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EXACT AND ASYMPTOTIC *n*-TUPLE LAWS AT FIRST AND LAST PASSAGE[1]

BY A. E. KYPRIANOU, J. C. PARDO AND V. RIVERO

University of Bath, University of Bath and Centro de Investigación en Matemáticas, A.C.

Understanding the space–time features of how a Lévy process crosses a constant barrier for the first time, and indeed the last time, is a problem which is central to many models in applied probability such as queueing theory, financial and actuarial mathematics, optimal stopping problems, the theory of branching processes, to name but a few. In Doney and Kyprianou [*Ann. Appl. Probab.* **16** (2006) 91–106] a new quintuple law was established for a general Lévy process at first passage below a fixed level. In this article we use the quintuple law to establish a family of related joint laws, which we call n-tuple laws, for Lévy processes, Lévy processes conditioned to stay positive and positive self-similar Markov processes at both first and last passage over a fixed level. Here the integer n typically ranges from three to seven. Moreover, we look at asymptotic overshoot and undershoot distributions and relate them to overshoot and undershoot distributions of positive self-similar Markov processes issued from the origin. Although the relation between the n-tuple laws for Lévy processes and positive self-similar Markov processes are straightforward thanks to the Lamperti transformation, by interplaying the role of a (conditioned) stable processes as both a (conditioned) Lévy processes and a positive self-similar Markov processes, we obtain a suite of completely explicit first and last passage identities for so-called Lamperti-stable Lévy processes. This leads further to the introduction of a more general family of Lévy processes which we call hypergeometric Lévy processes, for which similar explicit identities may be considered.

1. Introduction. This paper concerns the joint laws of overshoots and undershoots of Lévy processes at first and last upward passage times of a constant boundary leading to new general and explicit identities. We will therefore begin by introducing some necessary but standard notation.

In the sequel $X = \{X_t : t \geq 0\}$ will always denote a Lévy process defined on the filtered space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where the filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ is assumed to satisfy the usual assumptions of right continuity and completion. Its characteristic exponent will be given by $\Psi(\theta) := -\log \mathbb{E}(e^{i\theta X_1})$ and its jump measure by Π_X.

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Associated to the Lévy measure Π_X we define the left and right tail, $\overline{\Pi}_X^-$ and $\overline{\Pi}_X^+$, respectively, as follows:

$$\overline{\Pi}_X^-(x) = \Pi_X(-\infty, -x), \qquad \overline{\Pi}_X^+(x) = \Pi_X(x, \infty), \qquad x > 0.$$

We will work with the probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$ such that $\mathbb{P}_x(X_0 = x) = 1$ and $\mathbb{P}_0 = \mathbb{P}$. The probabilities $\{\widehat{\mathbb{P}}_x : x \in \mathbb{R}\}$ will be defined in a similar sense for the dual process, $-X$.

Denote by $\{(L_t^{-1}, H_t) : t \geq 0\}$ and $\{(\widehat{L}_t^{-1}, \widehat{H}_t) : t \geq 0\}$ the (possibly killed) bivariate subordinators representing the ascending and descending ladder processes. Write $\kappa(\alpha, \beta)$ and $\widehat{\kappa}(\alpha, \beta)$ for their joint Laplace exponents for $\alpha, \beta \geq 0$. For convenience we will write

$$\kappa(0, \beta) = q + \xi(\beta) = q + \mathrm{c}\beta + \int_{(0,\infty)} (1 - e^{-\beta x}) \Pi_H(dx),$$

where $q \geq 0$ is the killing rate of H so that $q > 0$ if and only if $\lim_{t\uparrow\infty} X_t = -\infty$, $\mathrm{c} \geq 0$ is the drift of H and Π_H is its jump measure. Similarly to Π_X we shall define $\overline{\Pi}_H(x) = \Pi_H(x, \infty)$. The quantity ξ is the Laplace exponent of a true subordinator. Similar notation will also be used for $\widehat{\kappa}(0, \beta)$ by replacing q, ξ, c and Π_H by $\widehat{q}$, $\widehat{\xi}$, $\widehat{\mathrm{c}}$ and $\Pi_{\widehat{H}}$. Note that necessarily $q\widehat{q} = 0$.

Associated with the ascending and descending ladder processes are the bivariate renewal functions V and $\widehat{V}$. The former is defined by

$$V(ds, dx) = \int_0^\infty dt \cdot \mathbb{P}(L_t^{-1} \in ds, H_t \in dx)$$

and taking double Laplace transforms shows that

$$\int_0^\infty \int_0^\infty e^{-\alpha s - \beta x} V(dx, ds) = \frac{1}{\kappa(\alpha, \beta)} \qquad \text{for } \alpha, \beta \geq 0, \tag{1.1}$$

with a similar definition and relation holding for $\widehat{V}$. These bivariate renewal measures are essentially the Green's measures of the ascending and descending ladder processes. With an abuse of notation we shall also write $V(dx)$ and $\widehat{V}(dx)$ for the marginal measures $V([0, \infty), dx)$ and $\widehat{V}([0, \infty), dx)$, respectively. (Since we shall never use the marginals $V(ds, [0, \infty))$ and $\widehat{V}(ds, [0, \infty))$ there should be no confusion.) Note that local time at the maximum is defined only up to a multiplicative constant. For this reason, the exponent κ can only be defined up to a multiplicative constant and hence the same is true of the measure V (and then obviously this argument applies to $\widehat{V}$).

Let

$$\overline{X}_t := \sup_{u \leq t} X_u$$

and define for each $x \in \mathbb{R}$,

$$\tau_x^+ = \inf\{t > 0 : X_t > x\} \quad \text{and} \quad \overline{G}_t = \sup\{s < t : X_s = \overline{X}_s\}.$$

A new identity was given in [16] for general Lévy processes which specifies at first passage over a fixed level the quintuple law of: the time of first passage relative to the time of the last maximum at first passage, the time of the last maximum at first passage, the overshoot at first passage, the undershoot at first passage and the undershoot of the last maximum at first passage. For sake of reference, the quintuple law is reproduced below.

THEOREM 1 (Doney and Kyprianou [16]). *Suppose that X is not a compound Poisson process. Then for each $x > 0$ we have on $u > 0$, $v \geq y$, $y \in [0, x]$, $s, t \geq 0$,*

$$\begin{aligned}\mathbb{P}(\tau_x^+ - \overline{G}_{\tau_x^+ -} \in dt, \overline{G}_{\tau_x^+ -} \in ds, X_{\tau_x^+} - x \in du, x - X_{\tau_x^+ -} \in dv, x - \overline{X}_{\tau_x^+ -} \in dy)\\ = V(ds, x - dy)\widehat{V}(dt, dv - y)\Pi_X(du + v),\end{aligned}$$

where the equality holds up to a multiplicative constant.

Many of the results in this paper will follow as a consequence, either as an application or by similar reasoning, of the above quintuple law. As mentioned earlier, we shall concentrate not only on the case of first passage above a fixed level, but also last passage above a fixed level. Additionally, limiting cases of such laws will also be on the agenda. Moreover, our reasoning permits us to deal with more than just Lévy processes and we consider also Lévy processes conditioned to stay positive as well as positive self-similar Markov processes. In all of the cases we consider, depending on the setting, it will be possible to produce joint laws of anywhere between three to seven variables associated with the passage problem. We therefore collectively refer to our results as the n-tuple laws.

The principal motivation for this work is the wide variety of applications that are connected to the first and last passage problem. In the theory of actuarial mathematics, the first passage problem is of fundamental interest with regard to the classical ruin problem and typically takes the form of the so-called expected discounted penalty function. The latter is also known in the actuarial community as the Gerber–Shiu function following the first article [19] of a series which has appeared in the actuarial literature. Recent literature, for example, [12], also cites interest in the last passage problem within the context of ruin problems. In the setting of financial mathematics, the first passage problem is of interest in the pricing of barrier options in markets driven by Lévy processes. In queueing theory, passage problems for Lévy processes play a central role in understanding the trajectory of the workload during busy periods as well as in relation to buffers. Many optimal stopping strategies also turn out to boil down to first passage problems; a classical example of which being McKean's optimal stopping problem [26]. It is not our purpose, however, to dwell on these applications as there is already much to say about the first and last passage problems as self-contained problems.

Let us conclude this section by outlining the remaining presentation of the paper. In the next section we present a family of three new quintuple laws. First, a quintuple law of a general class of Lévy processes conditioned to stay positive and issued from the origin which concerns overshoots and undershoots at last passage above a level $x > 0$. This quintuple law will follow from Theorem 1 as a natural consequence of the Tanaka path decomposition. Note that the latter originates from the theory of conditioned random walks, but thanks to [14] a version of the path decomposition is also available for Lévy processes conditioned to stay positive. The aforementioned quintuple law at last passage may then be used to construct two further septuple laws at last passage. One for a Lévy process conditioned to stay positive when issued from a positive position, and a second one for a Lévy process without conditioning. In Section 3 we turn return to a family of results concerning asymptotic overshoot–undershoot triple laws at first passage which improve on recent contributions in the literature. The improvements lie with the increased number of variables in the joint laws as well as, in some cases, the possibility of negative jumps in addition to positive jumps. These are then used to establish asymptotic overshoot–undershoot triple laws at last passage for Lévy processes and Lévy processes conditioned to stay positive. Proofs are given in Section 4. Next, in Section 5 we use ideas behind asymptotic overshoot–undershoot triple laws to examine the stationary nature of overshoot–undershoot triple laws for positive self similar Markov processes issued from the origin. Finally in Section 6 we consider some examples where the previously appearing identities become more explicit. Moreover we play off some of the results from the previous sections against one another and conclude with some explicit identities for Stable and Lamperti-stable processes.

2. Quintuple and septuple laws at last passage. We start with our first result which describes the quintuple law at last passage for a Lévy process conditioned to stay positive and issued from the origin. Henceforth we shall denote by $\mathbb{P}^{\uparrow}$ the law of $(X, \mathbb{P})$ conditioned to stay positive. This law can be constructed in several ways; see, for example, [8, 15]. However we will be specifically interested in Tanaka's construction of the law $\mathbb{P}^{\uparrow}$ as described in [14]. In the latter construction, which is valid for Lévy processes which do not drift to $-\infty$ and for which 0 is regular for $(-\infty, 0)$, the excursions from 0 of process $(X, \mathbb{P}^{\uparrow})$ reflected at its future infimum are those of $(X, \mathbb{P})$ reflected at its past supremum and time reversed. Moreover the closure of the set of zeros of the latter equals that of the former.

Let

$$\underrightarrow{X}_t = \inf\{X_s : s \geq t\}$$

be the future infimum of X,

$$\underrightarrow{G}_t = \sup\{s < t : X_s - \underrightarrow{X}_s = 0\}, \qquad \underrightarrow{D}_t = \inf\{s > t : X_s - \underrightarrow{X}_t = 0\}$$

are the left and right end points of the excursion of X from its future infimum straddling time t. Now define the last passage time

$$U_x = \sup\{s \geq 0 : X_t \leq x\}$$

and observe that U_x can be the left or right extrema of an excursion interval of the process conditioned to stay positive reflected at its future infimum. However, if x does not coincide with a point in $\{\underrightarrow{X}_t, t \geq 0\}^{\mathrm{cl}}$ then U_x corresponds to the left extrema of an excursion; in particular $U_x = \underrightarrow{G}_{U_x}$.

The quintuple law at last passage for Lévy processes conditioned to stay positive and issued from the origin reads as follows.

THEOREM 2. *Suppose that X is a Lévy process which does not drift to $-\infty$ and for which* 0 *is regular for* $(-\infty, 0)$. *For* $s, t \geq 0$, $0 < y \leq x$, $w \geq u > 0$,

$$\begin{aligned}
&\mathbb{P}^{\uparrow}(\underrightarrow{D}_{U_x} - U_x \in dt, U_x \in ds, \underrightarrow{X}_{U_x} - x \in du, x - X_{U_x-} \in dy, X_{U_x} - x \in dw) \\
&\qquad = V(ds, x - dy)\widehat{V}(dt, w - du)\Pi_X(dw + y),
\end{aligned}$$

where the equality hold up to a multiplicative constant.

PROOF. Suppose that $F : \mathbb{R}^5 \to \mathbb{R}^+$ is a measurable and bounded function such that $F(\cdot, \cdot, \cdot, \cdot, 0) = 0$. Thanks to Tanaka's path decomposition we may identify $\overline{G}_{\tau_x^+-} = U_x$, $\underrightarrow{D}_{U_x} = \tau_x^+$, $\underrightarrow{X}_{U_x} = X_{\tau_x^+}$, $X_{U_x-} = \overline{X}_{\tau_x^+-}$ and $X_{U_x} = \overline{X}_{\tau_x^+-} + X_{\tau_x^+} - X_{\tau_x^+-}$. Hence we may write directly the following identity:

$$\begin{aligned}
&\mathbb{E}^{\uparrow}\big(F(\underrightarrow{D}_{U_x} - U_x, U_x, \underrightarrow{X}_{U_x} - x, x - X_{U_x-}, X_{U_x} - x)\big) \\
&\quad = \mathbb{E}\big(F(\tau_x^+ - \overline{G}_{\tau_x^+-}, \overline{G}_{\tau_x^+-}, X_{\tau_x^+} - x, \\
&\qquad\qquad x - \overline{X}_{\tau_x^+-}, X_{\tau_x^+} - X_{\tau_x^+-} + \overline{X}_{\tau_x^+-} - x)\big) \\
&\quad = \mathbb{E}\big(F\big(\tau_x^+ - \overline{G}_{\tau_x^+-}, \overline{G}_{\tau_x^+-}, X_{\tau_x^+} - x, \\
&\qquad\qquad x - \overline{X}_{\tau_x^+-}, (X_{\tau_x^+} - x) + (x - X_{\tau_x^+-}) - (x - \overline{X}_{\tau_x^+-})\big)\big) \\
&\quad = \int_{(0,\infty)^5} V(ds, x - dy)\widehat{V}(dt, dv - y)\Pi_X(du + v) \\
&\qquad\qquad \times F(s, t, u, y, u + v - y)\mathbf{1}_{\{y \leq x \wedge v\}}.
\end{aligned}$$

The result follows by a change of variables $w = u + v - y$ in the above integral. Note in particular the assumption on F allows us to exclude from the expectation considerations corresponding to the Lévy process creeping upward; equivalently that $x \in \{\underrightarrow{X}_t, t \geq 0\}^{\mathrm{cl}}$. $\square$

As a consequence of the quintuple law in Theorem 2 we obtain the shortly following two corollaries which specify septuple laws at last passage for Lévy processes and for Lévy processes conditioned to stay positive when issued from a positive position. In both corollaries we use the notation

$$\underline{G}_t = \sup\{s < t : X_s - \underline{X}_s = 0\},$$

where

$$\underline{X}_t := \inf_{u \le t} X_u.$$

We also write $\mathbb{P}_z^{\uparrow}$ for the law of X conditioned to stay positive when issued from $z > 0$. It is known that the latter satisfies, for example, $\mathbb{P}_z^{\uparrow}(X_t \in dx) = \widehat{V}(z)^{-1}\widehat{V}(x)\mathbb{P}(X_t \in dx, \underline{X}_t > 0)$ where $x > 0$. Moreover, in the sense of weak convergence with respect to the Skorohod topology, $\lim_{z\downarrow 0}\mathbb{P}_z^{\uparrow} = \mathbb{P}^{\uparrow}$ when 0 is regular for $(0,\infty)$. See Chaumont and Doney [8] for full details.

COROLLARY 1. *Suppose that X is a Lévy process which does not drift to $-\infty$ and for which 0 is regular for $(-\infty, 0)$ as well as for $(0,\infty)$. For $t, x, z > 0, s > r > 0, 0 \le v \le z \wedge x, 0 < y \le x - v, w \ge u > 0$,*

$$\begin{aligned}\mathbb{P}_z^{\uparrow}(&\underline{G}_\infty \in dr, \underline{X}_\infty \in dv, \underrightarrow{D}_{U_x} - U_x \in dt, U_x \in ds,\\ &\underrightarrow{X}_{U_x} - x \in du, x - X_{U_x-} \in dy, X_{U_x} - x \in dw)\\ &= \widehat{V}(z)^{-1}\widehat{V}(dr, z - dv)V(ds - r, x - v - dy)\widehat{V}(dt, w - du)\Pi_X(dw + y),\end{aligned}$$

where the equality holds up to a multiplicative constant. Moreover, in the particular case that $z > x$

$$\mathbb{P}_z^{\uparrow}(U_x = 0, \underline{G}_\infty \in dr, \underline{X}_\infty \in dv) = \widehat{V}(z)^{-1}\widehat{V}(dr, z - dv)$$

for $r > 0$ and $v \in [x, z]$.

PROOF. The first part of the corollary is a direct consequence of Millar's result for splitting a Markov process at its infimum; cf. [27, 28]. Indeed, according to the latter, the postinfimum process is independent of the preinfimum process and, relative to the given space time point $(\underline{G}_\infty, \underline{X}_\infty)$ the postinfimum process has the law of $\mathbb{P}^{\uparrow}$. We should note that in Millar's description of the postinfimum process, the assumption that 0 is regular for $(0,\infty)$ means in particular that the process X is right continuous at times which belong to the set $\{t > 0 : X_t = \underline{X}_t\}$.

To compute the joint law of $(\underline{G}_\infty, \underline{X}_\infty)$, and thus complete the proof of the first part of the corollary, let $\mathbf{e}_q$ be an independent random variable which is exponentially distributed with rate $q > 0$. With the help of the compensation formula for the excursions of X away from $\underline{X}$ we have that for $r > 0$ and $v \in [0, z]$,

$$\begin{aligned}&\mathbb{P}_z^{\uparrow}(\underline{G}_\infty \in dr, \underline{X}_\infty \in dv)\\ &\quad = \lim_{q\downarrow 0}\widehat{V}(z)^{-1}\mathbb{E}_z\big(\mathbf{1}_{\{\underline{G}_{\mathbf{e}_q}\in dr, \underline{X}_{\mathbf{e}_q}\in dv\}}\widehat{V}(X_{\mathbf{e}_q})\mathbf{1}_{\{\underline{X}_{\mathbf{e}_q}\ge 0\}}\big) \qquad (2.1)\\ &\quad = \widehat{V}(z)^{-1}\lim_{q\downarrow 0}\mathbb{E}\big(\mathbf{1}_{\{\underline{G}_{\mathbf{e}_q}\in dr, z+\underline{X}_{\mathbf{e}_q}\in dv\}}\widehat{V}(v + X_{\mathbf{e}_q} - \underline{X}_{\mathbf{e}_q})\big).\end{aligned}$$

When X drifts to $+\infty$ the right-hand side above is well defined as $\widehat{V}(\infty) < \infty$ and is equal to

$$\widehat{V}(\infty)\widehat{V}^{-1}(z)\mathbb{P}(\underline{G}_\infty \in dr, z + \underline{X}_\infty \in dv) = \widehat{V}^{-1}(z)\widehat{V}(dr, z - dv).$$

Note that the last equality is consequence of the fact that the negative Wiener–Hopf factor takes the form

$$\mathbb{E}(e^{-\alpha \underline{G}_\infty - \beta \underline{X}_\infty}) = \frac{\widehat{\kappa}(0,0)}{\widehat{\kappa}(\alpha,\beta)}$$

the Laplace transform (1.1) and that $\widehat{V}(\infty) = 1/\widehat{\kappa}(0,0)$.

Henceforth we assume that X oscillates. Note that

$$\begin{aligned}
&\mathbb{E}\big(\mathbf{1}_{\{\underline{G}_{\mathbf{e}_q} \in dr, z + \underline{X}_{\mathbf{e}_q} \in dv\}} \widehat{V}(v + X_{\mathbf{e}_q} - \underline{X}_{\mathbf{e}_q})\big) \\
&\quad = \mathbb{E}\int_0^\infty q e^{-qt} \mathbf{1}_{\{\underline{G}_t \in dr, z + \underline{X}_t \in dv\}} \mathbf{1}_{\{\underline{X}_t = X_t\}} \widehat{V}(v)\, dt \qquad (2.2)\\
&\qquad + \mathbb{E}\sum_g \mathbf{1}_{\{\underline{G}_{g-} \in dr, z + \underline{X}_{g-} \in dv\}} \int_g^{d_g} q e^{-qt} \widehat{V}\big(v + \varepsilon_g(t)\big)\, dt,
\end{aligned}$$

where the sum is taken over all left end points, g, of excursions of X from its infimum $\underline{X}$, with corresponding excursion and right end point denoted by ε_g and d_g, respectively. Suppose that we call the two terms on the right-hand side of (2.2) A_q and B_q. Recalling that X oscillates, we have

$$\lim_{q\downarrow 0} A_q \le \lim_{q\downarrow 0} \widehat{V}(v)\mathbb{P}(\underline{G}_{\mathbf{e}_q} \in dr, z + \underline{X}_{\mathbf{e}_q} \in dv) = 0.$$

Appealing to the compensation formula for excursions we have

$$B_q = \mathbb{E}\bigg(\int_0^\infty \mathbf{1}_{\{\underline{G}_{s-} \in dr, z + \underline{X}_{s-} \in dv\}} e^{-qs}\, d\widehat{L}_s\bigg)\underline{n}\bigg(\int_0^\zeta q e^{-qu} \widehat{V}\big(v + \varepsilon(u)\big)\, du\bigg),$$

where ε is the generic excursion with life time ζ and $\underline{n}$ is the associated excursion measure. After a change of variables $s \mapsto \widehat{L}_t^{-1}$, the first term on the right-hand side above has a limit as $q \downarrow 0$ equal to $\widehat{V}(dr, z - dv)$. The second term on the other hand converges to a constant as $q \downarrow 0$ as we shall now explain. Note that it may be written in the form $\underline{n}(\widehat{V}(v + \varepsilon(\mathbf{e}_q))\mathbf{1}_{\{\mathbf{e}_q < \zeta\}})$ which, on the one hand is lower bounded by $\underline{n}(\widehat{V}(\varepsilon(\mathbf{e}_q))\mathbf{1}_{\{\mathbf{e}_q < \zeta\}})$ and, on the other, is upper bounded by $\underline{n}(\widehat{V}(v)\mathbf{1}_{\{\mathbf{e}_q < \zeta\}}) + \underline{n}(\widehat{V}(\varepsilon(\mathbf{e}_q))\mathbf{1}_{\{\mathbf{e}_q < \zeta\}})$. The latter bounds are, respectively, thanks to the monotonicity and subadditivity of the renewal function $\widehat{V}$. It is known (cf. [8, 32]) that $\widehat{V}$ is harmonic in the sense that $\mathbb{E}(\widehat{V}(z + X_t)\mathbf{1}_{\{z + \underline{X}_t \ge 0\}}) = \widehat{V}(z)$. Appealing to the description of the excursion measure $\underline{n}$ in Theorem 3 in [7] we find that

$$\underline{n}\big(\widehat{V}(\varepsilon_t), t < \zeta\big) = \mathbb{E}^{\uparrow}(1).$$

This in turn implies that $\underline{n}(\widehat{V}(\varepsilon(\mathbf{e}_q))\mathbf{1}_{\{\mathbf{e}_q<\zeta\}}) = 1$. At the same time, since X oscillates we have that $\underline{n}(\zeta = \infty) = 0$ and hence $\lim_{q\downarrow 0} \underline{n}(V(v)\mathbf{1}_{\{\mathbf{e}_q<\zeta\}}) = 0$. It follows that $\lim_{q\downarrow 0} B_q$ is proportional to $\widehat{V}(dr, z - dv)$. Referring back to (2.1) and (2.2) this completes the proof of the first part of the corollary.

The proof of the second part of the corollary is a direct consequence of the joint law of $(\underline{G}_\infty, \underline{X}_\infty)$. $\square$

REMARK 1. It is worth noting that contained in the proof of the above corollary is a generalization to Chaumont's law of the global infimum of a Lèvy process conditioned to stay positive (cf. Theorem 1 of [8] for its most general form). Namely, that, under the conditions of the above corollary, for $r \geq 0$ and $0 \leq v \leq z$

$$\mathbb{P}_z^{\uparrow}(\underline{G}_\infty \in dr, \underline{X}_\infty \in dv) = \frac{\widehat{V}(dr, z - dv)}{\widehat{V}(z)}.$$

COROLLARY 2. *Suppose that X is a Lévy process which drifts to ∞ and for which 0 is regular for both $(-\infty, 0)$ and $(0, \infty)$. For $t, x, v > 0, s > r > 0, 0 \leq y < x + v, w \geq u > 0$,*

$$\begin{aligned}
&\mathbb{P}(\underline{G}_\infty \in dr, -\underline{X}_\infty \in dv, \underrightarrow{D}_{U_x} - U_x \in dt, U_x \in ds,\\
&\qquad \underrightarrow{X}_{U_x} - x \in du, x - X_{U_x-} \in dy, X_{U_x} - x \in dw)\\
&\qquad = \widehat{V}(\infty)^{-1}\widehat{V}(dr, dv)V(ds - r, x + v - dy)\widehat{V}(dt, w - du)\Pi_X(dw + y),
\end{aligned}$$

where the equality holds up to a multiplicative constant.

PROOF. The corollary is again a consequence of Millar's result for splitting a Lévy process at its infimum. Specifically, the preinfimum and postinfimum processes are independent conditionally on the value of $(\underline{G}_\infty, -\underline{X}_\infty)$ and relative to the latter space–time point, the law of the postinfimum process is $\mathbb{P}^{\uparrow}$. Moreover, a computation similar in the spirit to (but much easier than) the proof of the previous corollary shows that the law of the pair $(\underline{G}_\infty, -\underline{X}_\infty)$ is given by $\widehat{V}(dr, dv)$. $\square$

3. Asymptotic triple laws at first and last passage times. We begin this section by returning to asymptotic overshoot–undershoot laws of Lévy processes at first and last passage. Related work on the forthcoming results can be found in [22] and [30]. In both of the aforementioned articles, two-dimensional asymptotic overshoot–undershoot laws were obtained. Here we address the case of three-dimensional overshoot–undershoot laws with the help of the following key observation.

For notational convenience frequently in this section we will denote the undershoots and overshoots at the first passage above a barrier as follows:

$$\mathcal{U}_x = x - \overline{X}_{\tau_x^+-}, \qquad \mathcal{V}_x = x - X_{\tau_x^+-}, \qquad \mathcal{O}_x = X_{\tau_x^+} - x, \qquad x > 0.$$

LEMMA 1. *For $u \le x$, $v \ge u$, $w \ge 0$ we have*

$$\mathbb{P}(\mathcal{U}_x > u, \mathcal{V}_x > v, \mathcal{O}_x > w) = \mathbb{P}(\mathcal{V}_{x-u} > v - u, \mathcal{O}_{x-u} > w + u).$$

PROOF. By virtue of the fact that τ_x^+ is a first passage time recall that $x - \overline{X}_{\tau_x^+} \le \mathcal{V}_x$. On the event $\{\mathcal{U}_x > u, \mathcal{V}_x > v, \mathcal{O}_x > w\}$ the interval $[x-u, x+w]$ does not belong to the range of $\overline{X}$. This implies that $\mathcal{O}_{x-u} > u + w$ and $\mathcal{V}_{x-u} > v - u$. Conversely if the latter two inequalities hold, then we may again claim that the interval $[x-u, x+w]$ does not belong to the range of $\overline{X}$. Since $\mathcal{U}_{x-u} \in [0, \mathcal{V}_{x-u}]$ it follows that $\mathcal{U}_x > u, \mathcal{V}_x > v, \mathcal{O}_x > w$. The reader is encouraged to accompany the proof with a sketch at which point the proof becomes completely transparent. $\square$

The above lemma tells us that studying the law of the triple $(x - \overline{X}_{\tau_x^+ -}, x - X_{\tau_x^+ -}, X_{\tau_x^+} - x)$ is equivalent to studying the law of the pair $(x - X_{\tau_x^+ -}, X_{\tau_x^+} - x)$. This is a recurrent idea appearing in the proof of the theorems below. We will also make repeated use in the aforementioned proofs of an important identity obtained by Vigon [34] that relates Π_H, the Lévy measure of the upward ladder height subordinator H, with that of the Lévy process X and $\widehat{V}$, the potential measure of the downward ladder height subordinator $\widehat{H}$. Specifically, defining $\overline{\Pi}_H(x) = \Pi_H(x, \infty)$, the identity states that

$$\overline{\Pi}_H(r) = \int_0^\infty \widehat{V}(dl)\overline{\Pi}_X^+(l+r), \qquad r > 0. \tag{3.1}$$

THEOREM 3. *Let X be a Lévy process that does not drift to $-\infty$.*

(i) *Assume that the law of X_1 is not arithmetic. The triple $(x - \overline{X}_{\tau_x^+ -}, x - X_{\tau_x^+ -}, X_{\tau_x^+} - x)$ converges weakly as $x \to \infty$ toward a nondegenerate random variable if and only if $\mu_+ := \mathbf{E}(H_1) < \infty$. In this case the limit law is given by*

$$\lim_{x\to\infty} \mathbb{P}(x - \overline{X}_{\tau_x^+ -} \in du, x - X_{\tau_x^+ -} \in dv, X_{\tau_x^+} - x \in dw)$$
$$= \frac{1}{\mu_+} du \widehat{V}(dv - u)\Pi_X(dw + v)1_{\{v \ge u \ge 0, w > 0\}}.$$

In particular,

$$\lim_{x\to\infty} \mathbb{P}(x - \overline{X}_{\tau_x^+ -} > u, x - X_{\tau_x^+ -} > v, X_{\tau_x^+} - x > w)$$
$$= \frac{1}{\mu_+}\int_0^{v-u} dy \int_{[y,\infty)} \widehat{V}(dl)\overline{\Pi}_X^+(w + l + v - y) \tag{3.2}$$
$$+ \frac{1}{\mu_+}\int_v^\infty dy \overline{\Pi}_H(w + y),$$

where $0 \le u \le v$, $w \ge 0$.

(ii) *If there exists a nondecreasing function* $b:(0,\infty)\to(0,\infty)$ *such that* $X_t/b(t)$ *converges weakly , as* $t\to\infty$, *toward a strictly stable random variable with index* $\alpha\in(0,2)$, *and positivity parameter* $\rho\in(0,1)$, *then*

$$\lim_{x\to\infty}\mathbb{P}\biggl(\frac{x-\overline{X}_{\tau_x^+-}}{x}\in du,\frac{x-X_{\tau_x^+-}}{x}\in dv,\frac{X_{\tau_x^+}-x}{x}\in dw\biggr)$$
$$=\frac{\sin(\alpha\rho\pi)}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))}\frac{(1-u)^{\alpha\rho-1}(v-u)^{\alpha(1-\rho)-1}}{(v+w)^{1+\alpha}}\,du\,dv\,dw$$

for $0\le u\le 1$, $v\ge u$, *and* $w>0$.

(iii) *Assume that* X *oscillates and that the mean of* $\widehat{H}_1$ *is finite. Suppose moreover that* $\overline{\Pi}_X^+:=\Pi_X^+(x,\infty)$ *is regularly varying at* ∞ *with index* $-1-\alpha$ *for some* $\alpha\in(0,1)$. *Then*

$$\lim_{x\to\infty}\mathbb{P}\biggl(\frac{x-\overline{X}_{\tau_x^+-}}{x}\in du,\frac{x-X_{\tau_x^+-}}{x}\in dv,\frac{X_{\tau_x^+}-x}{x}\in dw\biggr)$$
$$=\frac{\alpha(1+\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{1}{(1-u)^{1-\alpha}(v+w)^{2+\alpha}}\,du\,dv\,dw$$

for $0<u<1$, $v\ge u$ *and* $w>0$.

(iv) *Assume that* X *drifts to* ∞ *and that* $\overline{\Pi}_X^+$ *is regularly varying at* ∞ *with index* $-\alpha$ *for some* $\alpha\in(0,1)$. *Then*

$$\lim_{x\to\infty}\mathbb{P}\biggl(\frac{x-X_{\tau_x^+-}}{x}\in dv,\frac{X_{\tau_x^+}-x}{x}\in dw\biggr)$$
$$=\frac{\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{1}{(1-v)^{1-\alpha}(v+w)^{\alpha+1}}\,dv\,dw$$

for $w>0$ *and* $0<v<1$. *Furthermore*,

$$\frac{\overline{X}_{\tau_x^+-}-X_{\tau_x^+-}}{x}=\frac{x-X_{\tau_x^+-}}{x}-\frac{x-\overline{X}_{\tau_x^+-}}{x}\xrightarrow[x\to\infty]{\mathbb{P}}0.$$

REMARK 2. It is important to mention that the assumptions in Theorem 3 can be verified using only the characteristics of the underlying Lévy process X. According to a result due to Chow [13] necessary and sufficient conditions on X to be such that $\mathbf{E}(\widehat{H}_1)<\infty$, are either $0<\mathbf{E}(-X_1)\le\mathbf{E}|X_1|<\infty$ or $0=\mathbf{E}(-X_1)<\mathbf{E}|X_1|<\infty$ and

$$\int_{[1,\infty)}\biggl(\frac{x\overline{\Pi}_X^-(x)}{1+\int_0^x dy\int_y^\infty\overline{\Pi}_X^+(z)\,dz}\biggr)dx<\infty$$

with $\overline{\Pi}_X^-(x)=\Pi_X(-\infty,-x)$, $x>0$.

Observe that under such assumptions the Lévy process X does not drift to ∞, that is, $\liminf_{t\to\infty} X_t = -\infty$, $\mathbf{P}$-a.s. Kesten and Erickson's criteria state that X drift to ∞ if and only if

$$\int_{(-\infty,-1)} \left(\frac{|y|}{\overline{\Pi}_X^+(1) + \int_1^{|y|} \overline{\Pi}_X^+(z)\,dz} \right) \Pi_X(dy) < \infty = \int_1^\infty \overline{\Pi}_X^+(x)\,dx$$

or

$$0 < \mathbf{E}(X_1) \le \mathbf{E}|X_1| < \infty;$$

cf. [21] and [18]. (In fact, Chow, Kesten and Erickson proved the results above for random walks, its translation for real valued Lévy processes can be found in [17] and [33].) Moreover, a sufficient condition in terms of the tail Lévy measure of X for the hypothesis in (ii) in Theorem 3 to be satisfied can be found in Lemma 5 in [30].

REMARK 3. Under suitable hypotheses, which can be found in [30], and using very similar methods, it is possible to establish an analogue of the latter result when $x \to 0$. As this article is rather long already, and for sake of conciseness, we have chosen not to include a proof nor a statement.

A simple but interesting consequence of Theorems 2 and 3 are the following asymptotic triple laws for the overshoot and undershoot at the last passage above a barrier of a Lévy process conditioned to stay positive. We just state the result under the assumption that the process starts from 0, although, thanks to a simple application of Corollary 1, a similar result holds when the process starts from a strictly positive position. Moreover, using Corollary 2 it is also possible to establish from the following corollary the analogous result for the asymptotic law for the overshoot and undershoot at the last passage above a barrier of a Lévy process. We leave the details to the interested reader.

COROLLARY 3. *Suppose that X is a Lévy process which does not drift to $-\infty$ and for which 0 is regular for $(-\infty,0)$ and $(0,\infty)$. If the assumptions of Theorem 3(i) are satisfied then asymptotic three-dimensional law of overshoots and undershoot of a Lévy process conditioned to stay positive is given by*

$$\begin{aligned}\lim_{x\to\infty} &\mathbb{P}^{\uparrow}(x - X_{U_x-} \in dy, X_{U_x} - x \in du, \underrightarrow{X}_{U_x} - x \in dw) \\ &= \frac{dy}{\mu_+} \Pi_X(du + y) \widehat{V}(u - dw)\end{aligned}$$

for $y > 0$, $0 \le w \le u$.

If, respectively, the assumptions in (ii), (iii) *or* (iv) *in Theorem* 3 *are satisfied then*

$$\lim_{x\to\infty}\mathbb{P}^{\uparrow}\left(\frac{x-X_{U_x-}}{x}\in dy, \frac{X_{U_x}-x}{x}\in du, \frac{\underrightarrow{X}_{U_x}-x}{x}\in dw\right)$$
$$=\begin{cases}\dfrac{\sin(\alpha\rho\pi)\Gamma(\alpha+1)}{\pi\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))}\dfrac{(1-y)^{\alpha\rho-1}(u-w)^{\alpha(1-\rho)-1}}{(u+y)^{1+\alpha}}\,dy\,du\,dw, & \\ & \textit{in case (ii)},\\ \dfrac{\alpha(1+\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)}\dfrac{1}{(1-y)^{1-\alpha}(u+y)^{2+\alpha}}\,dy\,du\,dw, & \textit{in case (iii)},\\ \dfrac{\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)}\dfrac{1}{(1-y)^{1-\alpha}(u+y)^{1+\alpha}}\,dy\,du\,\delta_u(dw), & \textit{in case (iv)},\end{cases}$$

for $0<y<1, u\geq w>0$.

Theorem 3 excludes the case of a Lévy process that drifts to $-\infty$. In that case the process has a strictly positive probability of never crossing a given positive barrier and hence the overshoots and undershoot take the value ∞ on that event. Nevertheless, it is possible to establish similar results to those established in Theorem 3 conditionally on the event that the process reaches the level. That is the purpose of the following results which are in turn a generalization of the results in [22] where the asymptotic behavior of the overshoot and undershoot of a spectrally positive Lévy process has been studied. We will assume that the Lévy measure is subexponential. Analogous results for three-dimensional undershoot and overshoot laws in the case where the underlying Lévy process has a close to exponential Lévy measure have been obtained in [16].

Recall that a probability distribution function F over $[0,\infty)$ is said to be subexponential if the tail distribution, $\overline{F}(x):=1-F(x)$, $x\in\mathbb{R}$, satisfies that $\overline{F}(x)>0$, $x>0$, and

$$\lim_{x\to\infty}\frac{\overline{F^{*2}}(x)}{\overline{F}(x)}=2. \tag{3.3}$$

We will say that the Lévy measure $\Pi_X^+:=\Pi_X|_{[0,\infty)}$ is subexponential if the distribution of the probability measure $\Pi_X[1,\infty)^{-1}\Pi_X|_{[1,\infty)}$ is subexponential. We are interested in particular in two special cases of subexponential distributions: those which are regularly varying and those in the domain of attraction of a Gumbel distribution.

THEOREM 4. *Let X be a real valued Lévy process drifting to $-\infty$, with subexponential right tail Lévy measure, and such that the mean of $\widehat{H}_1$ is finite and put* $\mu_-=\mathbf{E}(\widehat{H}_1)$.

(i) *If* $\overline{\Pi}_X^+$ *is regularly varying at* ∞ *with index* $-1-\alpha$ *for some* $\alpha \in (0,1)$ *then*

$$\lim_{x\to\infty} \mathbb{P}\biggl(\frac{x-\overline{X}_{\tau_x^+-}}{x} \in du, \frac{-X_{\tau_x^+-}}{x} > v, \frac{X_{\tau_x^+}-x}{x} > w \Big| \tau_x^+ < \infty\biggr) \\ = \frac{1}{(1+v+w)^\alpha}\delta_1(du)$$

for $u, v, w \geq 0$. *Note in particular* (*with regard to the first element of the triple*) *that the limiting distribution is concentrated on* $\{1\} \times [0,\infty)^2$.

(ii) *Assume* $\overline{\Pi}_X^+$ *is in the maximum domain of attraction of the Gumbel distribution. Let* $a : \mathbb{R}^+ \to (0,\infty)$ *be a continuous and differentiable function such that*

$$a(x) \sim \int_x^\infty dy\, \overline{\Pi}_X^+(y)/\overline{\Pi}_X^+(x)$$

and $a'(x) \to 0$ *as* $x \to \infty$. *Then*

$$\lim_{x\to\infty} \mathbb{P}\biggl(\frac{x-\overline{X}_{\tau_x^+-}}{x} \in du, \frac{-X_{\tau_x^+-}}{a(x)} > v, \frac{X_{\tau_x^+}-x}{a(x)} > w \Big| \tau_x^+ < \infty\biggr) \\ = e^{-(v+w)}\delta_1(du)$$

for $u, v, w \geq 0$. *Note again in particular* (*with regard to the first element of the triple*) *that the limiting distribution is concentrated on* $\{1\} \times [0,\infty)^2$.

4. Proof of Theorems 3 and 4.

PROOF OF (i) IN THEOREM 3. We will prove that the Laplace transform of the triple law of overshoot and undershoots converges pointwise which is enough for proving the claimed weak convergence. From the quintuple law in Theorem 1 we obtain that the Laplace transform of the undershoots and overshoot of X can be written as

$$\begin{aligned}
&\mathbb{E}\bigl(\exp\{-\theta_1(x-\overline{X}_{\tau_x^+-}) - \theta_2(x-X_{\tau_x^+-}) - \theta_3(X_{\tau_x^+}-x)\}\bigr) \\
&\quad = \int_0^x V(dy)\int_0^\infty \widehat{V}(dl) \\
&\qquad \times \int_{z>x-y+l} \Pi_X(dz)\exp\bigl\{-\theta_1(x-y) - \theta_2(x-y+l) \\
&\qquad\qquad - \theta_3\bigl(z-(l+x-y)\bigr)\bigr\} \\
&\quad = \int_0^x V(dy)e^{-(\theta_1+\theta_2)(x-y)}\int_0^\infty \widehat{V}(dl)e^{-\theta_2 l} \\
&\qquad \times \int_{z>x-y+l} \Pi_X(dz)e^{-\theta_3(z-(l+x-y))}
\end{aligned}$$

for $x > 0$ and $\theta_1, \theta_2, \theta_3 \ge 0$. The proof will follow from an application of the version of the key renewal theorem appearing in Theorem 5.2.6 of [20] and the remark following it, applied to the renewal measure $V(dy)$ and the function

$$r \mapsto e^{-(\theta_1+\theta_2)r} \int_0^\infty \widehat{V}(dl) e^{-\theta_2 l} \int_{z>r+l} \Pi_X(dz) e^{-\theta_3(z-(l+r))}, \qquad r > 0. \tag{4.1}$$

To this end, observe that the measure $V(dy)$ is the renewal measure associated with the probability measure $\mathbb{P}(H_{\mathbf{e}} \in dy)$, where $\mathbf{e}$ is an independent exponential random variable with unit mean. An easy calculation using Laplace transforms shows that the random variable $H_{\mathbf{e}}$ is nonarithmetic, because X has the same property. Moreover, $\mathbb{E}(H_{\mathbf{e}}) = \mu_+ < \infty$; and the function defined in (4.1) is bounded above by the decreasing and integrable function

$$r \mapsto e^{-(\theta_1+\theta_2)r} \int_0^\infty \widehat{V}(dl) \overline{\Pi}_X^+(l+r), \qquad r > 0.$$

Note that the integrability of this function follows from (3.1) and the fact that, by assumption, the mean of H_1 is finite or equivalently $\overline{\Pi}_H$ is integrable.

We can hence apply the version of the renewal theorem appearing in [20], Theorem 5.2.6 and the remark following it, to deduce that for $\theta_1, \theta_2, \theta_3 \ge 0$,

$$\begin{aligned}
&\lim_{x\to\infty} \mathbb{E}\bigl(\exp\{-\theta_1(x - \overline{X}_{\tau_x^+ -}) - \theta_2(x - X_{\tau_x^+ -}) - \theta_3(X_{\tau_x^+} - x)\}\bigr) \\
&\quad = \frac{1}{\mu_+} \int_0^\infty dy \int_0^\infty \widehat{V}(dl) \\
&\qquad \times \int_{z>y+l} \Pi_X(dz) \exp\bigl\{-\theta_1 y - \theta_2(y+l) - \theta_3\bigl(z - (l+y)\bigr)\bigr\},
\end{aligned}$$

which implies the result. The formula for the asymptotic distribution of the three-dimensional law of overshoot and undershoots is obtained using elementary arguments.

To establish the other direction of the proof, we observe that if the triple of random variables $(x - \overline{X}_{\tau_x^+ -}, x - X_{\tau_x^+ -}, X_{\tau_x^+} - x)$ converges weakly as $x \to \infty$ to a nondegenerate random variable then $X_{\tau_x^+} - x$ also converges weakly to a nondegenerate random variable. It is known that this implies that the mean of H_1 is finite; see, for example, Theorem 8 and Lemma 7 in [17]. $\square$

PROOF OF (ii) IN THEOREM 3. The proof of this result is essentially an extension of the method of proof used in Theorem 2 in [30] so we will just provide an sketch of proof. Let $X^{(r)}$ be the Lévy process defined by $(X_{rt}/b(r), t \ge 0)$ and $\widetilde{X} = (\widetilde{X}_t, t \ge 0)$ be a strictly α-stable Lévy process.

By assumption $X_1^{(r)}$ converges weakly to $\widetilde{X}_1$, and it is well known that this implies that the process $X^{(r)}$ converges weakly toward $\widetilde{X}$. The undershoot and overshoot of $X^{(r)}$ are such that

$$(\mathcal{V}_1^{(r)}, \mathcal{O}_1^{(r)}) = \left(\frac{\mathcal{V}_{b(r)}}{b(r)}, \frac{\mathcal{O}_{b(r)}}{b(r)}\right), \qquad r > 0,$$

in the obvious notation. By Theorem 13.6.4 in [35] it follows that $(\mathcal{V}_1^{(r)}, \mathcal{O}_1^{(r)})$ converges weakly toward the undershoot and overshoot $(\widetilde{\mathcal{V}}_1, \widetilde{\mathcal{O}}_1)$ of $\widetilde{X}$ at level 1, which implies that

$$\left(\frac{\mathcal{V}_{b(r)}}{b(r)}, \frac{\mathcal{O}_{b(r)}}{b(r)}\right) \xrightarrow[r\to\infty]{D} (\widetilde{\mathcal{V}}_1, \widetilde{\mathcal{O}}_1).$$

Next we appeal to an argument similar to the one used in the proof of Theorem 2 in [30] to justify that

$$\left(\frac{x - X_{\tau_x^+ -}}{x}, \frac{X_{\tau_x^+} - x}{x}\right) \xrightarrow[r\to\infty]{D} (\widetilde{\mathcal{V}}_1, \widetilde{\mathcal{O}}_1).$$

The proof is based on the fact that the asymptotic inverse of b, say $b^{\leftarrow}$, is such that $b(b^{\leftarrow}(x)) \sim x$ as $x \to \infty$. Now using Lemma 1 we get that for $u \in [0, 1[$, $v \geq u$, $w > 0$,

$$\begin{aligned}
&\lim_{x\to\infty} \mathbb{P}\left(\frac{x - \overline{X}_{\tau_x^+ -}}{x} > u, \frac{x - X_{\tau_x^+ -}}{x} > v, \frac{X_{\tau_x^+} - x}{x} > w\right) \\
&\quad = \lim_{z\to\infty} \mathbb{P}\left(\frac{\mathcal{V}_z}{z} > \frac{v-u}{1-u}, \frac{\mathcal{O}_z}{z} > \frac{w+u}{1-u}\right) \qquad (4.2)\\
&\quad = \mathbb{P}\left(\widetilde{\mathcal{V}}_1 > \frac{v-u}{1-u}, \widetilde{\mathcal{O}}_1 > \frac{w+u}{1-u}\right).
\end{aligned}$$

To conclude the proof and for sake of reference we quote, from [16], the formula for the law of the undershoots–overshoot at level 1 for a stable process with index $\alpha \in (0, 2)$ and positivity parameter $\rho \in (0, 1)$,

$$\begin{aligned}
&\mathbb{P}(\widetilde{\mathcal{U}}_1 \in du, \widetilde{\mathcal{V}}_1 \in dv, \widetilde{\mathcal{O}}_1 \in dw) \\
&\quad = \frac{\sin(\alpha\rho\pi)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))} \qquad (4.3)\\
&\qquad \times \frac{(1-u)^{\alpha\rho-1}(v-u)^{\alpha(1-\rho)-1}}{(v+w)^{1+\alpha}}\, du\, dv\, dw
\end{aligned}$$

for $0 \leq u \leq 1$, $v \geq u$, and $w > 0$. From (4.2), (4.3) and elementary calculations we infer the required weak convergence. $\square$

PROOF OF (iii) IN THEOREM 3. The proof of (iii) and (iv) are based on the following basic identity which can easily be extracted from from the quintuple law in Theorem 1. For $v, w \geq 0$

$$\begin{aligned}\mathbb{P}(x - X_{\tau_x^+ -} > v, X_{\tau_x^+} - x > w) \\ = \int_0^x V(dy) \int_{[(v-(x-y))\vee 0,\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(w + l + (x - y)\big).\end{aligned} \tag{4.4}$$

From the basic identity (4.4) we have that for $a, b > 0$,

$$\begin{aligned}&\mathbb{P}(x - X_{\tau_x^+ -} > xb, X_{\tau_x^+} - x > ax) \\ &\quad = 1_{\{0<b\leq 1\}} \int_0^{(1-b)} V(xdy) \int_{[0,\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(ax + l + x(1 - y)\big) \\ &\qquad + 1_{\{0<b\leq 1\}} \int_{(1-b)}^1 V(xdy) \\ &\qquad \times \int_{[x(b-(1-y)),\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(ax + l + x(1 - y)\big) \\ &\qquad + 1_{\{b>1\}} \int_0^1 V(xdy) \int_{[x(b-(1-y)),\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(ax + l + x(1 - y)\big).\end{aligned} \tag{4.5}$$

Thanks to Theorem 3(a) in [30], under the assumptions in (iii), we have the estimate

$$\overline{\Pi}_H(x) \sim \frac{1}{\mu_-} \int_x^\infty \overline{\Pi}_X^+(z)\,dz, \qquad x \to \infty, \tag{4.6}$$

where $\mu_- = \mathbf{E}(\widehat{H}_1) < \infty$. By Karamata's theorem (see, e.g., Chapter 1 of [3]) and the assumption that $\overline{\Pi}_X^+$ is regularly varying at infinity with an index $-(1+\alpha)$, for $\alpha \in (0, 1)$, it follows that $\overline{\Pi}_H$ is regularly varying at infinity with index $-\alpha$. By Proposition 1.5 in [2] we have that

$$\overline{\Pi}_H(x)V[0, ax] \xrightarrow[x\to\infty]{} \frac{a^\alpha}{\alpha\Gamma(\alpha)\Gamma(1-\alpha)}, \qquad a > 0.$$

This implies the weak convergence of measures

$$\overline{\Pi}_H(x)V(x\,dy)1_{\{x\in(0,1]\}} \xrightarrow[x\to\infty]{\text{weakly}} \frac{y^{\alpha-1}}{\Gamma(\alpha)\Gamma(1-\alpha)} 1_{\{y\in(0,1]\}}\,dy. \tag{4.7}$$

We claim that the asymptotic results in (4.6) and (4.7) imply also that

$$\begin{aligned}&\frac{1}{\overline{\Pi}_H(x)} \int_{[x(b-(1-y)),\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(ax + l + x(1 - y)\big) \\ &\quad \xrightarrow[x\to\infty]{} (a + b)^{-\alpha},\end{aligned} \tag{4.8}$$

uniformly in $(a + b) \in [t, \infty)$, for every $t > 0$. To see this, we apply the renewal theorem to $\widehat{V}$ and use the estimate (4.6). More precisely, the renewal theorem implies that, for $h > 0$ and $\varepsilon > 0$, there exists a $t_0 > 0$ such that

$$\left|\widehat{V}[t, t+h) - \frac{h}{\mu_-}\right| < \frac{\varepsilon h}{\mu_-} \qquad \forall t \geq t_0.$$

Hence, for every ε, $h > 0$ and x large enough

$$\begin{aligned}
&\int_{[x(b-(1-y)),\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(ax + l + x(1-y)\big) \\
&\qquad = \int_{[x(b-(1-y)),\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(l - x\big(b - (1-y)\big) + x(a+b)\big) \\
&\qquad = \sum_{n=0}^{\infty} \int_{[x(b-(1-y))+nh, x(b-(1-y))+(n+1)h)} \widehat{V}(dl)\overline{\Pi}_X^+\big(l - x\big(b - (1-y)\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + x(a+b)\big) \\
&\qquad \leq \sum_{n=0}^{\infty} \widehat{V}\big[x\big(b-(1-y)\big) + nh, x\big(b-(1-y)\big) + (n+1)h\big)\overline{\Pi}_X^+ \\
&\qquad\qquad \times \big(nh + x(a+b)\big) \\
&\qquad \leq \frac{(1+\varepsilon)h}{\mu_-} \sum_{n=0}^{\infty} \overline{\Pi}_X^+\big(nh + x(a+b)\big) \\
&\qquad \leq \frac{(1+\varepsilon)}{\mu_-} \int_0^{\infty} \overline{\Pi}_X^+\big(z + x(a+b)\big)\,dz \\
&\qquad \sim (1+\varepsilon)\overline{\Pi}_H(x)(a+b)^{-\alpha} \qquad \text{as } x \to \infty,
\end{aligned}$$

where the final estimate follows from (4.6) and the regular variation of $\overline{\Pi}_H$. This implies that

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{1}{\overline{\Pi}_H(x)} \int_{[x(b-(1-y)),\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(l - x\big(b-(1-y)\big) + x(a+b)\big) \\
&\qquad \leq (a+b)^{-\alpha}
\end{aligned}$$

for all $a + b > 0$. The analog estimate for the limit inferior is obtained in a similar way thus justifying (4.8), but not uniformly in $(a+b) > t$ for every $t > 0$. The aforesaid uniformity in $(a+b)$ follows from the fact that as $\overline{\Pi}_H$ is regularly varying at infinity then

$$\lim_{x\to\infty} \frac{\overline{\Pi}_H(cx)}{\overline{\Pi}_H(x)} = c^{-\alpha} \qquad \text{uniformly in } c \in [t, \infty), \tag{4.9}$$

for each $t > 0$; see, for example, Theorem 1.5.2 on page 22 of [3]. Using the weak convergence in (4.7) and the uniformity in (4.9) we get that for $0 < b \leq 1$, $a > 0$, the first term in (4.5) tends as $x \to \infty$ toward

$$\begin{aligned}
&\int_0^{(1-b)} V(x\,dy)\int_{[0,\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(ax+l+x(1-y)\big)\\
&\quad= \int_0^{(1-b)} \overline{\Pi}_H(x)V(x\,dy)\frac{\overline{\Pi}_H(ax+x(1-y))}{\overline{\Pi}_H(x)}\\
&\quad\xrightarrow[x\to\infty]{} \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{(1-b)} y^{\alpha-1}\frac{1}{(a+(1-y))^\alpha}\,dy.
\end{aligned}$$

In addition, arguing as above, using instead of (4.9) the property (4.8), we may deal with the second and third terms in (4.5) as $x \to \infty$ and obtain for $0 < b \le 1$

$$\begin{aligned}
&\int_{1-b}^1 V(x\,dy)\int_{[x(b-(1-y)),\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(ax+l+x(1-y)\big)\\
&\quad\xrightarrow[x\to\infty]{} \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_{(1-b)}^1 y^{\alpha-1}\frac{1}{(a+b)^\alpha}\,dy
\end{aligned}$$

and for $b > 1$

$$\begin{aligned}
&\int_0^1 V(x\,dy)\int_{[x(b-(1-y)),\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(ax+l+x(1-y)\big)\\
&\quad\xrightarrow[x\to\infty]{} \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^1 y^{\alpha-1}\frac{1}{(a+b)^\alpha}\,dy,
\end{aligned}$$

respectively. Putting the three terms together back in (4.5) we get

$$\begin{aligned}
&\lim_{x\to\infty}\mathbb{P}\left(\frac{x-X_{\tau_x^+-}}{x} > b, \frac{X_{\tau_x^+}-x}{x} > a\right)\\
&\quad= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^1 y^{\alpha-1}\frac{1}{(a+(1-y)\vee b)^\alpha}\,dy
\end{aligned}$$

for $a, b > 0$. Taking derivatives we deduce that the weak limit, as $x \to \infty$, of the law of the couple $(\frac{x-X_{\tau_x^+-}}{x}, \frac{X_{\tau_x^+}-x}{x})$, has a density given by

$$\frac{(1+\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{(1-(1-v)_+^\alpha)}{(v+w)^{\alpha+2}}, \qquad v, w > 0, \tag{4.10}$$

where $z_+ = \max\{z, 0\}$. Lemma 1 and the identity (4.2) allows us to infer the weak convergence of the triplet $(\frac{x-\overline{X}_{\tau_x^+-}}{x}, \frac{x-X_{\tau_x^+-}}{x}, \frac{X_{\tau_x^+}-x}{x})$. Using (4.10) we deduce the form of the density for the asymptotic law for the overshoot and undershoots. $\square$

PROOF OF (iv) IN THEOREM 3. The proof is based on the identity (4.5) and the fact, proved in Theorem 3(b) and Corollary 2(b) in [30], that under our hypotheses

$$\overline{\Pi}_H(x) \sim \widehat{V}(\infty)\overline{\Pi}_X^+(x), \qquad x \to \infty.$$

This implies that for any $b, a > 0$ and $0 \leq y \leq 1$ such that $b - (1-y) > 0$,

$$
\begin{aligned}
&\frac{1}{\overline{\Pi}_H(x)} \int_{[x(b-(1-y)),\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(ax + l + x(1-y)\big) \\
&\quad = \frac{1}{\overline{\Pi}_H(x)} \int_{[x(b-(1-y)),\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(l - x\big(b-(1-y)\big) + x(a+b)\big) \\
&\quad \leq \frac{\overline{\Pi}_X^+(x(a+b))}{\overline{\Pi}_H(x)} \int_{[x(b-(1-y)),\infty)} \widehat{V}(dl) \\
&\quad \leq \frac{\overline{\Pi}_X^+(x(a+b))}{\overline{\Pi}_H(x)} \int_{[xb,\infty)} \widehat{V}(dl) \xrightarrow[x\to\infty]{} 0.
\end{aligned}
$$

Therefore, using the above estimate and that $\overline{\Pi}_H(x)V[0,xc] \to c^\alpha/\alpha\Gamma(\alpha)\Gamma(1-\alpha)$, as $x \to \infty$, uniformly for $c \in K$, K being any compact set in $(0,\infty)$ cf. [2], in the identity (4.5), we obtain that

$$
\begin{aligned}
&\lim_{x\to\infty} \mathbb{P}(x - X_{\tau_x^+ -} > xb, X_{\tau_x^+} - x > ax) \\
&\quad = 1_{\{0\leq b\leq 1\}} \lim_{x\to\infty} \int_0^{(1-b)} V(x\,dy) \\
&\qquad\qquad \times \int_{[0,\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(ax + l + x(1-y)\big).
\end{aligned} \tag{4.11}
$$

Moreover, it follows from (3.1) that for $a > 0$, $0 \leq b \leq 1$ and $y \in [0, 1-b]$,

$$
\begin{aligned}
&\overline{\Pi}_H\big(x\big(a + (1-y)\big)\big) \\
&\quad = \int_{[0,\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(ax + l + x(1-y)\big),
\end{aligned}
$$

and hence arguing as in the proof of Theorem 3(iii) we arrive at the identity

$$
\begin{aligned}
&\lim_{x\to\infty} \mathbb{P}(x - X_{\tau_x^+ -} > xb, X_{\tau_x^+} - x > ax) \\
&\quad = \lim_{x\to\infty} \int_0^{(1-b)} V(x\,dy) \int_{[0,\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(ax + l + x(1-y)\big) \\
&\quad = \lim_{x\to\infty} \int_0^{(1-b)} \overline{\Pi}_H(x)V(x\,dy) \frac{\overline{\Pi}_H(x(a+(1-y)))}{\overline{\Pi}_H(x)} \\
&\quad = \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^{(1-b)} y^{\alpha-1} \frac{1}{(a+(1-y))^\alpha}\,dy.
\end{aligned}
$$

Taking derivatives we obtain the form of the density of the asymptotic law for the overshoot and undershoot of X claimed in Theorem 3(iv).

The latter identity and Lemma 1 allow us to conclude that for $0 \le u \le v \le 1$, $w \ge 0$,

$$\begin{aligned}
&\lim_{x\to\infty} \mathbb{P}\left(\frac{x-\overline{X}_{\tau_x^+-}}{x} > u, \frac{x-X_{\tau_x^+-}}{x} > v, \frac{X_{\tau_x^+}-x}{x} > w\right) \\
&\quad = \lim_{z\to\infty} \mathbb{P}\left(\frac{\mathcal{V}_z}{z} > \frac{v-u}{1-u}, \frac{\mathcal{O}_z}{z} > \frac{w+u}{1-u}\right) \\
&\quad = \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^{(1-v)/(1-u)} y^{\alpha-1} \frac{1}{((1+w)/(1-u)-y)^\alpha}\, dy \\
&\quad = \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^{1-v} z^{\alpha-1} \frac{1}{(w+1-z)^\alpha}\, dz \\
&\quad = \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_{(v+w)/(1-v)}^{\infty} (1+y)^{-1} y^{-\alpha}\, dy.
\end{aligned}$$

The latter identity and the fact that $x - X_{\tau_x^+-} \ge x - \overline{X}_{\tau_x^+-}$, implies that in the present case the weak limit of $(x - X_{\tau_x^+-})/x$ equals that of $(x - \overline{X}_{\tau_x^+-})/x$. To see this, note that for $0 \le u < u + \varepsilon \le 1$,

$$\begin{aligned}
&\lim_{x\to\infty} \mathbb{P}\left(\frac{x-\overline{X}_{\tau_x^+-}}{x} \in (u, u+\varepsilon], \frac{x-X_{\tau_x^+-}}{x} > u+\varepsilon\right) \\
&\quad = \lim_{x\to\infty} \mathbb{P}\left(\frac{x-\overline{X}_{\tau_x^+-}}{x} > u, \frac{x-X_{\tau_x^+-}}{x} > u+\varepsilon\right) \\
&\qquad - \lim_{x\to\infty} \mathbb{P}\left(\frac{x-\overline{X}_{\tau_x^+-}}{x} > u+\varepsilon, \frac{x-X_{\tau_x^+-}}{x} > u+\varepsilon\right) \\
&\quad = 0.
\end{aligned}$$

This implies that for $0 \le u < u + \varepsilon \le 1$,

$$\lim_{x\to\infty} \mathbb{P}\left(\frac{x-X_{\tau_x^+-}}{x} \in (u, u+\varepsilon] \middle| \frac{x-\overline{X}_{\tau_x^+-}}{x} \in (u, u+\varepsilon]\right) = 1.$$

For $0 < \varepsilon < 1$, let n_ε be the largest integer such that $n_\varepsilon \varepsilon \le 1$. It follows that

$$\begin{aligned}
&\mathbb{P}\left(\frac{x-X_{\tau_x^+-}}{x} - \frac{x-\overline{X}_{\tau_x^+-}}{x} > \varepsilon\right) \\
&\quad = \sum_{k=0}^{n_\varepsilon} \mathbb{P}\left(\frac{x-X_{\tau_x^+-}}{x} - \frac{x-\overline{X}_{\tau_x^+-}}{x} > \varepsilon \middle| \frac{x-\overline{X}_{\tau_x^+-}}{x} \in \big(k\varepsilon, (k+1)\varepsilon \wedge 1\big]\right) \\
&\qquad \times \mathbb{P}\left(\frac{x-\overline{X}_{\tau_x^+-}}{x} \in \big(k\varepsilon, (k+1)\varepsilon \wedge 1\big]\right)
\end{aligned}$$

$$\leq \sum_{k=0}^{n_\varepsilon} \mathbb{P}\biggl(\frac{x - X_{\tau_x^+ -}}{x} \notin \bigl(k\varepsilon, (k+1)\varepsilon \wedge 1\bigr] \bigg| \frac{x - \overline{X}_{\tau_x^+ -}}{x} \in \bigl(k\varepsilon, (k+1)\varepsilon \wedge 1\bigr]\biggr)$$
$$\times \mathbb{P}\biggl(\frac{x - \overline{X}_{\tau_x^+ -}}{x} \in \bigl(k\varepsilon, (k+1)\varepsilon \wedge 1\bigr]\biggr)$$

and that the right-hand side tends to zero as $x \to \infty$. $\square$

PROOF OF (i) IN THEOREM 4. The proof of this result uses similar arguments to those used in the proof of (iii) in Theorem 3 so we will just provide the main steps of the proof. First with the help of Lemma 1, we have for $0 \leq u < 1$, $v > -1$, $w > 0$,

$$\begin{aligned}
&\mathbb{P}(x - \overline{X}_{\tau_x^+ -} > ux, -X_{\tau_x^+ -} > vx, X_{\tau_x^+} - x > wx, \tau_x^+ < \infty)\\
&\quad = \mathbb{P}\bigl(x(1-u) - X_{\tau_{x(1-u)}^+ -} > (v+1-u)x,\\
&\qquad\qquad X_{\tau_{x(1-u)}^+} - x(1-u) > (w+u)x, \tau_{(1-u)x}^+ < \infty\bigr)\\
&\quad = \int_0^{x(1-u)} V(dy) \int_{[(vx+y)\vee 0,\infty)} \widehat{V}(dl) \overline{\Pi}_X^+\bigl((w+1+v)x + l - (vx+y)\bigr).
\end{aligned}$$

Note that assumption (i) and the monotone density theorem for regularly varying functions imply that $\int_x^\infty \overline{\Pi}_X^+(z)\,dz$ is regularly varying with index $-\alpha$ so that Theorem 3 in [30] is applicable. The latter theorem together with Lemma 3.5 in [23] and our hypotheses imply that

$$\overline{\Pi}_H(x) \sim \frac{1}{\mu_-} \int_x^\infty \overline{\Pi}_X^+(z)\,dz, \qquad x \to \infty, \tag{4.12}$$

and

$$\lim_{x\to\infty} \frac{\mathbb{P}(\tau_x^+ < \infty)}{\overline{\Pi}_H(x)} = V(\infty). \tag{4.13}$$

Note that in the above application of Theorem 3 in [30] it is necessary to verity hypothesis (a-1). This boils down to checking that

$$\lim_{x\to\infty} \frac{\int_{x+t}^\infty \overline{\Pi}_X^+(y)\,dy}{\int_x^\infty \overline{\Pi}_X^+(y)\,dy} = 1, \qquad t \in \mathbb{R}.$$

However, this is a straightforward consequence of the fact that $\int_x^\infty \overline{\Pi}_X^+(z)\,dz$ is regularly varying at infinity.

We may now proceed to argue that, thanks to the regular variation of $\overline{\Pi}_H$ and a dominated convergence argument, for $0 \le u < 1$, $v > 0$, $w > 0$,

$$\begin{aligned}
&\mathbb{P}(x - \overline{X}_{\tau_x^+-} > ux, -X_{\tau_x^+-} > vx, X_{\tau_x^+} - x > wx | \tau_x^+ < \infty) \\
&\quad \sim \left(\frac{\overline{\Pi}_H(x)}{\mathbb{P}(\tau_x^+ < \infty)}\right)\left(\int_0^{x(1-u)} V(dy)\right)\left(\frac{1/\mu_- \int_{x(1+w+v)}^{\infty} \overline{\Pi}_X^+(z)\,dz}{\overline{\Pi}_H(x)}\right) \\
&\quad \sim (1 + w + v)^{-\alpha}
\end{aligned}$$

as $x \to \infty$. In principle, to complete the proof we are obliged to check convergence when $-1 < v < 0$. However, by noting that the established limiting triple law above is not a defective distribution, the proof is complete. $\square$

PROOF OF (ii) IN THEOREM 4. For the same reasons as in the proof of part (i) of Theorem 4 we will just make a sketch of proof as follows. Thanks to Lemma 1 and the quintuple law in Theorem 1 it follows that for $0 < u \le 1$, $w > 0$, $v \in \mathbb{R}$, such that $v + w > 0$,

$$\begin{aligned}
&\mathbb{P}\big(-\overline{X}_{\tau_x^+-} > -ux, -X_{\tau_x^+-} > a(x)v, \overline{X}_{\tau_x^+} - x > a(x)w, \tau_x^+ < \infty\big) \\
&\quad = \mathbb{P}\big(x - \overline{X}_{\tau_x^+-} > x(1-u), \\
&\qquad\qquad x - X_{\tau_x^+-} > a(x)v + x, \overline{X}_{\tau_x^+} - x > a(x)w, \tau_x^+ < \infty\big) \\
&\quad = \mathbb{P}\big(ux - X_{\tau_{ux}^+-} > va(x) + xu, \\
&\qquad\qquad \overline{X}_{\tau_{xu}^+} - xu > a(x)w + x(1-u), \tau_{xu}^+ < \infty\big) \qquad (4.14)\\
&\quad = \int_0^{xu} V(dy) \int_{[(va(x)+y)\vee 0,\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(wa(x) + x(1-u) \\
&\qquad\qquad + l + xu - y\big) \\
&\quad = \int_0^{xu} V(dy) \int_{[(va(x)+y)\vee 0,\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big((w+v)a(x) \\
&\qquad\qquad + x + l - \big(a(x)v + y\big)\big).
\end{aligned}$$

We recall that by Theorem 3 in [30] the assumption that Π is subexponential implies that Π_H is long tailed

$$\lim_{x\to\infty} \frac{\overline{\Pi}_H(x+t)}{\overline{\Pi}_H(x)} = 1 \qquad \text{for each } t \in \mathbb{R},$$

and moreover that (4.12) and (4.13) hold. Arguing as in the proof of (iii) in Theorem 3, when $va(x)+y>0$, we have that

$$\int_{[(va(x)+y)\vee 0,\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big((w+v)a(x)+x+l-\big(a(x)v+y\big)\big)$$
$$\sim \frac{1}{\mu_-}\int_{x+a(x)(v+w)}^{\infty} dl\overline{\Pi}_X^+(l), \qquad x\to\infty.$$

In the case where $a(x)v+y<0$, Vigon's identity (3.1) and long-tailed behavior imply that

$$\int_{[(va(x)+y)\vee 0,\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big((w+v)a(x)+x+l-\big(a(x)v+y\big)\big)$$
$$=\overline{\Pi}_H\big(wa(x)+x-y\big)\sim\overline{\Pi}_H\big(wa(x)+x\big)$$

for each y as $x\to\infty$.

Putting the pieces together it follows that for $0<u\le 1$, $v,w>0$

$$\mathbb{P}\big(-\overline{X}_{\tau_x^+-}>-ux, -X_{\tau_x^+-}>a(x)v, \overline{X}_{\tau_x^+}-x>a(x)w|\tau_x^+<\infty\big)$$
$$\sim\frac{\overline{\Pi}_H(x)}{\mathbb{P}(\tau_x^+<\infty)}V(\infty)\frac{1/\mu_-\int_{a(x)(v+w)+x}^{\infty}\overline{\Pi}_X^+(z)\,dz}{1/\mu_-\int_x^{\infty}\overline{\Pi}_X^+(z)\,dz}\sim e^{-(v+w)},$$

where the final estimate is obtained by L'Hôpital's rule using the fact that a is differentiable and $a'(x)\to 0$, as $x\to\infty$.

Once again it is not necessary to consider the case that $v<0$ as the triple law established above is not defective. Note in particular that in this setting the weak limit of $\overline{X}_{\tau_x^+-}/x$, conditionally on $\tau_x^+<\infty$, is 0 as $x\to\infty$. □

5. Triple and quadruple laws at first and last passage of positive self-similar Markov processes. The objective in this section is to bring some of the results from Sections 2 and 3 into the setting of positive self-similar Markov processes. Although this will be a relatively straightforward operation, it will allow us to construct many new explicit examples in the following section.

A positive Markov process $Y=(Y_t,t\ge 0)$ with càdlàg paths is a self-similar process if for every $k>0$ and every initial state $x\ge 0$ it satisfies the scaling property, that is, for some $\alpha>0$

$$\text{the law of } (kY_{k^{-\alpha}t}, t\ge 0) \text{ under } \mathbf{P}_x \text{ is } \mathbf{P}_{kx},$$

where $\mathbf{P}_x$ denotes the law of the process Y starting from $x\ge 0$. Here, we use the notation $Y^{(x)}$ or $(Y,\mathbf{P}_x)$ for a positive self-similar Markov process starting from $x\ge 0$. Well-known examples of such class of processes include Bessel processes, stable subordinators or more generally stable processes conditioned to stay positive.

Lamperti [25] proved that there is a bijective correspondence between the class of positive self-similar Markov processes that never hit the state 0 and the class of Lévy processes which do not drift to $-\infty$. More precisely, let $Y^{(x)}$ be a self-similar Markov process started from $x > 0$ that fulfills the scaling property for some $\alpha > 0$, then

$$Y_t^{(x)} = x \exp\{X_{\theta(tx^{-\alpha})}\}, \qquad 0 \le t \le x^{\alpha} I(X), \tag{5.1}$$

where

$$\theta_t = \inf\{s \ge 0 : I_s(X) > t\}, \qquad I_s(X) = \int_0^s \exp\{\alpha X_u\}\, du,$$
$$I(X) = \lim_{t \to +\infty} I_t(X),$$

and X is a Lévy process starting from 0 which does not drift to $-\infty$ and whose law does not depend on $x > 0$, here denoted by $\mathbb{P}$. This is the so-called Lamperti representation.

Recall that H denotes the ascending ladder height process associated to X and V its corresponding bivariate renewal function. Similarly, $\widehat{V}$ denotes the bivariate renewal function associated to the descending ladder processes and Π_X the Lévy measure of X.

Caballero and Chaumont [5] studied the problem of when an entrance law at 0 for $(Y, \mathbf{P}_x)$ can be defined. In particular, the authors in [5] gave necessary and sufficient conditions for the weak convergence of $Y^{(x)}$ on the Skorokhod's space, as x goes to 0, toward a nondegenerate process, that we will denote by $Y^{(0)}$ on some occasions and $(Y, \mathbf{P}_0)$ on others. The limit process $Y^{(0)}$ is a positive self-similar Markov process which starts from 0 continuously, it fulfills the Feller property on $[0, \infty)$ and possesses the same transition functions as $Y^{(x)}$, $x > 0$.

According to Caballero and Chaumont [5], necessary and sufficient conditions for the weak convergence of $Y^{(x)}$ on the Skorokhod's space are: X is not arithmetic, $\mu_+ := \mathbb{E}(H_1) < \infty$ and

$$\mathbb{E}\left(\log^+ \int_0^{\tau_1^+} \exp\{\alpha X_s\}\, ds\right) < \infty, \tag{5.2}$$

where τ_x^+ is the first passage time above $x \ge 0$. Recently, Chaumont et al. [10] proved that the additional hypothesis (5.2) is always satisfied whenever the Lévy process X is not arithmetic and $\mu_+ < \infty$.

For $b > x \ge 0$, we set

$$T_b^{(x)} = \inf\{t \ge 0 : Y_t^{(x)} \ge b\} \quad \text{and} \quad M_t^{(x)} = \sup_{0 \le s \le t} Y_s^{(x)}.$$

The first result of this section consist of computing the law of the triplet $(M^{(x)}_{T_b^{(x)}-}, Y^{(x)}_{T_b^{(x)}-}, Y^{(x)}_{T_b^{(x)}})$ and may be considered as a corollary to Theorem 1. Recall that we drop the dependency on x in the aforementioned random variable when the point of issue is indicated in the measure.

COROLLARY 4. *For $0 < x < b$, we have on $u \in [x, b)$, $0 < v \le u$ and $w > b$*

$$\begin{aligned}
&\mathbf{P}_x(M_{T_b-} \in du, Y_{T_b-} \in dv, Y_{T_b} \in dw) \\
&\quad = V\big(\log(b/x) + du/u\big)\widehat{V}\big(\log(u/b) - dv/v\big)\Pi_X\big(dw/w - \log(v/b)\big),
\end{aligned} \tag{5.3}$$

where the equality holds up to a multiplicative constant.

Moreover, if X is not arithmetic and $\mu_+ < \infty$, we have on $0 < v \le u < b$ and $w > b$,

$$\begin{aligned}
&\mathbf{P}_0(M_{T_b-} < u, Y_{T_b-} < v, Y_{T_b} > w) \\
&\quad = \frac{1}{\mu_+}\int_0^{\log(u/v)} dy \int_{[y,\infty)} \widehat{V}(dl)\overline{\Pi}_X^+\big(\log(w/v) + l - y\big) \\
&\qquad + \frac{1}{\mu_+}\int_{\log(b/v)}^{\infty} dy \overline{\Pi}_H\big(\log(w/b) + y\big).
\end{aligned} \tag{5.4}$$

PROOF. Suppose that $F : \mathbb{R}_+^3 \to \mathbb{R}_+$ is a measurable and bounded function such that $F(\cdot, \cdot, b) = 0$. From the Lamperti representation, it is clear that

$$\begin{aligned}
&\mathbf{E}_x(F(M_{T_b-}, Y_{T_b-}, Y_{T_b})) \\
&\quad = \mathbb{E}\Big(F(x\exp\{\overline{X}_{\tau^+_{\log b/x}-}\}, x\exp\{X_{\tau^+_{\log b/x}-}\}, x\exp\{X_{\tau^+_{\log b/x}}\})\Big) \\
&\quad = \mathbb{E}\Big(F\Big(be^{\overline{X}_{\tau^+_{\log b/x}-}-\log(b/x)}, be^{X_{\tau^+_{\log b/x}-}-\log(b/x)}, be^{X_{\tau^+_{\log b/x}}-\log(b/x)}\Big)\Big).
\end{aligned}$$

Therefore, the identity (5.3) follows using Theorem 1 and straightforward computations.

Now, let $F : \mathbb{R}_+^3 \to \mathbb{R}_+$ be a bounded and continuous function such that $F(\cdot, \cdot, b) = 0$ and suppose that X is not arithmetic and $\mu_+ < \infty$. From the scaling property, we have that for every positive real constant c,

$$\mathbf{E}_0(F(M_{T_b-}, Y_{T_b-}, Y_{T_b})) = \mathbf{E}_0(F(c^{-1}M_{T_{cb}-}, c^{-1}Y_{T_{cb}-}, c^{-1}Y_{T_{cb}})).$$

Let $0 < \varepsilon < cb$, hence

$$\begin{aligned}
&\mathbf{E}_0(F(c^{-1}M_{T_{cb}-}, c^{-1}Y_{T_{cb}-}, c^{-1}Y_{T_{cb}})) \\
&\quad = \mathbf{E}_0\big(F(c^{-1}M_{T_{cb}-}, c^{-1}Y_{T_{cb}-}, c^{-1}Y_{T_{cb}}), T_\varepsilon < T_{bc}\big) \\
&\qquad + \mathbf{E}_0\big(F(c^{-1}M_{T_{cb}-}, c^{-1}Y_{T_{cb}-}, c^{-1}Y_{T_{cb}}), T_\varepsilon = T_{cb}\big).
\end{aligned} \tag{5.5}$$

From the Markov property and the Lamperti representation, the first term of the right-hand side of (5.5) satisfies that

$$\begin{aligned}
&\mathbf{E}_0\big(F(c^{-1}M_{T_{cb}-}, c^{-1}Y_{T_{cb}-}, c^{-1}Y_{T_{cb}}), T_\varepsilon < T_{bc}\big) \\
&\quad = \int_\varepsilon^{cb} \mathbf{P}_0(Y_{T_\varepsilon} \in dx)\mathbf{E}_x(F(c^{-1}M_{T_{cb}-}, c^{-1}Y_{T_{cb}-}, c^{-1}Y_{T_{cb}}))
\end{aligned} \tag{5.6}$$

$$= \int_{\varepsilon}^{cb} \mathbf{P}_0(Y_{T_\varepsilon} \in dx)\mathbb{E}\Big(F\Big(be^{\overline{X}_{\tau^+_{\log cb/x}-}-\log(cb/x)},$$
$$be^{X_{\tau^+_{\log cb/x}-}-\log(cb/x)}, be^{X_{\tau^+_{\log cb/x}}-\log(cb/x)}\Big)\Big).$$

On the other hand, since F is bounded by some positive constant, say $k > 0$, and the scaling property, we get

$$0 \le \mathbf{E}_0\big(F(c^{-1}M_{T_{cb}-}, c^{-1}Y_{T_{cb}-}, c^{-1}Y_{T_{cb}}), T_\varepsilon = T_{cb}\big) \le k\mathbf{P}_0(T_{c^{-1}\varepsilon} = T_b).$$

Hence, the second term of the right-hand side of (5.5) goes to 0, as c tends to ∞, since $\lim_{c\to\infty} \mathbf{P}_0(T_{c^{-1}\varepsilon} = T_b) = 0$. From identity (5.6), Theorem 3 part (i) and the dominated convergence theorem, we deduce

$$\lim_{c\to\infty} \mathbf{E}_0\big(F(c^{-1}M_{T_{cb}-}, c^{-1}Y_{T_{cb}-}, c^{-1}Y_{T_{cb}}), T_\varepsilon < T_{bc}\big)$$
$$= \lim_{c\to\infty} \mathbb{E}\Big(F\Big(be^{\overline{X}_{\tau^+_{\log cb/x}-}-\log(cb/x)},$$
$$be^{X_{\tau^+_{\log cb/x}-}-\log(cb/x)}, be^{X_{\tau^+_{\log cb/x}}-\log(cb/x)}\Big)\Big).$$

Putting the pieces together, we conclude that

$$\mathbf{E}_0(F(M_{T_b-}, Y_{T_b-}, Y_{T_b}))$$
$$= \lim_{y\to\infty} \mathbb{E}\Big(F\Big(be^{\overline{X}_{\tau^+_{\log y}-}-\log y}, be^{X_{\tau^+_{\log y}-}-\log y}, be^{X_{\tau^+_{\log y}}-\log y}\Big)\Big).$$

The identity (5.4) follows from (3.2) after some straightforward computations. $\square$

Let $x \ge 0$ and take $b > 0$. We set

$$\sigma_b^{(x)} = \sup\{s \ge 0 : Y_s^{(x)} \le b\} \quad \text{and} \quad J_t^{(x)} = \inf_{s\ge t} Y_s^{(x)}.$$

The next result deals with the computation of the law of $(1/J_0^{(x)}, Y^{(x)}_{\sigma_b^{(x)}-}, Y^{(x)}_{\sigma_b^{(x)}}, J^{(x)}_{\sigma_b^{(x)}})$, for $x > 0$.

COROLLARY 5. *Suppose that the underlying Lévy process X is regular for both $(0,\infty)$ and $(-\infty, 0)$. For $x, b > 0$, we have on $v \ge x^{-1} \vee b^{-1}$, $v^{-1} < y < b$ and $b < u \le w < \infty$*

$$\begin{aligned} &\mathbf{P}_x(1/J_0 \in dv, Y_{\sigma_b-} \in dy, Y_{\sigma_b} \in dw, J_{\sigma_b} \in du) \\ &\quad = \widehat{V}(dv/v)V\big(\log(bv) + dy/y\big) \\ &\qquad \times \Pi_X\big(dw/w - \log(y/b)\big)\widehat{V}\big(\log(w/b) - du/u\big), \end{aligned} \tag{5.7}$$

where the equality holds up to a multiplicative constant.

PROOF. We first suppose that $0 < b \le x$ and take $F : \mathbb{R}_+^4 \to \mathbb{R}_+$ a measurable and bounded function such that $F(\cdot, \cdot, \cdot, b) = 0$. From the Lamperti representation, it is clear that

$$\begin{aligned}
&\mathbf{E}_x\big(F(1/J_0, Y_{\sigma_b -}, Y_{\sigma_b}, J_{\sigma_b})1_{\{J_0 < b\}}\big)\\
&\quad= \mathbb{E}\big(F(x^{-1}e^{-\underline{X}_\infty}, xe^{X_{U_{\log b/x}-}}, xe^{X_{U_{\log b/x}}}, xe^{\underset{\rightarrow}{X}_{U_{\log b/x}}})1_{\{\underline{X}_\infty < \log b/x\}}\big)\\
&\quad= \mathbb{E}\big(F(x^{-1}e^{-\underline{X}_\infty}, xe^{\underline{X}_\infty}e^{\tilde{X}_{\tilde{U}_{\log b/x - \underline{X}_\infty}-}},\\
&\qquad\qquad xe^{\underline{X}_\infty}e^{\tilde{X}_{\tilde{U}_{\log b/x - \underline{X}_\infty}}}, xe^{\underline{X}_\infty}e^{\underset{\rightarrow}{\tilde{X}}_{\tilde{U}_{\log b/x - \underline{X}_\infty}}})1_{\{\underline{X}_\infty < \log b/x\}}\big),
\end{aligned}$$

where $\tilde{X} = (X_{G_\infty + t} - X_{G_\infty}, t \ge 0)$, G_∞ is the time when the process X reaches is global minimum and $\tilde{U}_x$ denotes the last passage time of $\tilde{X}$ above the level x. In particular, note that when 0 is regular for $(-\infty, 0)$ and for $(0, \infty)$ the following identity holds $X_{G_\infty} = \underline{X}_\infty$.

On the other hand, from Millar's path decomposition (see Theorem 3.1 in [28]) we deduce that

$$\begin{aligned}
&\mathbb{E}\Big(F\Big(x^{-1}e^{-\underline{X}_\infty}, xe^{\underline{X}_\infty}e^{\tilde{X}_{\tilde{U}_{\log b/x - \underline{X}_\infty}-}},\\
&\qquad xe^{\underline{X}_\infty}e^{\tilde{X}_{\tilde{U}_{\log b/x - \underline{X}_\infty}}}, xe^{\underline{X}_\infty}e^{\underset{\rightarrow}{\tilde{X}}_{\tilde{U}_{\log b/x - \underline{X}_\infty}}}\Big)1_{\{\underline{X}_\infty < \log b/x\}}\Big)\\
&\quad= \int_{\log x/b}^\infty \widehat{V}(dv)\mathbb{E}^\uparrow\Big(F\Big(x^{-1}e^v, xe^{-v}e^{X_{U_{\log b/x + v}-}},\\
&\qquad\qquad xe^{-v}e^{X_{U_{\log b/x + v}}}, xe^{-v}e^{\underset{\rightarrow}{X}_{U_{\log b/x + v}}}\Big)\Big).
\end{aligned}$$

Then our assertion follows from Theorem 2 and straightforward computations.

The case when $0 < x < b$ is much simpler, since we do not need to decompose the process at its global infimum. We may proceed as above, using Lamperti representation and then Corollary 2 to get (5.7). $\square$

Finally, we are interested in computing the law of $(Y^{(0)}_{\sigma_b^{(0)}-}, Y^{(0)}_{\sigma_b^{(0)}}, J^{(0)}_{\sigma_b^{(0)}})$. The distribution of $Y^{(0)}_{\sigma_b^{(0)}-}$ has been recently characterized in [11], as follows: $b^{-1}Y^{(0)}_{\sigma_b^{(0)}-} \overset{(d)}{=} e^{-\mathfrak{U}Z}$, where $\mathfrak{U}$ and Z are independent random variables, $\mathfrak{U}$ is uniformly distributed over [0, 1] and the law of Z is given by

$$\mathbb{P}(Z > u) = \frac{1}{\mu_+}\int_u^\infty s\Pi_H(ds), \qquad u \ge 0.$$

Before we state the last result of this section, let us recall a path decomposition, introduced in [11], of the positive self-similar Markov process $(Y, \mathbf{P}_0)$ time-reversed at last passage which is associated to the Lévy process X. Fix a decreasing sequence $(x_n, n \geq 0)$ of positive real numbers which tends to 0. From Corollary 1 in [11], we have

$$\left(Y^{(0)}_{(\sigma^{(0)}_{x_n}-t)-}, 0 \leq t \leq \sigma^{(0)}_{x_n} - \sigma^{(0)}_{x_{n+1}}\right) = \left(\Gamma_n \exp\{X^n_{\theta^n(t/\Gamma_n)}\}, 0 \leq t \leq H_n\right), \qquad n \geq 0,$$

where the processes $X^n, n \geq 0$ are mutually independent and have the same law as $\widehat{X} = -X$. Moreover the sequence $(X^n, n \geq 0)$ is independent of $Y_{\sigma^{(0)}_{x_0}-} := \Gamma_0$ and

$$\begin{cases} \theta^n(t) = \inf\left\{s : \int_0^s \exp\{X^n_u\}\,du \geq t\right\}, \\ H_n = \Gamma_n \int_0^{\tau^n(\log(x_{n+1}/\Gamma_n))} \exp\{X^n_s\}\,ds, \\ \Gamma_{n+1} = \Gamma_n \exp\{X^n_{\tau^n(\log(x_{n+1}/\Gamma_n))}\}, \qquad n \geq 0, \\ \tau^n(z) = \inf\{t : X^n_t \leq z\}. \end{cases}$$

Moreover for each n, Γ_n is independent of X^n and

$$x_n^{-1}\Gamma_n \overset{(d)}{=} x_1\Gamma_1. \tag{5.8}$$

PROPOSITION 1. *Let $b > 0$ and assume that X is not arithmetic and $\mu_+ < \infty$. Then the following identity holds*:

$$\begin{aligned} &\mathbb{P}_0(Y_{\sigma_b^-} < \omega, Y_{\sigma_b} > v, J_{\sigma_b} > u) \\ &\quad = \frac{1}{\mu_+}\int_0^{\log(v/u)} dx \int_{[x,\infty)} \widehat{V}(dl)\overline{\Pi}^+_X\big(\log(v/\omega) + l - x\big) \\ &\qquad + \frac{1}{\mu_+}\int_{\log(v/b)}^{\infty} dx\,\overline{\Pi}_H\big(\log(b/\omega) + x\big), \end{aligned}$$

where $0 < \omega \leq b \leq u \leq v$.

PROOF. Take a decreasing sequence $(x_n, n \geq 0)$ of positive real numbers converging to 0 and such that $x_0 > b$ and $x_1 = b$. By the scaling property, it is clear that for each $n \geq 1$,

$$\begin{aligned} &x_n^{-1} J^{(0)}_{\sigma^{(0)}_{x_n}} \overset{(d)}{=} x_1^{-1} J^{(0)}_{\sigma^{(0)}_{x_1}}, \qquad x_n^{-1} Y^{(0)}_{\sigma^{(0)}_{x_n}} \overset{(d)}{=} x_1^{-1} Y^{(0)}_{\sigma^{(0)}_{x_1}} \quad \text{and} \\ &x_n^{-1} Y^{(0)}_{\sigma^{(0)}_{x_n}-} \overset{(d)}{=} x_1^{-1} Y^{(0)}_{\sigma^{(0)}_{x_1}-}. \end{aligned} \tag{5.9}$$

Now using the path decomposition of the process $Y^{(0)}$ reversed at $\sigma^{(0)}_{x_0}$ described above, we deduce that the first identity in law in (5.9) can be written as follows:

$$x_n^{-1}\Gamma_{n-1} e^{\underline{X}^{n-1}_{\tau^{n-1}(\log(x_n/\Gamma_{n-1}))-}} \overset{(d)}{=} x_1^{-1} J^{(0)}_{\sigma^{(0)}_{x_1}},$$

where $\underline{X}_t^{n-1} = \inf_{0\le u\le t} X_u^{n-1}$. Similarly, we have that

$$x_n^{-1}\Gamma_{n-1}e^{X^{n-1}_{\tau^{n-1}(\log(x_n/\Gamma_{n-1}))-}} \stackrel{(d)}{=} x_1^{-1}Y^{(0)}_{\sigma^{(0)}_{x_1}} \quad \text{and} \quad x_n^{-1}\Gamma_n \stackrel{(d)}{=} x_1^{-1}Y^{(0)}_{\sigma^{(0)}_{x_1}-}.$$

Recall that $x_1^{-1}Y^{(0)}_{\sigma^{(0)}_{x_1}-} = e^{-\mathfrak{U}Z}$. By the independence of X^{n-1} and Γ_{n-1} and the identity (5.8), we deduce

$$x_1^{-1}J^{(0)}_{\sigma^{(0)}_{x_1}} \stackrel{(d)}{=} e^{\underline{X}^{n-1}_{\tau^{n-1}(\log(x_n/\Gamma_{n-1}))-}-\log(x_n/\Gamma_{n-1})} \stackrel{(d)}{=} e^{\underline{\widehat{X}}_{\widehat{\tau}(\log(x_n/x_{n-1})-\mathfrak{U}Z)-}-\log(x_n/x_{n-1})+\mathfrak{U}Z}.$$

From the same arguments as above, we have that

$$x_1^{-1}Y^{(0)}_{\sigma^{(0)}_{x_1}} \stackrel{(d)}{=} e^{X^{n-1}_{\tau^{n-1}(\log(x_n/\Gamma_{n-1}))-}-\log(x_n/\Gamma_{n-1})} \stackrel{(d)}{=} e^{\widehat{X}_{\widehat{\tau}(\log(x_n/x_{n-1})-\mathfrak{U}Z)-}-\log(x_n/x_{n-1})+\mathfrak{U}Z}.$$

Then by taking $x_n = be^{-n^2}$ for $n \ge 2$, we deduce from the above equalities that $\log(x_1^{-1}J^{(0)}_{\sigma^{(0)}_{x_1}})$ and $\log(x_1^{-1}X^{(0)}_{\sigma^{(0)}_{x_1}})$ have the same limit as the limit undershoot of the processes $(\underline{\widehat{X}}_t, t \ge 0)$ and X, respectively, that is,

$$\underline{\widehat{X}}_{\widehat{\tau}(x)-} - x \to \log\Bigl(x_1^{-1}J^{(0)}_{\sigma^{(0)}_{x_1}}\Bigr) \quad \text{and} \quad x - \widehat{X}_{\widehat{\tau}(x)-} \to \log\Bigl(x_1^{-1}Y^{(0)}_{\sigma^{(0)}_{x_1}}\Bigr),$$

in law as x tends to $-\infty$. Hence, Theorem 3 part (i) gives us the desired result. □

6. Some explicit examples. We conclude our exposition by offering a number of explicit examples. A significant number of these examples are the result of interplaying the role of a stable process until it first exits $(0,\infty)$, and conditioned versions thereof, as both are self-similar Markov process as well as (Doob h-transforms of) a Lévy process. In this respect, the routine calculations in the previous section will prove to have been very useful. Note, it is straightforward to check that all the Lévy processes mentioned below are regular for both $(0,\infty)$ and $(-\infty, 0)$.

6.1. *Conditioned stable processes and last passage times.* Suppose that X is a stable Lévy process with index $\alpha \in (0,2)$, that is, a Lévy process satisfying the scaling property with index α. It is known that its Lévy measure is given by

$$\Pi_X(dx) = 1_{\{x>0\}}\frac{c_+}{x^{1+\alpha}}\,dx + 1_{\{x<0\}}\frac{c_-}{x^{1+\alpha}}\,dx,$$

where c_+ and c_- are two nonnegative real numbers (see, e.g., [1]). To avoid trivialities, we assume $c_+ > 0$.

It is known (cf. Bertoin [1]) that the ladder process H of a stable process of index α is a stable subordinator with index $\alpha\rho$, where $\rho = \mathbb{P}(X_1 \ge 0)$ (positivity parameter) and, hence, up to a multiplicative constant $\kappa(0,\beta) = \beta^{\alpha\rho}$ for $\beta \ge 0$. In a similar way, up to a multiplicative constant $\widehat{\kappa}(0,\beta)=\beta^{\alpha(1-\rho)}$. From (1.1) it can easily be shown that (up to a multiplicative constant)

$$V(dx)=\frac{x^{\alpha\rho-1}}{\Gamma(\alpha\rho)}\,dx \quad \text{and} \quad \widehat{V}(dx)=\frac{x^{\alpha(1-\rho)-1}}{\Gamma(\alpha(1-\rho))}\,dx.$$

The form of the law of the triple law of undershoots and overshoot for a stable process can be read from (4.3).

Marginalizing the quintuple law for the stable process conditioned to stay positive (see Theorem 2), we now obtain the following new identity.

COROLLARY 6. *For $0<y\le x$ and $0<u\le w$,*

$$\begin{aligned}&\mathbb{P}^{\uparrow}(\underrightarrow{X}_{U_x}-x\in du, x-X_{U_x-}\in dy, X_{U_x}-x\in dw)\\&\quad=\frac{\sin(\pi\alpha\rho)}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))}\frac{(x-y)^{\alpha\rho-1}(w-u)^{\alpha(1-\rho)-1}}{(w+y)^{\alpha+1}}\,du\,dy\,dw.\end{aligned}$$

Note that the normalizing constant above is chosen to make the density on the right-hand side a distribution. In particular, stable processes do not creep and hence from Tanaka's path construction, we deduce that it is not necessary to take care of an atom on the event $\{X_{U_x}=x\}$.

When the stable process conditioned to stay positive starts from $z>0$, Corollary 1 give us the following identity.

COROLLARY 7. *For $x>0$, $0<v<z\wedge x$, $0<y\le x-v$ and $0<u\le w$,*

$$\begin{aligned}&\mathbb{P}_z^{\uparrow}(\underline{X}_{\infty}\in dv, \underrightarrow{X}_{U_x}-x\in du, x-X_{U_x-}\in dy, X_{U_x}-x\in dw)\\&\quad=K_1(x,z)\frac{(z-v)^{\alpha(1-\rho)-1}(x-v-y)^{\alpha\rho-1}(w-u)^{\alpha(1-\rho)-1}}{z^{\alpha(1-\rho)}(w+y)^{\alpha+1}}\,dv\,du\,dy\,dw.\end{aligned}$$

The normalizing constant $K_1(x,z)$ (which depends on x and z) makes the right-hand side of the previous identity a distribution and following a quadruple integral can be shown to be

$$K_1(x,z)=\frac{\sin(\pi\alpha\rho)}{\pi}\frac{\alpha(1-\rho)\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))}\left(1-\left(1-\frac{z\wedge x}{z}\right)^{\alpha(1-\rho)}\right)^{-1}.$$

6.2. *Lamperti-stable processes*: I. A particular family of Lévy processes which will be of interest to us in this and subsequent examples are Lamperti-stable process with characteristics (ϱ,β,γ) where $\varrho\in(0,2)$ and $\beta,\gamma\le\varrho+1$. Such Lévy process have no Gaussian component and their Lévy measure is of the type

$$1_{\{x>0\}}\frac{c_+e^{\beta x}}{(e^x-1)^{1+\varrho}}\,dx+1_{\{x<0\}}\frac{c_-e^{-\gamma x}}{(e^{-x}-1)^{1+\varrho}}\,dx, \tag{6.1}$$

where c_+ and c_- are two nonnegative real numbers. We refer to [6] for a proper definition and [4, 9] for more details in what follows. We also mention the work of [24] in which a larger class of Lévy processes (called the β-class) is defined. We shall predominantly be concerned with the case that $c_+ > 0$. In the forthcoming text we shall also make reference to Lamperti-stable subordinators with characteristics (ϱ, γ). In that case we mean a (possibly killed) subordinator which has no drift term and Lévy measure of the form

$$1_{\{x>0\}} c \frac{e^{\gamma x}}{(e^x - 1)^{1+\varrho}}\, dx \tag{6.2}$$

for $c > 0$, $\gamma \leq 1 + \varrho$ and $\varrho \in (0, 1)$.

Lamperti-stable processes occur naturally when considering an α-stable process conditioned to stay positive. Indeed, the latter processes are self-similar and never hit the origin and hence respect the Lamperti representation (5.1). More formally (keeping with the same notation as in the previous subsection) when X is issued from $x > 0$ we may write

$$X_t = x \exp\{\xi^{\uparrow}_{\theta(tx^{-\alpha})}\}, \tag{6.3}$$

where for $t > 0$,

$$\theta(t) = \inf\left\{s \geq 0 : \int_0^s \exp\{\alpha \xi^{\uparrow}_u\}\, du \geq t\right\}.$$

Moreover, the process $\xi^{\uparrow} = (\xi^{\uparrow}_t, t \geq 0)$ is a Lévy process started from 0 whose law does not depend on $x > 0$ and in Caballero and Chaumont [4] it was shown that $\xi^{\uparrow}$ is a Lamperti-stable process with characteristics given by $\varrho = \alpha$, $\beta = \alpha(1-\rho) + 1$ and $\gamma = \alpha\rho$, where ρ is the positivity parameter of the associated stable process.

Our objective in this section is to offer some explicit identities for the process $\xi^{\uparrow}$. In that case the first and last passage times, that is, $\tau^+_\cdot$ and $U_\cdot$, as well as the notation for the running maximum and minimum should be understood accordingly. We recall that the stable process conditioned to stay positive drifts to $+\infty$, from the Lamperti representation (6.3) we deduce that the process $\xi^{\uparrow}$ also drifts to $+\infty$. The law of the overall infimum of $\xi^{\uparrow}$ has been computed in Proposition 2 of [4] (see also Corollary 2 in [9]) which is given by

$$\mathbb{P}(-\underline{\xi}^{\uparrow}_{\infty} \leq z) = (1 - e^{-z})^{\alpha(1-\rho)} \qquad \text{for all } z \geq 0,$$

which implies, by Proposition VI.17 in [1], that the renewal function $\widehat{V}$ can be represented as follows:

$$\widehat{V}(z) = \widehat{V}(\infty)(1 - e^{-z})^{\alpha(1-\rho)} \qquad \text{for all } z \geq 0. \tag{6.4}$$

It is well known that $\widehat{V}$ is unique up to a multiplicative constant which depends on the normalization of local time of $\xi^{\uparrow}$ at its infimum. Without loss of generality we may therefore assume in the forthcoming analysis that $\widehat{V}(\infty)$, which is equal to the reciprocal of killing rate of the descending ladder height process, may be taken identically equal to 1. In this respect we shall also assume that $c_+ = 1$.

With these assumptions in place, we find by (1.1) that

$$\widehat{\kappa}(0,\lambda) = \frac{1}{\Gamma(\alpha(1-\rho)+1)} \frac{\Gamma(\alpha(1-\rho)+1+\lambda)}{\Gamma(\lambda+1)} \qquad \text{for all } \lambda \geq 0.$$

Then, according to Corollary 1 in [6], the descending ladder height subordinator $\widehat{H}$ is a killed Lamperti-stable subordinator with characteristics $(\alpha(1-\rho), 0)$, that is, a subordinator whose Lévy measure is given by

$$\Pi_{\widehat{H}}(dx) = \frac{\sin(\pi\alpha(1-\rho))}{\pi} \frac{dx}{(e^x-1)^{1+\alpha(1-\rho)}}, \qquad x > 0,$$

and killed at unit rate.

On the other hand from (3.1), we have that the Lévy measure of the upward ladder height subordinator H satisfies

$$\overline{\Pi}_H(x) = \alpha(1-\rho)\int_0^\infty dy(1-e^{-y})^{\alpha(1-\rho)-1}e^{-y} \times \int_{x+y}^\infty \frac{e^{(\alpha(1-\rho)+1)u}}{(e^u-1)^{\alpha+1}}\,du, \qquad x > 0.$$

Performing the above integral, we get

$$\overline{\Pi}_H(x) = \frac{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho)+1)}{\Gamma(\alpha+1)} \frac{1}{(e^x-1)^{\alpha\rho}},$$

which implies that H is a subordinator whose Lévy measure is given by

$$\Pi_H(dx) = \frac{\Gamma(\alpha\rho+1)\Gamma(\alpha(1-\rho)+1)}{\Gamma(\alpha+1)} \frac{e^x}{(e^x-1)^{1+\alpha\rho}}\,dx,$$

that is to say a Lamperti-stable subordinator with characteristics $(\alpha\rho, 1)$. Since the stable process conditioned to stay positive does not creep, we deduce that $\xi^\uparrow$ does not creep either and from Theorem VI.19 in [1] the subordinator H has no drift. Hence from Corollary 1 in [6], the Laplace exponent of H is as follows:

$$\kappa(0,\lambda) = \frac{\pi}{\sin(\pi\alpha\rho)} \frac{\Gamma(\alpha(1-\rho)+1)}{\Gamma(\alpha+1)} \frac{\Gamma(\lambda+\alpha\rho)}{\Gamma(\lambda)}, \qquad \lambda \geq 0,$$

which implies, from (1.1) , that

$$V(dx) = \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho)+1)}(1-e^{-x})^{\alpha\rho-1}\,dx. \tag{6.5}$$

It is important to note that the above discussion provides a new explicit example of the spatial Wiener–Hopf factorization which we formally state as a proposition.

PROPOSITION 2. *For any Lamperti-stable process $\xi^{\uparrow}$ with characteristics $(\alpha, \alpha(1-\rho)+1, \alpha\rho)$, its characteristic exponent, $\Psi_{\xi^{\uparrow}}(\lambda) := -\log \mathbb{E}(e^{i\lambda\xi_1^{\uparrow}})$, enjoys the following Wiener–Hopf factorization*:

$$\begin{aligned}\Psi_{\xi^{\uparrow}}(\lambda) &= \frac{\pi}{\sin(\pi\alpha\rho)\Gamma(\alpha+1)} \frac{\Gamma(-i\lambda+\alpha\rho)\Gamma(\alpha(1-\rho)+1+i\lambda)}{\Gamma(-i\lambda)\Gamma(i\lambda+1)} \\ &= \frac{\pi}{\sin(\pi\alpha\rho)} \frac{\Gamma(\alpha(1-\rho)+1)}{\Gamma(\alpha+1)} \frac{\Gamma(-i\lambda+\alpha\rho)}{\Gamma(-i\lambda)} \\ &\quad \times \frac{1}{\Gamma(\alpha(1-\rho)+1)} \frac{\Gamma(\alpha(1-\rho)+1+i\lambda)}{\Gamma(i\lambda+1)}\end{aligned}$$

for $\lambda \in \mathbb{R}$ where the first equality holds up to a multiplicative constant.

Note that the above factorization also provides an alternative way of computing the characteristic exponent of such Lamperti-stable processes to the methods employed, for example, in [6] and [29]. This factorization should also be seen as a special case of the Wiener–Hopf factorization of the β-class of Lévy processes appearing in the concurrent work of Kuznetsov [24].

Now that we are in possession of the potential measures V and $\widehat{V}$, we may marginalize the quintuple law at first passage times (Theorem 1) and obtain a new identity for the Lamperti process $\xi^{\uparrow}$ which is given below.

COROLLARY 8. *For $y \in [0, x]$, $v \geq y$ and $u > 0$,*

$$\begin{aligned}&\mathbb{P}(\xi^{\uparrow}_{\tau_x^+} - x \in du, x - \xi^{\uparrow}_{\tau_x^+ -} \in dv, x - \overline{\xi}^{\uparrow}_{\tau_x^+ -} \in dy) \\ &\quad = \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))} (1-e^{-x+y})^{\alpha\rho-1}(1-e^{-v+y})^{\alpha(1-\rho)-1} \\ &\quad\quad \times e^{-v+y} e^{(\alpha(1-\rho)+1)(u+v)} (e^{u+v}-1)^{-\alpha-1}\, dy\, dv\, du.\end{aligned}$$

Similarly, from Corollary 2, we obtain a quadruple law for the last passage time of $\xi^{\uparrow}$.

COROLLARY 9. *For $v > 0$, $0 \leq y < x + v$, $w \geq u > 0$,*

$$\begin{aligned}&\mathbb{P}(-\underline{\xi}^{\uparrow}_{\infty} \in dv, \xi^{\uparrow}_{\underrightarrow{U}_x} - x \in du, x - \xi^{\uparrow}_{U_x -} \in dy, \xi^{\uparrow}_{U_x} - x \in dw) \\ &\quad = \frac{\alpha(1-\rho)\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))} (1-e^{-x-v+y})^{\alpha\rho-1} \\ &\quad\quad \times \big((1-e^{-v})(1-e^{-w+u})\big)^{\alpha(1-\rho)-1} e^{-v-w+u} e^{(\alpha(1-\rho)+1)(y+w)} \\ &\quad\quad \times (e^{y+w}-1)^{-1-\alpha}\, dv\, dy\, dw\, du.\end{aligned}$$

Further, we may compute the triple law at last passage times for the Lamperti-stable $\xi^{\uparrow}$ conditioned to stay positive starting from 0 with the help of Theorem 2.

COROLLARY 10. *For* $0 < y \le x$ *and* $0 < u \le w$

$$\begin{aligned}
&\mathbb{P}^{\uparrow}(\underset{\rightarrow}{\xi}^{\uparrow}_{U_x} - x \in du, x - \xi^{\uparrow}_{U_x-} \in dy, \xi^{\uparrow}_{U_x} - x \in dw)\\
&\quad = \frac{\sin(\pi\alpha\rho)}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))}(1-e^{-x+y})^{\alpha\rho-1}(1-e^{-w+u})^{\alpha(1-\rho)-1}\\
&\qquad \times e^{-w+u}e^{(\alpha(1-\rho)+1)(y+w)}(e^{y+w}-1)^{-\alpha-1}\,du\,dy\,dw.
\end{aligned}$$

Moreover, when the Lamperti-stable process conditioned to stay positive starts from a positive state, Corollary 1 give us the following explicit identity.

COROLLARY 11. *For* $x > 0$, $0 < v < z \wedge x$, $0 < y \le x - v$ *and* $0 < u \le w$

$$\begin{aligned}
&\mathbb{P}^{\uparrow}_z(\underline{\xi}^{\uparrow}_{\infty} \in dv, \underset{\rightarrow}{\xi}^{\uparrow}_{U_x} - x \in du, x - \xi^{\uparrow}_{U_x-} \in dy, \xi^{\uparrow}_{U_x} - x \in dw)\\
&\quad = K_2(x,y)(1-e^{-x+v+y})^{\alpha\rho-1}\big((1-e^{-z+v})(1-e^{-w+u})\big)^{\alpha(1-\rho)-1}\\
&\qquad \times e^{(\alpha(1-\rho)+1)(y+w)}e^{-z-w+v+u}(e^{w+y}-1)^{-\alpha-1}\,dv\,du\,dy\,dw.
\end{aligned}$$

The normalizing constant $K_2(x,z)$ (*which depends on* x *and* z) *makes the right-hand side of the previous identity a distribution and following a quadruple intergal can be shown to be*

$$K_2(x,z) = \frac{\sin(\pi\alpha\rho)}{\pi}\frac{\alpha(1-\rho)\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))}\left(1-\left(0 \vee \frac{1-e^{-z+x}}{1-e^{-z}}\right)^{\alpha(1-\rho)}\right)^{-1}.$$

Finally, we note that the process $\xi^{\uparrow}$ is not arithmetic and that

$$\mu_+ = \kappa'(0,0^+) = \frac{\pi}{\sin(\pi\alpha\rho)}\frac{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho)+1)}{\Gamma(\alpha+1)} < \infty.$$

Therefore, from Theorem 3(i), the random variable $(x - \overline{\xi}^{\uparrow}_{\tau_x^+-}, x - \xi^{\uparrow}_{\tau_x^+-}, \xi^{\uparrow}_{\tau_x^+} - x)$ converges weakly toward a nondegenerate random variable which is given in the next corollary.

COROLLARY 12. *For* $0 \le u \le v$, $w \ge 0$.

$$\begin{aligned}
&\lim_{x\to\infty}\mathbb{P}(x - \overline{\xi}^{\uparrow}_{\tau_x^+-} > u, x - \xi^{\uparrow}_{\tau_x^+-} > v, \xi^{\uparrow}_{\tau_x^+} - x > w)\\
&\quad = \frac{\sin(\pi\alpha\rho)}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))}\int_0^{v-u}dy\int_y^{\infty}dz\,e^{-z}(1-e^{-z})^{\alpha(1-\rho)-1}\\
&\qquad\qquad \times \int_{\omega+z+v-y}^{\infty}\frac{e^{(\alpha(1-\rho)+1)l}}{(e^l-1)^{\alpha+1}}\,dl\\
&\qquad + \frac{\sin(\pi\alpha\rho)}{\pi}\int_v^{\infty}\frac{e^{-\alpha\rho(\omega+y)}}{(1-e^{-(\omega+y)})^{\alpha\rho}}\,dy.
\end{aligned}$$

6.3. *Conditioned stable processes and first passage times*. In this example, we are interested in computing the triple law at first passage times of stable processes conditioned to stay positive. Note that the results in Section 2 do not cover this eventuality. However, thanks to the Lamperti transformation, we can recover the required identities from some of the conclusions in the previous subsection. To this end we keep with our earlier notation so that X is a stable process of index $\alpha \in (0,2)$ enjoying positive jumps.

Taking note of the the form of the Lévy measure of $\xi^{\uparrow}$ and the renewal functions (6.4) and (6.5), after some algebra, we get from Corollary 4 the following result.

COROLLARY 13. *Let $b > x > 0$. For $u \in [0, b-x]$, $v \in [u, b)$ and $y > 0$,*

$$\begin{aligned}
&\mathbb{P}_x^{\uparrow}(b - \overline{X}_{\tau_b^+ -} \in du, b - X_{\tau_b^+ -} \in dv, X_{\tau_b^+} - b \in dy)\\
&\quad = \frac{\sin(\pi\alpha\rho)}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))}\\
&\qquad \times \frac{(b-x-u)^{\alpha\rho-1}(v-u)^{\alpha(1-\rho)-1}(b-v)^{\alpha\rho}(y+b)^{\alpha(1-\rho)}}{(b-u)^{\alpha}(y+v)^{\alpha+1}}\,du\,dv\,dy.
\end{aligned}$$

We obtain similarly from Corollary 4 the following formula for the stable process conditioned to stay positive starting from 0:

COROLLARY 14. *For $u \in [0, b]$, $v \in [0, u]$, $w > b > 0$*

$$\begin{aligned}
&\mathbb{P}^{\uparrow}(\overline{X}_{\tau_b^+ -} < u, X_{\tau_b^+ -} < v, X_{\tau_b^+} > w)\\
&\quad = \frac{\sin(\pi\alpha\rho)}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))}\\
&\qquad \times \int_0^{\log(u/v)} dy \int_y^{\infty} dl\,(1-e^{-l})^{\alpha(1-\rho)-1}e^{-l}\\
&\qquad \times \int_{\log(w/v)+l-y}^{\infty} \frac{e^{-\alpha\rho x}}{(1-e^{-x})^{\alpha+1}}\,dx\\
&\qquad + \frac{\sin(\pi\alpha\rho)}{\pi}\int_{\log(b/v)}^{\infty}\frac{b^{\alpha\rho}\,dy}{(e^y w - b)^{\alpha\rho}}.
\end{aligned}$$

6.4. *Lamperti-stable processes*: II. Next we return to Lamperti-stable processes and make use of some of the results in the previous section to push further more explicit identities. To this end, recall that the law of a stable process conditioned to stay positive at time $t > 0$ when issued from $x > 0$ is defined via the transformation

$$\mathbb{P}_x^{\uparrow}(X_t \in dz) = \left(\frac{z}{x}\right)^{\alpha(1-\rho)}\mathbb{P}_x(X_t \in dz, t < \tau_0^-), \qquad t \ge 0, z > 0, \tag{6.6}$$

where $\tau_0^- = \inf\{t > 0 : X_t \leq 0\}$. It is well known that by the optional sampling theorem the latter identity extends to finite stopping times and hence

$$\begin{aligned}
&\mathbb{P}_x^{\uparrow}(b - \overline{X}_{\tau_b^+ -} \in du, b - X_{\tau_b^+ -} \in dv, X_{\tau_b^+} - b \in dy)\\
&\quad = \left(\frac{b+y}{x}\right)^{\alpha(1-\rho)} \mathbb{P}_x(b - \overline{X}_{\tau_b^+ -} \in du, b - X_{\tau_b^+ -} \in dv,\\
&\qquad\qquad X_{\tau_b^+} - b \in dy, \tau_b^+ < \tau_0^-).
\end{aligned}$$

Taking account of the identity established in Corollary 13, we deduce the following new identity which extends the main result of Rogozin [31].

COROLLARY 15. *For $u \in [0, b - x]$, $v \in [u, b)$ and $y > 0$,*

$$\begin{aligned}
&\mathbb{P}_x(b - \overline{X}_{\tau_b^+ -} \in du, b - X_{\tau_b^+ -} \in dv, X_{\tau_b^+} - b \in dy, \tau_b^+ < \tau_0^-)\\
&\quad = \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))}\\
&\qquad \times \frac{x^{\alpha(1-\rho)}(b-x-u)^{\alpha\rho-1}(v-u)^{\alpha(1-\rho)-1}(b-v)^{\alpha\rho}}{(b-u)^{\alpha}(y+v)^{\alpha+1}}\, du\, dv\, dy.
\end{aligned} \tag{6.7}$$

Having established the above corollary, we may now use it to extract even more identities for Lamperti-stable processes. To begin with, we will follow the same line of reasoning used in Theorem 2 in [9] in order to get a similar identity for the Lamperti-stable $\xi^{\uparrow}$. To this end, we set for $-\infty < u \leq 0 < b < \infty$,

$$T_b^{\uparrow +} = \inf\{t \geq 0 : \xi_t^{\uparrow} \geq b\} \quad \text{and} \quad T_u^{\uparrow -} = \inf\{t \geq 0 : \xi_t^{\uparrow} \leq u\}.$$

From the Lamperti representation of $(X, \mathbb{P}^{\uparrow})$ and identity (6.6), we get for $0 < \theta \leq \phi < b - u$ and $\eta > 0$,

$$\begin{aligned}
&\mathbb{P}(b - \overline{\xi^{\uparrow}}_{T_b^{\uparrow +} -} < \theta, b - \xi^{\uparrow}_{T_b^{\uparrow +} -} > \phi, \xi^{\uparrow}_{T_b^{\uparrow +}} - b < \eta, T_b^{\uparrow +} < T_u^{\uparrow -})\\
&\quad = \mathbb{P}_1^{\uparrow}(e^b - \overline{X}_{\tau_{e^b}^+ -} < e^b - e^{b-\theta},\\
&\qquad\qquad e^b - X_{\tau_{e^b}^+ -} > e^b - e^{b-\phi}, X_{\tau_{e^b}^+} - e^b < e^{\eta+b} - e^b, \tau_{e^b}^+ < \tau_{e^u}^-)\\
&\quad = \int_0^{e^b - e^{b-\theta}} dx \int_{e^b - e^{b-\phi}}^{e^b - e^u} dy\\
&\qquad \times \int_0^{e^{\eta+b} - e^b} dz (z + e^b)^{\alpha(1-\rho)} \mathbb{P}_1(e^b - \overline{X}_{\tau_{e^b}^+ -} \in dx, e^b - X_{\tau_{e^b}^+ -} \in dy,\\
&\qquad\qquad X_{\tau_{e^b}^+} - e^b \in dz, \tau_{e^b}^+ < \tau_{e^u}^-)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{e^b - e^{b-\theta}} dx \int_{e^b - e^{b-\phi}}^{e^b - e^u} dy \\
&\quad \times \int_0^{e^{\eta+b} - e^b} dz (z + e^b)^{\alpha(1-\rho)} \mathbb{P}_{1-e^u}(h - \overline{X}_{\tau_h^+ -} \in dx, h - X_{\tau_h^+ -} \in dy, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad X_{\tau_h^+} - h \in dz, \tau_h^+ < \tau_0^-),
\end{aligned}
$$

where $h = e^b - e^u$. From the identity (6.7) and some straightforward computations, we obtain the following identity for $\xi^\uparrow$, which generalizes Theorem 2 in [9].

COROLLARY 16. *For $\theta \in [0, b]$, $\theta \le \phi < b - u$ and $\eta > 0$*

$$
\begin{aligned}
&\mathbb{P}(b - \overline{\xi^\uparrow}_{T_b^{\uparrow+} -} \in d\theta, b - \xi^\uparrow_{T_b^{\uparrow+} -} \in d\phi, \xi^\uparrow_{T_b^{\uparrow+}} - b \in d\eta, T_b^{\uparrow+} < T_u^{\uparrow-}) \\
&\quad = \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))} e^{b(\alpha(1-2\rho)+1)} (1 - e^u)^{\alpha(1-\rho)} \\
&\qquad \times e^{-\theta-\phi} e^{(\alpha(1-\rho)+1)\eta} (e^{b-\theta} - 1)^{\alpha\rho - 1} (e^{-\theta} - e^{-\phi})^{\alpha(1-\rho)-1} \\
&\qquad \times (e^{b-\phi} - e^u)^{\alpha\rho} (e^{b-\theta} - e^u)^{-\alpha} (e^\eta - e^{-\phi})^{-\alpha-1} \, d\theta \, d\phi \, d\eta.
\end{aligned}
$$

According to Caballero and Chaumont [4], stable processes when initiated from a positive position and killed at τ_0^- are also positive self-similar Markov processes. Such processes also enjoy a transformation of the kind (5.1), but the underling Lévy process in the transformation is killed at an independent and exponentially distributed time. In the case at hand, the underlying Lévy process is a Lamperti-stable process with characteristics $(\alpha, 1, \alpha)$ and the killing rate is $c_-\alpha^{-1}$. Let us denote the latter process by ξ^* and set for $-\infty < u \le 0 < b < \infty$,

$$
T_b^{*+} = \inf\{t \ge 0 : \xi_t^* \ge b\} \quad \text{and} \quad T_u^{*-} = \inf\{t \ge 0 : \xi_t^* \le u\}.
$$

Similar arguments to those used above give us the following new identity for ξ^*, which generalize Theorem 3 in [9].

COROLLARY 17. *For $\theta \in [0, b]$, $\theta \le \phi < b - u$ and $\eta > 0$,*

$$
\begin{aligned}
&\mathbb{P}(b - \overline{\xi^*}_{T_b^{*+} -} \in d\theta, b - \xi^*_{T_b^{*+} -} \in d\phi, \xi^*_{T_b^*} - b \in d\eta, T_b^{*+} < T_u^{*-}) \\
&\quad = \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))} \\
&\qquad \times e^{b(1-\alpha\rho)} (1 - e^u)^{\alpha(1-\rho)} e^{-\theta-\phi+\eta} \\
&\qquad \times (e^{b-\theta} - 1)^{\alpha\rho-1} (e^{-\theta} - e^{-\phi})^{\alpha(1-\rho)-1} \\
&\qquad \times (e^{b-\phi} - e^u)^{\alpha\rho} (e^{b-\theta} - e^u)^{-\alpha} (e^\eta - e^{-\phi})^{-\alpha-1} \, d\theta \, d\phi \, d\eta.
\end{aligned}
$$

Finally, we consider the stable process X conditioned to hit 0 continuously. This process is defined as a Doob h-transform with respect to the function $h(x) = \alpha(1-\rho)x^{\alpha(1-\rho)-1}$ which is excessive for the killed stable process at τ_0^-. Moreover, the latter process is also a positive self-similar Markov process. According to Caballero and Chaumont [4] a Lamperti transformation of the kind (5.1) exists where the underlying Lévy process, denoted by $\xi^{\downarrow}$, is a Lamperti-stable process with characteristics $(\alpha, \alpha(1-\rho), \alpha\rho+1)$.

We set for $-\infty < u \le 0 < b < \infty$

$$T_b^{\downarrow +} = \inf\{t \ge 0 : \xi_t^{\downarrow} \ge b\} \quad \text{and} \quad T_u^{\downarrow -} = \inf\{t \ge 0 : \xi_t^{\downarrow} \le u\}.$$

The following new identity for $\xi^{\downarrow}$ follows in a similar spirit to the calculations for $\xi^{\uparrow}$ and generalizes Theorem 4 in [9].

COROLLARY 18. *For $\theta \in [0, b]$, $\theta \le \phi < b - u$ and $\eta > 0$,*

$$\begin{aligned}
&\mathbb{P}(b - \overline{\xi^{\downarrow}}_{T_b^{\downarrow+}-} \in d\theta, b - \xi^{\downarrow}_{T_b^{\downarrow+}-} \in d\phi, \xi^{\downarrow}_{T_b^{\downarrow+}} - b \in d\eta, T_b^{\downarrow+} < T_u^{\downarrow-}) \\
&\qquad = \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))} \\
&\qquad\quad \times e^{\alpha(1-2\rho)b}(1-e^u)^{\alpha(1-\rho)} e^{-\theta-\phi} \\
&\qquad\quad \times e^{\alpha(1-\rho)\eta}(e^{b-\theta}-1)^{\alpha\rho-1}(e^{-\theta}-e^{-\phi})^{\alpha(1-\rho)-1} \\
&\qquad\quad \times (e^{b-\phi}-e^u)^{\alpha\rho}(e^{b-\theta}-e^u)^{-\alpha}(e^{\eta}-e^{-\phi})^{-\alpha-1}\, d\theta\, d\phi\, d\eta.
\end{aligned}$$

6.5. *Lamperti-stable subordinators, philanthropy and hypergeometric Lévy processes.* Here we will use the, previously unexploited, theory of philanthropy due to Vigon [33] in order to construct a new family of Lévy processes, which we shall denote hypergeometric Lévy processes, for which we may compute triple laws at first and last passage times.

According to Vigon's theory of philanthropy, a subordinator is called philanthropist if its Lévy measure has a decreasing density on $\mathbb{R}_+$. Moreover, given any two subordinators H and $\widehat{H}$ which are philanthropists there exist a Lévy process X such that H and $\widehat{H}$ have the same law as the ascending and descending ladder height processes of X, respectively. Moreover, the Lévy measure of X satisfies the following identity:

$$\overline{\Pi}_X^+(x) = \int_0^\infty \Pi_H(x+du)\overline{\Pi}_{\widehat{H}}(u) + \widehat{\delta}\pi_H(x) + \widehat{k}\,\overline{\Pi}_H(x), \qquad x > 0, \tag{6.8}$$

where $(\widehat{k}, \widehat{\delta}, \Pi_{\widehat{H}})$ are the characteristics of $\widehat{H}$, Π_H denotes the Lévy measure of H and π_H its corresponding density. By symmetry, an obvious analog of (6.8) holds for the negative tail $\overline{\Pi}_X^-(x) := \Pi_X(-\infty, x)$, $x < 0$.

Recall that a Lamperti-stable subordinator with characteristics (ϱ, γ) is a (possibly killed) subordinator with no drift component and Lévy measure given by

(6.2). From the form of the latter it is clear that Lamperti-stable subordinators are philanthropists. For simplicity, in what follows we will assume that the constant $c = 1$.

Let $\widehat{H}$ be a Lamperti-stable subordinator with characteristics (ϱ, β) which is killed at rate

$$\frac{\Gamma(1-\varrho)}{\varrho}\frac{\Gamma(1-\beta+\varrho)}{\Gamma(1-\beta)},$$

and H a Lamperti subordinator with characteristics $(\gamma, 1)$ with no killing, where $\varrho, \gamma \in (0, 1)$ and $\beta \le 1$. Let us denote by X the Lévy process whose ascending and descending ladder height processes have the same law as H and $\widehat{H}$, respectively. From (6.8), the Lévy measure of X is such that

$$\begin{aligned}\overline{\Pi}^{+}_{X}(x) &= \int_x^\infty \frac{e^u}{(e^u-1)^{\gamma+1}}\int_{u-x}^\infty \frac{e^{\beta z}}{(e^z-1)^{\varrho+1}}\,dz\,du \\ &\quad + \frac{\Gamma(1-\varrho)}{\gamma\varrho}\frac{\Gamma(1-\beta+\varrho)}{\Gamma(1-\beta)}(e^x-1)^{-\gamma}.\end{aligned}$$

Applying the binomial expansion twice, we obtain that

$$\begin{aligned}&\int_x^\infty \frac{e^u}{(e^u-1)^{\gamma+1}}\int_{u-x}^\infty \frac{e^{\beta z}}{(e^z-1)^{\varrho+1}}\,dz\,du \\ &\quad = \frac{1}{(\varrho+1-\beta)}\sum_{n,k=0}^{\infty}\frac{(\varrho+1)_n(\varrho+1-\beta)_n}{n!(\varrho+2-\beta)_n} \\ &\qquad\qquad \times \frac{(\gamma+1)_k(\gamma+1+\varrho+n-\beta)_k e^{-x(\gamma+k)}}{(\gamma+1+\varrho+n-\beta)k!(\gamma+2+\varrho+n-\beta)_k},\end{aligned}$$

where $(z)_n = \Gamma(z+n)/\Gamma(z)$, $z \in \mathbb{C}$. Putting the pieces together, we may write the Lévy measure of X as follows:

$$\begin{aligned}\overline{\Pi}^{+}_{X}(x) &= \frac{1}{(\varrho+1-\beta)} \\ &\quad \times \sum_{n,k=0}^{\infty}\frac{(\varrho+1)_n(\varrho+1-\beta)_n}{n!(\varrho+2-\beta)_n} \\ &\qquad \times \frac{(\gamma+1)_k(\gamma+1+\varrho+n-\beta)_k e^{-x(\gamma+k)}}{(\gamma+1+\varrho+n-\beta)k!(\gamma+2+\varrho+n-\beta)_k} \\ &\quad + \frac{\Gamma(1-\varrho)}{\gamma\varrho}\frac{\Gamma(1-\beta+\varrho)}{\Gamma(1-\beta)}(e^x-1)^{-\gamma}.\end{aligned}$$

For simplicity, we denote by f for the density of Lévy measure Π_X on $\mathbb{R}_+$ which can be obtained by differentiating the above expression.

It is important to note that the process X has no Gaussian component and that Π_X also satisfies

$$\overline{\Pi}_X^-(x) = \sum_{n,k=0}^{\infty} \frac{(\gamma)_n}{\gamma n!} \frac{(\varrho+1)_k(\gamma+\varrho+1-\beta+n)_k}{k!(\gamma+2+\varrho-\beta+n)_k(\gamma+\varrho+1-\beta+n)} e^{-x(\varrho+1+k-\beta)}$$

for $x < 0$. Moreover the process X drift to ∞, when $\beta < 1$, and oscillates, when $\beta = 1$. In the latter case, the form of the Lévy measure of X is much simpler and is given by

$$\overline{\Pi}_X^+(x) = \frac{\Gamma(\gamma+\varrho)\Gamma(1-\varrho)}{\varrho\Gamma(\gamma+1)} \frac{1}{(e^x-1)^\gamma(1-e^{-x})^\varrho}, \qquad x > 0.$$

We call the process X a hypergeometric Lévy process with characteristics (ϱ, γ, β). When the characteristics of the hypergeometric process are such that $\varrho = \alpha(1-\rho), \gamma = \alpha\rho$ and $\beta = 0$, the process X is the Lamperti-stable process with characteristics $(\alpha, \alpha(1-\rho)+1, \alpha\rho)$ studied in Section 6.2.

From Corollary 1 in [6], we know that the Laplace exponent of $\widehat{H}$ satisfies

$$\widehat{\kappa}(0,\lambda) = \frac{\Gamma(1-\varrho)}{\varrho} \frac{\Gamma(\lambda+1-\beta+\varrho)}{\Gamma(\lambda+1-\beta)}, \qquad \lambda \geq 0,$$

and from (1.1), we deduce that

$$\widehat{V}(dx) = \frac{\varrho \sin(\pi\varrho)}{\pi} e^{(\beta-1)x}(1-e^{-x})^{\varrho-1}\,dx.$$

Similarly for the subordinator H, we have

$$\kappa(0,\lambda) = \frac{\Gamma(1-\gamma)}{\gamma} \frac{\Gamma(\lambda+\gamma)}{\Gamma(\lambda)}, \qquad \lambda \geq 0,$$

and

$$V(dx) = \frac{\gamma \sin(\pi\gamma)}{\pi}(1-e^{-x})^{\gamma-1}\,dx.$$

Hence we identify the following Wiener–Hopf factorization which generalizes Proposition 2.

PROPOSITION 3. *For any hypergeometric Lévy process X with characteristics (ϱ, γ, β), its characteristic exponent, $\Psi_X(\lambda) = -\log \mathbb{E}(e^{i\lambda X_1})$, enjoys the following Wiener–Hopf factorization*:

$$\begin{aligned}\Psi_X(\lambda) &= \frac{\Gamma(1-\gamma)\Gamma(1-\varrho)}{\varrho\gamma} \frac{\Gamma(-i\lambda+\gamma)\Gamma(i\lambda+1-\beta+\varrho)}{\Gamma(-i\lambda)\Gamma(i\lambda+1-\beta)} \\ &= \frac{\Gamma(1-\gamma)}{\gamma} \frac{\Gamma(-i\lambda+\gamma)}{\Gamma(-i\lambda)} \times \frac{\Gamma(1-\varrho)}{\varrho} \frac{\Gamma(i\lambda+1-\beta+\varrho)}{\Gamma(i\lambda+1-\beta)},\end{aligned}$$

where the first equality hold up to a multiplicative constant.

Marginalizing the quintuple law at first passage times (Theorem 1), we obtain one of but many identities for the hypergeometric Lévy process X.

COROLLARY 19. *For $y \in [0, x]$, $v \geq y$ and $u > 0$,*

$$\begin{aligned}
&\mathbb{P}(X_{\tau_x^+} - x \in du, x - X_{\tau_x^+ -} \in dv, x - \overline{X}_{\tau_x^+ -} \in dy) \\
&\quad = \varrho\gamma \frac{\sin(\pi\varrho)\sin(\pi\gamma)}{\pi^2}(1 - e^{-x+y})^{\gamma-1}(1 - e^{-v+y})^{\rho-1} \\
&\qquad \times e^{(\beta-1)(v-y)} f(u+v)\, dy\, dv\, du.
\end{aligned}$$

We leave the reader to amuse him/herself with some of the other related examples which can be obtained from earlier results in this paper.

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A. E. KYPRIANOU
J. C. PARDO
DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF BATH
CLAVERTON DOWN
BATH, BA2 7AY
UNITED KINGDOM
E-MAIL: a.kyprianou@bath.ac.uk
jcpm20@bath.ac.uk

V. RIVERO
CENTRO DE INVESTIGACIÓN
EN MATEMÁTICAS, A.C.
CALLE JALISCO S/N. 36240
GUANAJUATO
MÉXICO
E-MAIL: rivero@cimat.mx