as identifying some new integral identities for special functions. For example:

$$\left(\frac{W_{-1}(-t)}{-t}\right)^r = \mathrm{e}^{-rW_{-1}(-t)} = -\int_{-r}^{\infty} r\frac{(w+r)^{w-1}}{\Gamma(1+w)}t^w \mathrm{d}w,$$

and

$$g(x;\tfrac{3}{2}) = x^{-\frac{5}{2}}g(x^{-\frac{3}{2}};\tfrac{2}{3}) = \frac{1}{\sqrt{3\pi}x}\mathrm{e}^{-\frac{2}{27}x^3}W_{\frac{1}{2},\frac{1}{6}}\left(\frac{4}{27}x^3\right),$$

where $W_{a,b}$ is a Whittaker function.