Lecture Notes on the Spin and Loop O(n) models

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Contents

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1 Introduction

The classical spin $O(n)$ model is a model on a d-dimensional lattice in which a vector on the $(n-1)$-dimensional sphere is assigned to every lattice site and the vectors at adjacent sites interact ferromagnetically via their inner product. Special cases include the Ising model ($n = 1$), the XY model ($n = 2$) and the Heisenberg model ($n = 3$). We discuss questions of long-range order (spontaneous magnetization) and decay of correlations in the spin $O(n)$ model for different combinations of the lattice dimension d and the spin dimension n. Among the topics presented are the Mermin-Wagner theorem, the Berezinskii-Kosterlitz-Thouless transition, the infra-red bound and Polyakov's conjecture on the two-dimensional Heisenberg model.

The loop $O(n)$ model is a model for a random configuration of disjoint loops on the hexagonal lattice. The model is parameterized by a loop weight $n \geq 0$ and an edge weight $x \geq 0$. Special cases include self-avoiding walk ($n = 0$), the Ising model ($n = 1$), critical percolation ($n = x = 1$), dimer model ($n = 1, x = \infty$), integer-valued ($n = 2$) and tree-valued (integer $n >= 3$) Lipschitz functions and the hard hexagon model ($n = \infty$). The object of study in the model is the typical structure of loops. We will review the connection of the model with the spin $O(n)$ model and discuss its conjectured phase diagram, emphasizing the many open problems remaining. We then elaborate on recent results for the self-avoiding walk case and for large values of n.

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2 The Spin $O(n)$ model

2.1 Definitions

Let $n \geq 1$ be an integer and let $G = (V(G), E(G))$ be a finite graph. A *configuration* of the *spin $O(n)$ model* (also called *n-vector model*) on G is an assignment $\sigma : V(G) \to \mathbb{S}^{n-1}$ of spins to each vertex of G, where $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ is the $(n-1)$-dimensional unit sphere (simply $\{-1, 1\}$ if $n = 1$). We write

$$\Omega := (\mathbb{S}^{n-1})^{V(G)}$$

for the space of configurations. At inverse temperature $\beta \in [0, \infty)$, configurations are randomly chosen from the probability measure $\mu_{G,n,\beta}$ given by

$$d\mu_{G,n,\beta}(\sigma) := \frac{1}{Z^{\text{spin}}_{G,n,\beta}} \exp\left[\beta \sum_{\{u,v\} \in E(G)} \langle \sigma_u, \sigma_v \rangle\right] d\sigma, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$, the *partition function* $Z^{\text{spin}}_{G,n,\beta}$ is given by

$$Z^{\text{spin}}_{G,n,\beta} := \int_\Omega \exp\left[\beta \sum_{\{u,v\}\in E(G)} \langle \sigma_u, \sigma_v \rangle\right] d\sigma \tag{2}$$

and $d\sigma$ is the uniform probability measure on Ω (i.e., the product measure of the uniform distributions on $\mathbb{S}^{n-1}$ for each vertex in G).

Special cases of the model have names of their own:

- When $n = 1$, spins take values in $\{-1, 1\}$ and the model becomes the famous *Ising model*.
- When $n = 2$, spins take values in the unit circle and the model is called the *XY model* or the *plane rotator model*.
- When $n = 3$, spins take values in the two-dimensional sphere $\mathbb{S}^2$ and the model is called the *Heisenberg model*.
- In a sense, as n tends to infinity the model approaches the *Berlin-Kac spherical model* (which will not be discussed in these notes), see [6, 12, 22] and [4, Chapter 5].

We will sometimes discuss a more general model, in which we replace the inner product in (1) by a function of that inner product. In other words, when the energy of a configuration is measured with a more general pair interaction term. Precisely, given a measurable function $U : [-1, 1] \to \mathbb{R}$, termed the *potential function*, we define the *spin $O(n)$ model with potential U* to be the probability measure $\mu_{G,n,U}$ over configurations $\sigma : V(G) \to \mathbb{S}^{n-1}$ given by

$$d\mu_{G,n,U}(\sigma) := \frac{1}{Z^{\text{spin}}_{G,n,U}} \exp\left[-\sum_{\{u,v\}\in E(G)} U(\langle \sigma_u, \sigma_v \rangle)\right] d\sigma, \tag{3}$$

where the partition function $Z^{\text{spin}}_{G,n,U}$ is defined analogously to (2). Of course, for this to be well defined (i.e., to have finite $Z^{\text{spin}}_{G,n,U}$) some restrictions need to be placed on U but this will always be the case in the models discussed in these notes (*** reference later remark? ***).

One may also impose *boundary conditions* on the model, where the values of certain spins are pre-specified. In addition, one may consider a more general model by adding an *external magnetic field*, in which a vector $s \in \mathbb{R}^n$ is specified and a term of the form $\sum_{v\in V(G)} \langle \sigma_v, s \rangle$ is added to the exponent in the definition of the density (1). We will, however, focus on the version of the models given above.

The graph G is typically taken to be a portion of a d-dimensional lattice (possibly with periodic boundary conditions). When discussing the spin $O(n)$ model in these notes we mostly take

$$G = \mathbb{T}^d_L,$$

where $\mathbb{T}^d_L$ denotes the *d-dimensional discrete torus of side length* $2L$ defined as follows: The vertex set of $\mathbb{T}^d_L$ is

$$V(\mathbb{T}^d_L) := \{-L + 1, -L + 2, \ldots, L - 1, L\}^d \tag{4}$$

and a pair $u, v \in V(\mathbb{T}_L^d)$ is adjacent, written $\{u, v\} \in E(\mathbb{T}_L^d)$, if u and v are equal in all but one coordinate and differ by exactly 1 modulo $2L$ in that coordinate. We write $\|x - y\|_1$ for the *graph distance in* $\mathbb{T}_L^d$ of two vertices $x, y \in V(\mathbb{T}_L^d)$ (for brevity, we suppress the dependence on L in this notation).

The results presented below should admit analogues if the graph G is changed to a different d-dimensional lattice graph with appropriate boundary conditions. However, the presented proofs sometimes require the presence of symmetries in the graph G.

2.2 Main results and conjectures

We will focus on the questions of existence of long-range order and decay of correlations in the spin $O(n)$ model. To this end we shall study the correlation

$$\rho_{x,y} := \mathbb{E}(\langle \sigma_x, \sigma_y \rangle)$$

for a configuration σ randomly chosen from the spin $O(n)$ model at inverse temperature β and two vertices $x, y \in V(\mathbb{T}_L^d)$ with large graph distance $\|x - y\|_1$. This correlation is *always* non-negative (*** ref, also monotonicity in β? ***), but its magnitude behaves very differently for different combinations of the spatial dimension d, spin dimension n and inverse temperature β. The following list summarizes the main results and conjectures. We use the notation $c_\beta, C_\beta, c_{n,\beta}, \ldots$ to denote positive constants whose value depends only on the parameters given in the subscript (and is always independent of the lattice size L) and may change from line to line.

High temperatures and spatial dimension $d = 1$. All the models exhibit *exponential decay of correlations* at high temperature. Precisely, there exists a $\beta_0(d, n) > 0$ such that

$$d, n \geq 1,\ \beta < \beta_0(d, n): \quad \rho_{x,y} \leq C_{d,n,\beta} \exp(-c_{d,n,\beta} \|x - y\|_1) \quad \text{for all } x, y \in V(\mathbb{T}_L^d).$$

This is a relatively simple fact and the main interest is in understanding the behavior at low temperatures. In one spatial dimension ($d = 1$) the exponential decay persists at all positive temperatures. That is,

$$d = 1,\ n \geq 1,\ \beta \in [0, \infty): \quad \rho_{x,y} \leq C_{n,\beta} \exp(-c_{n,\beta} \|x - y\|_1) \quad \text{for all } x, y \in V(\mathbb{T}_L^1).$$

The Ising model $n = 1$. The Ising model exhibits a *phase transition* in all dimensions $d \geq 2$ at some critical temperature $\beta_c(d)$. The transition is from a regime with exponential decay of correlations,

$$d \geq 2,\ n = 1,\ \beta < \beta_c(d): \quad \rho_{x,y} \leq C_{d,\beta} \exp(-c_{d,\beta} \|x - y\|_1) \quad \text{for all } x, y \in V(\mathbb{T}_L^d)$$

to a regime with *long-range order*, also called *spontaneous magnetization*, which is characterized by

$$d \geq 2,\ n = 1,\ \beta > \beta_c(d): \quad \rho_{x,y} \geq c_{d,\beta} \quad \text{for all } x, y \in V(\mathbb{T}_L^d).$$

The behavior of the model at the critical temperature, when $\beta = \beta_c(d)$, is a rich source of study with many mathematical features, including a conformally-invariant scaling limit in two dimensions, but its treatment lies beyond the scope of these notes. We mention only that it is proved (see Aizenman, Duminil-Copin, Sidoravicius [2] and references within) that the model does not exhibit long-range order at its critical point in all dimensions $d \geq 2$. Moreover, in dimension $d = 2$ it is known that correlations *decay polynomially* (*** detail ***), as discovered by Onsager [19] in his famous solution of the two-dimensional Ising model.

The Mermin-Wagner theorem: No continuous symmetry breaking in $2d$. Perhaps surprisingly, the behavior of the two-dimensional model when $n \geq 2$, so that the spin space $\mathbb{S}^{n-1}$ has a continuous symmetry, is quite different from that of the Ising model. The *Mermin-Wagner theorem* [18] asserts that in this case there is no phase with long-range order at any inverse temperature β. Quantifying the rate at which correlations decay has been the focus of much research along the years (*** ref - Fisher-Jasnow, Pfister, Dobrushin-Shlosman, Hohenberg, Frohlich-Spencer, Polyakov, Frohlich-Pfister ***) and is still not completely understood. Improving on earlier bounds, McBryan and Spencer [17] showed in 1977 that the decay occurs at least at a polynomial rate,

$$d = 2,\ n \geq 2,\ \beta \in [0, \infty): \quad \rho_{x,y} \leq C_{n,\beta} \|x - y\|_1^{-c_{n,\beta}} \quad \text{for all } x, y \in V(\mathbb{T}_L^2). \tag{5}$$

The sharpness of this bound is discussed in the next paragraphs.

The Berezinskii-Kosterlitz-Thouless transition for the $2d$ XY Model. It was predicted by Berezinskii [5] and by Kosterlitz and Thouless [13, 14] that the XY model ($n = 2$) in two spatial dimensions should indeed exhibit polynomial decay of correlations at low temperatures. Thus the model undergoes a phase transition (of a different nature than that of the Ising model) from a phase with exponential decay of correlations to a phase with polynomial decay of correlations. This transition is called the Berezinskii-Kosterlitz-Thouless transition. The existence of the transition has been proved mathematically in the celebrated work of Fröhlich and Spencer [11], who show that there exists a β_1 for which

$$d = 2,\ n = 2,\ \beta > \beta_1: \quad \rho_{x,y} \geq c_\beta \|x - y\|_1^{-C_\beta} \quad \text{for all } x, y \in V(\mathbb{T}_L^2). \tag{6}$$

(*** check exact statement ***).

A rigorous proof of the bound (6) is beyond the scope of these notes. In Section 2.7 we present a heuristic discussion of the transition highlighting the role of *vortices* - cycles of length 4 in $\mathbb{T}_L^2$ on which the configuration completes a full rotation. We then proceed to present a beautiful result of Aizenman [1], following Patrascioiu and Seiler [20], who showed that correlations decay at most polynomially fast in the spin $O(2)$ model with potential U, for certain potentials U for which vortices are deterministically excluded.

Polyakov's conjecture for the $2d$ Heisenberg model. Polyakov [21] predicted in 1975 that the spin $O(n)$ model with $n \geq 3$ should exhibit exponential decay of correlations in two dimensions at any temperature. That is, that there is no phase transition of the Berezinskii-Kosterlitz-Thouless type in the Heisenberg model and the spin $O(n)$ models with larger n. This prediction may be stated precisely as

$$d = 2,\ n \geq 3,\ \beta \in [0, \infty): \quad \rho_{x,y} \leq C_{n,\beta} \exp(-c_{n,\beta} \|x - y\|_1) \quad \text{for all } x, y \in V(\mathbb{T}_L^2).$$

Giving a mathematical proof of this statement remains one of the major challenges of the subject. The best known results in this direction are by Kupiainen [15] who performed a $1/n$-expansion as n tends to infinity.

The infra-red bound: Long-range order in dimensions $d \geq 3$. In three and higher spatial dimensions, the spin $O(n)$ model exhibits long-range order at sufficiently low temperatures for all n. This was first established by Fröhlich, Simon and Spencer [10] in 1976 who introduced the powerful method of the *infra-red bound*, and applied it to the analysis of the spin $O(n)$ and other models (*** check statements there ***). They prove that correlations do not decay at temperatures below a threshold $\beta_1(d,n)^{-1}$, at least in the following averaged sense,

$$d \geq 3,\ n \geq 1,\ \beta > \beta_1(d,n): \quad \frac{1}{|V(\mathbb{T}_L^d)|^2} \sum_{x,y \in V(\mathbb{T}_L^d)} \rho_{x,y} \geq c_{d,n,\beta}.$$

The proof uses the reflection symmetries of the underlying lattice, relying on the tool of *reflection positivity.*

2.3 High-temperature expansion

At infinite temperature ($\beta = 0$) the models are completely disordered as all spins are independent of one another. At high-temperature (when β is sufficiently small), the same type of behavior persists and the models remain in a disordered phase. Specifically, we show that in this regime the models exhibit exponential decay of correlations as stated in the previous section.

We begin by expanding the partition function of the model on an arbitrary finite graph in the following manner. Denoting $f_\beta(s,t) := \exp\big[\beta\big(\langle s,t\rangle + 1\big)\big] - 1$ for $s,t \in \mathbb{S}^{n-1}$, we have

$$\begin{aligned} Z^{\text{spin}}_{G,n,\beta} &= \int_\Omega \prod_{\{u,v\}\in E(G)} \exp\left[\beta\left\langle \sigma_u,\sigma_v\right\rangle\right] d\sigma = e^{-\beta|E(G)|} \int_\Omega \prod_{\{u,v\}\in E(G)} \exp\left[\beta\left(\left\langle \sigma_u,\sigma_v\right\rangle + 1\right)\right] d\sigma \\ &= e^{-\beta|E(G)|} \int_\Omega \prod_{\{u,v\}\in E(G)} \big(1 + f_\beta(\sigma_u,\sigma_v)\big) d\sigma = e^{-\beta|E(G)|} \sum_{E \subset E(G)} \int_\Omega \prod_{\{u,v\}\in E} f_\beta(\sigma_u,\sigma_v) d\sigma. \end{aligned}$$

Exercise. Verify the last equality in the above expansion by showing that for any $(x_e)_{e\in\mathcal{E}}$,

$$\prod_{e\in\mathcal{E}} (1 + x_e) = \sum_{E\subset\mathcal{E}} \prod_{e\in E} x_e.$$

Thus, we have

$$Z^{\text{spin}}_{G,n,\beta} = e^{-\beta|E(G)|} \sum_{E\subset E(G)} Z(E), \tag{7}$$

where

$$Z(E) := \int_\Omega \prod_{\{u,v\}\in E} f_\beta(\sigma_u,\sigma_v) d\sigma. \tag{8}$$

Since f_β is non-negative, we may interpret (7) as prescribing a probability measure on (spanning) subgraphs of G, where the subgraph $(V(G), E)$ has probability proportional to $Z(E)$. Furthermore, given such a subgraph, we may interpret (8) as prescribing a probability measure on spin configurations σ, whose density with respect to $d\sigma$ is proportional to

$$Z(E,\sigma) := \prod_{\{u,v\}\in E} f_\beta(\sigma_u,\sigma_v).$$

Remark 2.1. *For the Ising model ($n = 1$), the above joint distribution on the graph $(V(G), E)$ and spin configuration σ is called the* Edwards-Sokal coupling *(*** ref ***). Here, the marginal probability of E is proportional to*

$$q^{N(E)} p^{|E|} (1-p)^{|E(\mathbb{T}_L^d)| \setminus |E|} \quad \textit{with } q = 2 \textit{ and } p = 1 - \exp(-2\beta), \tag{9}$$

where $N(E)$ stands for the number of connected components in $(V(G), E)$. Moreover, given E, σ is sampled by independently assigning to the vertices in each connected component of $(V(G), E)$ the same spin value, picked uniformly from $\{-1, 1\}$. The marginal distribution (9) *of E is the famous* Fortuin-Kasteleyn (FK) random cluster model*, which makes sense also for other values of p and q (*** ref. Grimmett book? ***). Both the Edwards-Sokal coupling and the FK model are available also for the more general* Potts models*.*

*The Edwards-Sokal coupling immediately implies that, for the Ising model, the correlation $\rho_{x,y} = \mathbb{E}(\sigma_x \sigma_y)$ equals the probability that x is connected to y in the graph $(V(G), E)$. In particular, $\rho_{x,y}$ is non-negative and, as connectivity probabilities in the FK model (with $q \geq 1$) are non-decreasing with p (*** Grimmett random cluster, Theorem 3.21 ***), it follows also that $\rho_{x,y}$ is non-decreasing with the inverse temperature β. A version of the Edwards-Sokal coupling and FK model for the spin $O(n)$ models with $n \geq 2$ will be used in Section *** (*** section on Aizenman's result ***).*

Remark 2.2. *Conditioned on E, the spin configuration σ may be seen as a sample from the spin $O(n)$ model on the graph $(V(G), E)$ with potential $U(x) := -\log(\exp(\beta(1+x)) - 1)$. That is, conditioned on E, the distribution of σ is given by $\mu_{(V(G),E),n,U}$.*

It follows from the last remark that, conditioned on E,

If $x \in V(G)$ then σ_x is distributed uniformly on $\mathbb{S}^{n-1}$.
If $x, y \in V(G)$ are not connected in $(V(G), E)$ then σ_x and σ_y are independent.

Hence, we deduce that $\mathbb{E}(\langle \sigma_x, \sigma_y \rangle \mid E) = 0$ when x and y are not connected in $(V(G), E)$. Since $|\langle \sigma_x, \sigma_y \rangle| \leq 1$, we obtain

$$|\rho_{x,y}| \leq \mathbb{P}(x \text{ and } y \text{ are connected in } (V(G), E)),$$

where E a random subset of $E(G)$ chosen according to the above probability measure. Thus, to establish the decay of correlations, we must show that long connections in E are very unlikely. We first show that

$$\text{For any } e \in E(G) \text{ and } E_0 \subset E(G) \setminus \{e\}, \mathbb{P}(e \in E \mid E \setminus \{e\} = E_0) \leq 1 - e^{-2\beta}. \tag{10}$$

Indeed,

$$\mathbb{P}(e \in E \mid E \setminus \{e\} = E_0) = \frac{Z(E_0 \cup \{e\})}{Z(E_0 \cup \{e\}) + Z(E_0)} = \frac{1}{1 + \frac{Z(E_0)}{Z(E_0 \cup \{e\})}},$$

and denoting $e = \{u, v\}$ and noting that $f_\beta(s,t) \leq \exp(2\beta) - 1$,

$$\frac{Z(E_0 \cup \{e\})}{Z(E_0)} = \frac{\int_\Omega Z(E_0 \cup \{e\}, \sigma) d\sigma}{\int_\Omega Z(E_0, \sigma) d\sigma} = \frac{\int_\Omega Z(E_0, \sigma) f_\beta(\sigma_u, \sigma_v) d\sigma}{\int_\Omega Z(E_0, \sigma) d\sigma} \leq e^{2\beta} - 1.$$

Repeated applications of (10) now imply that the probability that E contains any fixed k edges is exponentially small in k. Namely,

$$\text{For any } e_1, \ldots, e_k \in E(G),\ \mathbb{P}(e_1, \ldots, e_k \in E) \le \left(1 - e^{-2\beta}\right)^k.$$

We now specialize to the case $G = \mathbb{T}_L^d$ (in fact, the only property of $\mathbb{T}_L^d$ we use is that its maximum degree is $2d$). Since the event that x and y are connected in $(V(G), E)$ implies the existence of a simple path in E of some length $k \ge \|x - y\|_1$ starting at x, and since the number of such paths is at most $2d(2d-1)^{k-1} \le 2(2d-1)^k$, we obtain

$$\begin{aligned}\mathbb{P}(x \text{ and } y \text{ are connected in } (V(G), E)) &\le \sum_{k=\|x-y\|_1}^{\infty} 2(2d-1)^k(1-e^{-2\beta})^k \\ &\le C_{d,\beta}\Big((2d-1)(1-e^{-2\beta})\Big)^{\|x-y\|_1},\end{aligned}$$

when $(2d-1)(1-e^{-2\beta}) < 1$. Thus, we have established that

$$|\rho_{x,y}| \le C_{d,\beta} \exp\left(-c_{d,\beta}\|x-y\|_1\right) \quad \text{when } \beta < \frac{1}{2}\log\left(\frac{2d-1}{2d-2}\right).$$

Remark 2.3. *This gives exponential decay in dimension $d \ge 2$ whenever $\beta \le d/4$ and in one dimension for all finite β.*

2.4 Low-temperature Ising model

One could approach the low-temperature Ising model by expanding the partition function in a similar manner as before. This would lead to a representation of random even subgraphs (subgraphs in which the degrees of all vertices are even) (*** reference remark in loop $O(n)$ section? ***). Here, however, we choose a slightly different approach, partially for variety.

Let G be a finite connected graph and let $x, y \in V(G)$ be two vertices. We begin by noting that in the Ising model, since spins take values in $\{1, -1\}$, we may write the correlations in the following form:

$$\rho_{x,y} = \mathbb{E}(\sigma_x\sigma_y) = \mathbb{P}(\sigma_x = \sigma_y) - \mathbb{P}(\sigma_x \ne \sigma_y) = 1 - 2\mathbb{P}(\sigma_x \ne \sigma_y)$$

Thus, to establish a lower bound on the correlation we must provide an upper bound the probability that the spins at x and y are different. To this end, we require some definitions. Given a set of vertices $A \subset V(G)$, we denote the *edge-boundary* of A, the set of edges in $E(G)$ with precisely one endpoint in A, by ∂A. A *contour* is a set of edges $\gamma \subset E(G)$ such that $\gamma = \partial A$ for some $A \subset V(G)$ satisfying that both A and $A^c := V(G) \setminus A$ are induced connected (non-empty) subgraphs of G. Thus, a contour can be identified with a partition of the set of vertices of G into two connected sets. We say that γ separates two vertices x and y if they belong to different sets of the corresponding partition. The length of a contour is the number of edges it contains.

Exercise. A set of edges γ is a contour if and only if γ is cutset (i.e., the removal of γ disconnects the graph) which is minimal with repsect to inclusion (i.e., no proper subset of γ is also a cutset).

Let σ be a spin configuration. We say that γ is an interface (with respect to σ) if γ is a contour separating x and y such that

$$\sigma_u \neq \sigma_v \quad \text{for all } \{u,v\} \in \gamma.$$

The first step in the proof is the following observation:

$$\text{If } \sigma_x \neq \sigma_y \text{ then there exists an interface.}$$

Indeed, if $\sigma_x \neq \sigma_y$ then the connected component of $\{u \in V(G) : \sigma_u = \sigma_x\}$ containing x, which we denote by B, does not contain y. Hence, if we denote the connected component of B^c containing y by A, then $\gamma := \partial A$ is a contour separating x and y. Moreover, it is easy to check that $\sigma_u = \sigma_x$ and $\sigma_v = \sigma_y$ for all $\{u,v\} \in \gamma$ such that $u \in A^c$ and $v \in A$, so that γ is an interface.

Next, we show that for any fixed contour γ of length k,

$$\mathbb{P}(\gamma \text{ is an interface}) \leq e^{-2\beta k}.$$

To see this, let $\{A, A^c\}$ be the partition corresponding to γ and, given a spin configuration σ, consider the modified spin configuration σ' in which the spins in A are flipped, i.e.,

$$\sigma'_u := \begin{cases} -\sigma_u & \text{if } u \in A \\ \sigma_u & \text{if } u \in A^c \end{cases}.$$

Observe that if γ is an interface with respect to σ then

$$\sum_{\{u,v\} \in E(G)} \sigma'_u \sigma'_v - \sum_{\{u,v\} \in E(G)} \sigma_u \sigma_v = \sum_{\{u,v\} \in \gamma} (\sigma'_u \sigma'_v - \sigma_u \sigma_v) = 2|\gamma|.$$

Thus, denoting $F := \{\sigma \in \Omega : \gamma \text{ is an interface with respect to } \sigma\}$ and noting that $\sigma \mapsto \sigma'$ is injective on Ω (in fact, an involution of Ω), we have

$$\begin{aligned} \mathbb{P}(\gamma \text{ is an interface}) &= \frac{\sum_{\sigma \in F} \exp\left[\beta \sum_{\{u,v\} \in E(G)} \sigma_u \sigma_v\right]}{\sum_{\sigma \in \Omega} \exp\left[\beta \sum_{\{u,v\} \in E(G)} \sigma_u \sigma_v\right]} \\ &\leq \frac{\sum_{\sigma \in F} \exp\left[\beta \sum_{\{u,v\} \in E(G)} \sigma_u \sigma_v\right]}{\sum_{\sigma \in F} \exp\left[\beta \sum_{\{u,v\} \in E(G)} \sigma'_u \sigma'_v\right]} = e^{-2\beta|\gamma|}. \end{aligned}$$

The final ingredient in the proof is an upper bound on the number of contours of a given length. For this, we henceforth restrict ourselves to the case $G = \mathbb{T}_L^d$, for which we use the following fact:

The number of contours of length k separating two given vertices is at most $\exp(C_d k)$.

The proof of this fact consists of the following two lemmas.

Lemma 2.1. *Let γ be a set of edges and consider the graph $\mathcal{G}_\gamma$ on γ in which two edges $e, f \in \gamma$ are adjacent if the $(d-1)$-dimensional faces corresponding to e and f share a common $(d-2)$-dimensional face. If γ is a contour then either $\mathcal{G}_\gamma$ is connected or it has precisely two connected components, each of which has size at least L^{d-1}.*

Although intuitively clear, the proof of the above lemma is not completely straightforward. (*** even for $\mathbb{Z}^d$ this is not straightforward, while for $\mathbb{T}_L^d$ there is an additional topological complication ***) We refer the reader to *** [23] *** for a proof.

Lemma 2.2. *Let $\mathcal{G}$ be a graph with maximum degree Δ. The number of connected subsets of $V(\mathcal{G})$ which have size k and contain a given vertex is at most $a(\Delta)^k$, where $a(\Delta)$ is a positive constant depending only on Δ.*

This lemma has several simple proofs. One may for instance use a depth-first-search algorithm to provide a proof with the constant $a(\Delta) = \Delta^2$. We refer the reader to [7, Chapter 45] for a proof yielding the constant $a(\Delta) = e(\Delta - 1)$ (which is optimal as can be seen by considering the case when $\mathcal{G}$ is a regular tree).

Exercise. Deduce the fact from the two lemmas.

Finally, putting everything together, when $\beta \geq C_d$, we obtain

$$\begin{aligned}\mathbb{P}(\sigma_x \neq \sigma_y) \leq \mathbb{P}(\text{there exists an interface}) &\leq \sum_{\substack{\gamma \text{ contour} \\ \text{separating } x \text{ and } y}} \mathbb{P}(\gamma \text{ is an interface}) \\ &\leq \sum_{k=1}^{\infty} e^{C_d k} e^{-2\beta k} \leq C_d e^{-2\beta}.\end{aligned}$$

Thus, in terms of correlations, we have established that

$$\rho_{x,y} \geq 1 - C_d e^{-2\beta} \geq c_{d,\beta} \quad \text{when } \beta \geq C_d.$$

Remark 2.4. *Specializing Lemma 2.2 to the relevant graph in our situation, one may obtain an improved and explicit bound of* $\exp(Ck\log(d+1)/d)$ *on the number of contours of length k separating two given vertices [16, 3]. This gives that $\beta_c(d) \leq C\log(d+1)/d$. In fact, the correct asymptotic value is $\beta_c(d) \sim 1/2d$.*

(*** Further discussion on this in long-range order section. ***)

2.5 No long-range order in two dimensional models with continuous symmetry - the Mermin-Wagner theorem

In this section we establish polynomial decay of correlations for the two-dimensional spin $O(n)$ model with $n \geq 2$ at any inverse temperature. The proof applies in the generality of the spin $O(n)$ model with potential U, where U satisfies certain assumptions, and it is convenient to present it in this context, to highlight the core parts of the argument. The fact that there is no long-range order was first established by Mermin and Wagner (*** ref ***), with later works providing upper bounds on the rate of decay of the correlations (*** ref ***). The following theorem was first proved by Dobrushin and Shlosman (*** ref ***), following a proof by McBryan and Spencer (*** ref ***) for the special case of the standard XY model (with a technique relying on the density being an analytic function).

Theorem 2.3. *Let $U : [-1, 1] \to \mathbb{R}$ be a C^2 function. Let $n \geq 2$. Suppose that $\sigma : V(\mathbb{T}_L^2) \to \mathbb{S}^{n-1}$ is randomly sampled from the two-dimensional spin $O(n)$ model with potential U (see (3)). Then there exist $C_{n,U}, c_{n,U} > 0$ such that*

$$|\rho_{x,y}| = |\mathbb{E}(\langle \sigma_x, \sigma_y \rangle)| \leq C_{n,U} \|x - y\|_1^{-c_{n,U}} \quad \textit{for all } x, y \in V(\mathbb{T}_L^2). \tag{11}$$

The proof presented below combines elements of the Dobrushin-Shlosman (*** ref ***) and Pfister (*** ref ***) approaches to the Mermin-Wagner theorem. The idea to combine the approaches is introduced in a forthcoming paper of Gagnebin, Miłoś and Peled (*** ref ***), where it is pushed further to prove polynomial decay of correlations for *any* potential U satisfying only very mild integrability conditions. The work (*** ref GMP ***) relies further on ideas used by (*** ISV, MP, Richthammer ***).

For simplicity, we will prove Theorem 2.3 in the special case that $n = 2$, $x = (0, 0)$ and $y = (2^m, 0)$ for some integer $m \geq 0$ (assuming, implicitly, that $L \geq 2^m$). We briefly explain the necessary modifications for the general case after the proof.

Fix a C^2 function $U : [-1, 1] \to \mathbb{R}$ satisfying $U(x) = U(-x)$. Suppose that $\sigma : V(\mathbb{T}_L^2) \to \mathbb{S}^1$ is randomly sampled from the two-dimensional spin $O(2)$ model with potential U. It is convenient to parametrize configurations differently: Identifying $\mathbb{S}^1$ with the unit circle in the complex plane, we consider the angle θ_v that each vector σ_v forms with respect to the x-axis. Precisely, for the rest of the argument we let $\theta : V(\mathbb{T}_L^2) \to [-\pi, \pi)$ be randomly sampled from the probability density

$$t(\phi) := \frac{1}{Z} \exp\left[- \sum_{\{u,v\} \in E(G)} U(\cos(\phi_u - \phi_v)) \right] \prod_{v \in V(\mathbb{T}_L^2)} \mathbf{1}_{(\phi_v \in [-\pi, \pi))}, \tag{12}$$

where Z is a normalization constant. One checks simply that then (σ_v) is equal in distribution to $(\exp(i\theta_v))$. Thus, with our choice of the vertices x and y, the estimate (11) that we would like to prove becomes

$$|\rho_{(0,0),(2^m,0)}| = |\mathbb{E}(\cos(\theta_{(0,0)} - \theta_{(2^m,0)}))| \leq C_{n,U} \cdot 2^{-c_{n,U} \cdot m}. \tag{13}$$

Step 1: Product of conditional correlations. We start by pointing out a conditional independence property inherent in the distribution of θ, which is a consequence of the domain Markov property and the fact that the interaction term in (12) depends only on the difference of angles in ϕ (the gradient of ϕ). This part of the argument is inspired by the technique of Dobrushin and Shlosman (*** ref ***).

Define a vector-valued function g on $\mathbb{R}^{V(\mathbb{T}_L^2)}$ by

$$g(\phi) = (\phi_u - \phi_v \colon u, v \in V(\mathbb{T}_L^2),\ \exists\, 0 \leq k \leq m - 1,\ \|u\|_1 = \|v\|_1 = 2^k),$$

so that $g(\phi)$ contains, for every $0 \leq k \leq m - 1$, the information on the difference of angles in ϕ for vertices in a given 'layer' at radius 2^k from the origin.

Proposition 2.4. *Conditioned on $g(\theta)$, the random variables*

$$\theta_{(0,0)} - \theta_{(1,0)},\ \theta_{(1,0)} - \theta_{(2,0)},\ \theta_{(2,0)} - \theta_{(4,0)},\ \ldots,\ \theta_{(2^{m-1},0)} - \theta_{(2^m,0)}$$

are independent.

This proposition allows us to reexpress the quantity of interest to us in the following way:

$$\mathbb{E}(\cos(\theta_{(0,0)}-\theta_{(2^m,0)})) = \Re\mathbb{E}\left(e^{i\left(\theta_{(0,0)}-\theta_{(2^m,0)}\right)}\right) = \Re\mathbb{E}\left(e^{i\left(\theta_{(0,0)}-\theta_{(1,0)}\right)}\prod_{k=0}^{m-1}e^{i\left(\theta_{(2^k,0)}-\theta_{(2^{k+1},0)}\right)}\right)$$
$$= \Re\mathbb{E}\left(\mathbb{E}\left(e^{i\left(\theta_{(0,0)}-\theta_{(1,0)}\right)}\mid g(\theta)\right)\prod_{k=0}^{m-1}\mathbb{E}\left(e^{i\left(\theta_{(2^k,0)}-\theta_{(2^{k+1},0)}\right)}\mid g(\theta)\right)\right) \quad (14)$$

which will be the starting point for our next step.

The proposition is somewhat intuitive, though a formal proof seems to necessitate some technicalities. We proceed to explain the idea of proof but the reader who feels comfortable with the proposition may wish to skip directly to the next step.

Idea of proof of Proposition 2.4. It suffices to show that for each $0 \le \ell \le m-1$, the random variable $\theta_{(2^\ell,0)} - \theta_{(2^{\ell+1},0)}$ is conditionally independent of the random variables

$$\theta_{(0,0)} - \theta_{(1,0)},\ (\theta_{(2^k,0)} - \theta_{(2^{k+1},0)})_{0\le k<\ell}. \quad (15)$$

given $g(\theta)$.

For a subset $V \subseteq V(\mathbb{T}_L^2)$, we define the random vectors

$$\theta(V) = (\theta_v\colon v \in V),$$
$$\nabla\theta(V) = (\theta_u - \theta_v\colon u, v \in V).$$

Fix $0 \le \ell \le m-1$ and define the subsets of vertices

$$A := \{v \in V(\mathbb{T}_L^2)\colon \|v\|_1 \le 2^\ell\},$$
$$B := \{v \in V(\mathbb{T}_L^2)\colon \|v\|_1 = 2^\ell\}.$$

As the random variable $\theta_{(2^\ell,0)}-\theta_{(2^{\ell+1},0)}$ is a function of $\nabla\theta(A^c\cup B)$ (we write A^c for $V(\mathbb{T}_L^2)\setminus A$) and the random variables in (15) are functions of $\theta(A)$, it suffices to show that

$$\text{conditioned on } g(\theta),\ \nabla\theta(A^c \cup B) \text{ is independent of } \theta(A). \quad (16)$$

We will show the stronger statement that

$$\text{conditioned on } \nabla\theta(B),\ \nabla\theta(A^c \cup B) \text{ is independent of } \theta(A). \quad (17)$$

The fact that (17) implies (16) is a special case of the following exercise.

Exercise. Suppose X, Y, Z are random variables satisfying that X is conditionally independent of Y given Z. Then for every two measurable functions f, g, X is conditionally independent of Y, given $(Z, f(X), g(Y))$.

In our case, $X = \nabla\theta(A^c \cup B)$, $Y = \theta(A)$, $Z = \nabla\theta(B)$, each coordinate of $g(\theta)$ is either a function of X or a function of Y and all the coordinates of Z are coordinates of $g(\theta)$.

We proceed to prove (17). Fix an event E in the sigma algebra generated by $\nabla\theta(A^c\cup B)$. We need to show that

$$\mathbb{P}(E \mid \sigma(\nabla\theta(B), \theta(A))) = \mathbb{P}(E \mid \nabla\theta(B)), \quad \text{almost surely.} \quad (18)$$

As $\nabla\theta(B)$ is a function of $\theta(A)$, we have

$$\mathbb{P}(E \mid \sigma(\nabla\theta(B), \theta(A))) = \mathbb{P}(E \mid \theta(A)), \quad \text{almost surely.} \tag{19}$$

The *domain Markov property* and the definitions of A, B and E now imply that,

$$\mathbb{P}(E \mid \theta(A)) = \mathbb{P}(E \mid \theta(B)), \quad \text{almost surely.} \tag{20}$$

Additionally, the fact that the interaction term in the density (12) depends only on the gradient of ϕ implies that (θ_v) has the same distribution as $(\theta_v + \alpha \pmod{2\pi})$ for any fixed $\alpha \in [0, 2\pi)$ (the mapping $x \mapsto x \pmod{2\pi}$ yields a value in $[-\pi, \pi)$ obtained by subtracting $k \cdot 2\pi$ from x for some integer k). Together with the fact that the event E depends only on the gradient of θ this yields that

$$\mathbb{P}(E \mid \theta(B)) = \mathbb{P}(E \mid \nabla\theta(B)), \quad \text{almost surely.} \tag{21}$$

The formal verification of (20) and (21) is left as a further *exercise* to the reader. Finally, (18) is a consequence of (19), (20) and (21). $\square$

Step 2: Upper bound on the conditional correlations. In this step we estimate the individual conditional expectations in (14), proving that there exists an absolute constant $\varepsilon > 0$ for which

$$\text{almost surely,} \quad \left|\mathbb{E}\left(e^{i\left(\theta_{(2^k,0)} - \theta_{(2^{k+1},0)}\right)} \mid g(\theta)\right)\right| \le 1 - \varepsilon \quad \text{for all } 0 \le k \le m-1, \tag{22}$$

immediately implying the required bound (13) as, from (14),

$$\begin{aligned}\left|\mathbb{E}(\cos(\theta_{(0,0)} - \theta_{(2^m,0)}))\right| &\le \mathbb{E}\left(\left|\mathbb{E}\left(e^{i\left(\theta_{(0,0)} - \theta_{(1,0)}\right)} \mid g(\theta)\right)\right| \cdot \prod_{k=0}^{m-1} \left|\mathbb{E}\left(e^{i\left(\theta_{(2^k,0)} - \theta_{(2^{k+1},0)}\right)} \mid g(\theta)\right)\right|\right) \\ &\le \mathbb{E}\left(1 \cdot (1-\varepsilon)^m\right) = (1-\varepsilon)^m.\end{aligned}$$

This part of the argument is inspired by the technique of Pfister (*** ref ***), and the variants used in (*** MP, Richthammer ***). The idea of introducing a *spin wave* which rotates slowly (our function τ below and its property (27)) is at the heart of the Mermin-Wagner theorem.

Write dm_{g_0} for the lower-dimensional Lebesgue measure supported on the affine subspace of $\mathbb{R}^{V(\mathbb{T}_L^2)}$ where $g(\theta) = g_0$. Standard facts (following from Fubini's theorem) imply that conditioned on $g(\theta) = g_0$, for almost every value of g_0, the density of θ exists with respect to dm_{g_0} and is of the form (as in (12))

$$\begin{aligned} t_{g_0}(\phi) &= \frac{1}{Z_{g_0}} \exp\left[-\sum_{\{u,v\} \in E(G)} U(\cos(\phi_u - \phi_v))\right] \prod_{v \in V(\mathbb{T}_L^2)} \mathbf{1}_{(\phi_v \in [-\pi,\pi))} \\ &= \frac{1}{Z_{g_0}} \exp\left[-\sum_{\{u,v\} \in E(G)} \tilde{U}(\phi_u - \phi_v)\right] \prod_{v \in V(\mathbb{T}_L^2)} \mathbf{1}_{(\phi_v \in [-\pi,\pi))}, \end{aligned}$$

where we define

$$\tilde{U}(\alpha) := U(\cos(\alpha))$$

and note that $\tilde{U} : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic C^2 function. In particular,

$$\tilde{U}(x+\delta) \le \tilde{U}(x) + \tilde{U}'(x)\delta + \frac{\sup_y \tilde{U}''(y)}{2}\delta^2 \quad \text{for all } x, \delta \in \mathbb{R}. \tag{23}$$

Fix $0 \le k \le m-1$. Define a function $\tau : V(\mathbb{T}_L^2) \to \mathbb{R}$ by

$$\tau_v := \begin{cases} 0 & \|v\|_1 \le 2^k \\ \frac{\|v\|_1}{2^k} - 1 & 2^k \le \|v\|_1 \le 2^{k+1} \\ 1 & \|v\|_1 \ge 2^{k+1} \end{cases} \tag{24}$$

and define for each $\phi : V(\mathbb{T}_L^2) \to [-\pi, \pi)$ its perturbations $\phi^+, \phi^- : V(\mathbb{T}_L^2) \to [-\pi, \pi)$ by

$$\phi_v^+ := \phi_v + \tau_v \pmod{2\pi}, \quad \phi_v^- := \phi_v - \tau_v \pmod{2\pi}. \tag{25}$$

We shall need the following two properties of τ:

$$g(\phi^+) = g(\phi^-) = g(\phi) \quad \text{for every } \phi : V(\mathbb{T}_L^2) \to \mathbb{R}, \tag{26}$$

$$\sum_{u,v \in E(\mathbb{T}_L^2)} (\tau_u - \tau_v)^2 \le C \tag{27}$$

for some absolute constant C.

The following is the key calculation of the proof. For every $\phi : V(\mathbb{T}_L^2) \to [-\pi, \pi)$, setting $g_0 := g(\phi)$,

$$\begin{aligned} \sqrt{t_{g_0}(\phi^+)t_{g_0}(\phi^-)} &= \frac{1}{Z_{g_0}} \exp\Big[-\frac{1}{2}\sum_{\{u,v\}\in E(G)} \tilde{U}(\phi_u - \phi_v + \tau_u - \tau_v) + \tilde{U}(\phi_u - \phi_v - \tau_u + \tau_v))\Big] \\ &\overset{(23)}{\ge} \frac{1}{Z_{g_0}} \exp\Big[-\sum_{\{u,v\}\in E(G)} \tilde{U}(\phi_u - \phi_v) - \frac{\sup_y U''(y)}{2}\sum_{\{u,v\}\in E(G)} (\tau_u - \tau_v)^2\Big] \overset{(27)}{\ge} c \cdot t_{g_0}(\phi) \end{aligned} \tag{28}$$

for an absolute constant $c > 0$.

We wish to convert the inequality (28) into an inequality of probabilities rather than densities. To this end define, for $a \in \mathbb{R}$,

$$E_a := \left\{\phi : V(\mathbb{T}_n^2) \to [-\pi, \pi) \colon \left|\Re e^{i(\phi_{(2^k,0)} - \phi_{(2^{k+1},0)} - a)}\right| \ge \frac{9}{10}\right\}. \tag{29}$$

and, for almost every g_0 with respect to the distribution of $g(\theta)$,

$$I_{a,g_0} := \int_{E_a} \sqrt{t_{g_0}(\phi^+)t_{g_0}(\phi^-)} dm_{g_0}(\phi).$$

On the one hand, by (28),

$$I_{a,g_0} \ge c\int_{E_a} t_{g_0}(\phi) dm_{g_0}(\phi) = c \cdot \mathbb{P}\left(\theta \in E_a \mid g(\theta) = g_0\right). \tag{30}$$

On the other hand, the Cauchy-Schwartz inequality and a change of variables using (25) and (26) yields

$$\begin{aligned} I_{a,g_0} &\le \left(\int_{E_a} t_{g_0}(\phi^+)dm_{g_0}(\phi)\int_{E_a} t_{g_0}(\phi^-)dm_{g_0}(\phi)\right)^{\frac{1}{2}} \\ &= \left(\mathbb{P}(\theta-\tau\in E_a \mid g(\theta)=g_0)\cdot\mathbb{P}(\theta+\tau\in E_a \mid g(\theta)=g_0)\right)^{\frac{1}{2}}. \end{aligned} \tag{31}$$

Putting together (30) and (31) and recalling (24) and (29) we obtain that, almost surely,

$$\begin{aligned} &\left(\mathbb{P}\left(\left|\Re e^{i(\phi_{(2^k,0)}-\phi_{(2^{k+1},0)}+1-a)}\right|\ge\frac{9}{10}\mid g(\theta)\right)\cdot\mathbb{P}\left(\left|\Re e^{i(\phi_{(2^k,0)}-\phi_{(2^{k+1},0)}-1-a)}\right|\ge\frac{9}{10}\mid g(\theta)\right)\right)^{\frac{1}{2}} \\ &\ge c\cdot\mathbb{P}\left(\left|\Re e^{i(\phi_{(2^k,0)}-\phi_{(2^{k+1},0)}-a)}\right|\ge\frac{9}{10}\mid g(\theta)\right). \end{aligned}$$

As this inequality holds for any $a\in\mathbb{R}$, it implies that, conditioned on $g(\theta)$, $e^{i\left(\theta_{(2^k,0)}-\theta_{(2^{k+1},0)}\right)}$ cannot be concentrated around any one value, proving the inequality (22) that we wanted to show.

(*** add remark about modifications for general $n\ge 2$ and other vertices x,y ***)

2.6 Long-range order in dimensions $d\ge 3$ - the infra-red bound

In this section we prove that the spin $O(n)$ model in spatial dimensions $d\ge 3$ exhibits long-range order at sufficiently low temperatures. This was first proved by Fröhlich, Simon and Spencer [10] who introduced the method of the *infra-red bound* to this end. To our knowledge, this method remains the only known technique for establishing the result. In our exposition, we follow closely the excellent 'Marseille notes' of Daniel Ueltschi (*** ref, lecture 2, part 2 ***), with some modifications suggested by the chapter on reflection positivity in the superb book in preparation by Friedli and Velenik (*** ref, Chapter 10 ***). We prove the following result.

Theorem 2.5. *Let $d\ge 3$ and $n\ge 1$. There exists $\beta_1(d,n)<\infty$ and such that the following holds. Suppose $\sigma: V(\mathbb{T}_L^d)\to\mathbb{S}^{n-1}$ is randomly sampled from the d-dimensional spin $O(n)$ model at inverse temperature $\beta>\beta_1(d,n)$. Then there exists $c_{d,n,\beta}>0$ for which*

$$\frac{1}{|V(\mathbb{T}_L^d)|^2}\sum_{x,y\in V(\mathbb{T}_L^d)}\mathbb{E}(\langle\sigma_x,\sigma_y\rangle)\ge c_{d,n,\beta}.$$

Of course, the Ising model case $n=1$ has already been discussed in Section 2.4 so that our main interest is in the case of continuous spins, when $n\ge 2$. (*** mention that the actual bound we get on the critical β is close to $1/2d$. Of the same order of magnitude as the high-temperature expansion for large d, and in fact with the sharp constant (*** find ref ***). The low-temperature Ising section only shows that the critical temperature $\beta_c(d)<c\log(d)/d$, and this is sharp there up to the value of c as it prohibits also minority percolation (*** ref? ***) ***).

Recall that $\Omega = (\mathbb{S}^{n-1})^{V(\mathbb{T}_L^d)}$ denotes the space of configurations of the spin $O(n)$ model on $\mathbb{T}_L^d$. A key part of the argument is the study of the function $Z : (\mathbb{R}^n)^{V(\mathbb{T}_L^d)} \to \mathbb{R}$ defined by

$$Z(\tau) := \int_\Omega \exp\left(-\frac{\beta}{2} \sum_{\{u,v\}\in E(\mathbb{T}_L^d)} \|\sigma_u + \tau_u - \sigma_v - \tau_v\|_2^2\right) d\sigma$$

where $\|\cdot\|_2$ denotes the Euclidean norm of a vector. As $\|\sigma_v\|_2^2 = 1$ at each vertex v since $\sigma \in \Omega$, the function $Z(\tau)$ at the zero function $\tau = 0$ is closely related to the partition function of the spin $O(n)$ model (see (2)),

$$Z(0) = e^{2d|V(\mathbb{T}_L^d)|} \cdot Z^{\text{spin}}_{\mathbb{T}_L^d,n,\beta}. \tag{32}$$

The main step in the proof of Theorem 2.5 is the verification of the following *Gaussian domination* inequality,

$$Z(\tau) \le Z(0) \quad \text{for all } \tau : V(\mathbb{T}_L^d) \to \mathbb{R}^n. \tag{33}$$

This is achieved using the method of *reflection positivity* as detailed below.

Step 1: Reflection Positivity. Define a *reflection operation* (across edges) on the vertices of $\mathbb{T}_L^d$ by $\theta : V(\mathbb{T}_L^d) \to V(\mathbb{T}_L^d)$,

$$\theta((v_1, v_2, \dots, v_d)) = (-v_1 + 1, v2, \dots, v_d).$$

Geometrically, the reflection is done across the hyperplane orthogonal to the x-axis which passes through the edges between x-coordinate 0 and x-coordinate 1 (or equivalently, the hyperplane passing through the edges between x-coordinate L and x-coordinate $-L+1$). One may similarly consider reflections through other planes orthogonal to one of the coordinate axes, however, for concreteness, we focus on the reflection above. Correspondingly to this choice, we split the vertices of the torus into the 'left' and 'right' halves,

$$V_0 := \{v = (v_1, v_2, \dots, v_d) \in V(\mathbb{T}_L^d) \colon v_1 \le 0\},$$
$$V_1 := \{v = (v_1, v_2, \dots, v_d) \in V(\mathbb{T}_L^d) \colon v_1 \ge 1\},$$

so that $\theta(V_0) = V_1$. We may also correspondingly write functions $\tau : V(\mathbb{T}_L^d) \to \mathbb{R}^n$ as $\tau = (\tau_0, \tau_1)$ with $\tau_i : V_i \to \mathbb{R}^n$ for $i \in \{0, 1\}$ (but will write $Z(\tau_0, \tau_1)$ instead of $Z((\tau_{0,1}))$ for brevity). The operation θ then naturally extends to such functions, mapping $(\mathbb{R}^n)^{V_i}$ to $(\mathbb{R}_n)^{V_{1-i}}$ by

$$\theta(\tau_i)(v) := \tau_i(\theta(v)), \quad i \in \{0, 1\}.$$

Proposition 2.6. *(Reflection positivity) For all $\tau_i : V_i \to \mathbb{R}^n$, $i \in \{0, 1\}$, we have*

$$Z(\tau_0, \tau_1)^2 \le Z(\tau_0, \theta(\tau_0)) \cdot Z(\theta(\tau_1), \tau_1).$$

Proof. Similarly to the previous definitions, we let also $\Omega_i = (\mathbb{S}^{n-1})^{V_i}$, $i \in \{0,1\}$, with the corresponding uniform measures $d\sigma_i$, $i \in \{0,1\}$. Then, letting $\tau = (\tau_0, \tau_1)$,

$$Z(\tau) = \int_{\Omega_0} \int_{\Omega_1} \exp\left(f_0(\sigma_0, \tau_0) + f_1(\sigma_1, \tau_1) - \frac{\beta}{2} \sum_{\substack{\{u,v\} \in E(\mathbb{T}_L^d) \\ u \in V_0, v \in V_1}} \|\sigma_u + \tau_u - \sigma_v - \tau_v\|_2^2 \right) d\sigma_0 d\sigma_1 \tag{34}$$

where f_0 and f_1 take care of the sum over edges both of whose endpoints are in V_0 or V_1, respectively. The main step of the proof is to make the above integrand linear in the (τ_i) by introducing certain Gaussian random variables, a trick sometimes known as the Hubbard-Stratonovich transformation. Recall that if $X = (X_1, \ldots, X_n)$ is a vector of independent standard normal random variables and $a \in \mathbb{C}^n$ ($\mathbb{C}$ stands for the complex numbers) then

$$\mathbb{E}(\exp(\langle a, X \rangle)) = \exp\left(\frac{1}{2} \sum_{j=1}^{n} a_j^2 \right). \tag{35}$$

Now introduce for each edge $\{u,v\} \in E(\mathbb{T}_L^d)$, $u \in V_0, v \in V_1$, an independent Gaussian vector $X_{\{u,v\}}$ with the distribution of X. The equality (35) along with Fubini's theorem allow to rewrite (34) as

$$\begin{aligned} Z(\tau) &= \mathbb{E}\Bigg(\int_{\Omega_0} \int_{\Omega_1} \exp\left(f_0(\sigma_0, \tau_0) + f_1(\sigma_1, \tau_1) + i\sqrt{\beta} \sum_{\substack{\{u,v\} \in E(\mathbb{T}_L^d) \\ u \in V_0, v \in V_1}} \langle \sigma_u + \tau_u - \sigma_v - \tau_v, X_{\{u,v\}} \rangle \right) d\sigma_0 d\sigma_1 \Bigg) \\ &= \mathbb{E}\Bigg(\int_{\Omega_0} \exp\left(f_0(\sigma_0, \tau_0) + i\sqrt{\beta} \sum_{\substack{\{u,v\} \in E(\mathbb{T}_L^d) \\ u \in V_0, v \in V_1}} \langle \sigma_u + \tau_u, X_{\{u,v\}} \rangle \right) d\sigma_0 \cdot \\ &\qquad \int_{\Omega_1} \exp\left(f_1(\sigma_1, \tau_1) + i\sqrt{\beta} \sum_{\substack{\{u,v\} \in E(\mathbb{T}_L^d) \\ u \in V_0, v \in V_1}} \langle \sigma_v + \tau_v, X_{\{u,v\}} \rangle \right) d\sigma_1 \Bigg). \end{aligned}$$

Now, using the Cauchy-Schwartz inequality on the outer expectation shows that

$$\begin{aligned} Z(\tau)^2 \le\ & \mathbb{E}\left(\left| \int_{\Omega_0} \exp\left(f_0(\sigma_0, \tau_0) + i\sqrt{\beta} \sum_{\substack{\{u,v\} \in E(\mathbb{T}_L^d) \\ u \in V_0, v \in V_1}} \langle \sigma_u + \tau_u, X_{\{u,v\}} \rangle \right) d\sigma_0 \right|^2 \right) \cdot \\ & \mathbb{E}\left(\left| \int_{\Omega_1} \exp\left(f_1(\sigma_1, \tau_1) + i\sqrt{\beta} \sum_{\substack{\{u,v\} \in E(\mathbb{T}_L^d) \\ u \in V_0, v \in V_1}} \langle \sigma_v + \tau_v, X_{\{u,v\}} \rangle \right) d\sigma_1 \right|^2 \right). \end{aligned} \tag{36}$$

Let us analyze further the first expectation. We have

$$\left|\int_{\Omega_0} \exp\left(f_0(\sigma_0,\tau_0) + i\sqrt{\beta}\sum_{\substack{\{u,v\}\in E(\mathbb{T}_L^d)\\ u\in V_0, v\in V_1}} \left\langle \sigma_u + \tau_u, X_{\{u,v\}}\right\rangle\right) d\sigma_0\right|^2$$

$$= \int_{\Omega_0}\int_{\Omega_0} \exp\left(f_0(\sigma_0,\tau_0) + f_0(\sigma_0',\tau_0) + i\sqrt{\beta}\sum_{\substack{\{u,v\}\in E(\mathbb{T}_L^d)\\ u\in V_0, v\in V_1}} \left\langle \sigma_u + \tau_u - \sigma'_{\theta(v)} - \tau_{\theta(v)}, X_{\{u,v\}}\right\rangle\right) d\sigma_0 d\sigma_0'.$$

Thus, identifying σ_0' with the configuration $\theta(\sigma_0')$ in Ω_1 we conclude that

$$\mathbb{E}\left(\left|\int_{\Omega_0} \exp\left(f_0(\sigma_0,\tau_0) + i\sqrt{\beta}\sum_{\substack{\{u,v\}\in E(\mathbb{T}_L^d)\\ u\in V_0, v\in V_1}} \left\langle \sigma_u + \tau_u, X_{\{u,v\}}\right\rangle\right) d\sigma_0\right|^2\right) = Z(\tau_0, \theta(\tau_0)).$$

Doing a similar analysis for the second expectation in (36) finishes the proof. □

Step 2: Gaussian Domination. In this step we prove the Gaussian domination inequality (33) as a consequence of the reflection positivity Proposition 2.6.

Observe first that the function $Z(\cdot)$ has a maximum. Indeed, this follows simply from its definition as $Z(\cdot)$ is continuous, $Z(\tau) = Z(\tau + c)$ for any constant c and $Z(\tau)$ tends to zero when the difference of any two coordinates in τ tends to infinity. For each $\tau : V(\mathbb{T}_L^d) \to \mathbb{R}^n$ define $k(\tau)$ to be the number of edges $\{u,v\} \in E(\mathbb{T}_L^d)$ for which $\tau_u \neq \tau_v$. Let k_0 be the minimum of $k(\tau)$ over all τ which maximize $Z(\cdot)$. It suffices to show that $k_0 = 0$ as then there is a maximizer τ of $Z(\cdot)$ with all coordinates equal, whence $Z(\tau) = Z(0)$.

The proof is by contradiction. Suppose that $k_0 > 0$ and let τ be a maximizer of $Z(\tau)$ having $k(\tau) = k_0$. By rotating and translating the torus $\mathbb{T}_L^d$ if necessary, we may assume without loss of generality that there exists an edge $\{u,v\} \in E(\mathbb{T}_L^d)$ with $u \in V_0$, $v \in V_1$ on which $\tau_u \neq \tau_v$. Now, by Proposition 2.6,

$$Z(\tau_0,\tau_1)^2 \leq Z(\tau_0,\theta(\tau_0))Z(\theta(\tau_1),\tau_1)$$

from which it follows that both $(\tau_0, \theta(\tau_0))$ and $(\theta(\tau_1), \tau_1)$ are maximizers of $Z(\cdot)$. It is now straightforward to check that either $k((\tau_0,\theta(\tau_0))) < k(\tau)$ or $k((\theta(\tau_1),\tau_1)) < k(\tau)$, establishing the required contradiction.

Step 3: Infra-red bound. In this step we prove the following bound on the Fourier transform of the correlation function. Write $(\mathbb{T}_L^d)^*$ for the dual torus to $\mathbb{T}_L^d$, whose vertex set is $\frac{\pi}{L}\{-L+1, -L+2, \ldots, L\}^d$.

Proposition 2.7. *(Infra-red bound) For each $k \in V((\mathbb{T}_L^d)^*) \setminus \{0\}$,*

$$\frac{1}{|V(\mathbb{T}_L^d)|}\sum_{u,v\in V(\mathbb{T}_L^d)} e^{i\langle k, v-u\rangle}\mathbb{E}\left(\langle \sigma_u, \sigma_v\rangle\right) \leq \frac{n}{2\beta\left(\sum_{j=1}^d (1-\cos(k_j))\right)}.$$

The name of the bound stems from the fact that we may use translation invariance to rewrite the left-hand side as

$$\sum_{v\in V(\mathbb{T}_L^d)} e^{i\langle k,v\rangle}\mathbb{E}\left(\langle \sigma_{\mathbf{0}},\sigma_v\rangle\right),$$

where $\mathbf{0} := (0,\dots,0)$, whence we recognize the Fourier transform (in the group $\mathbb{T}_L^d$) of the correlation function $v\mapsto \mathbb{E}\left(\langle \sigma_{\mathbf{0}},\sigma_v\rangle\right)$. The above bound thus shows that the Fourier spectrum of the correlation function does not place much mass on non-zero frequencies, corresponding to our goal of showing that the correlation function correlates with a constant function.

Proof of Proposition 2.7. The Gaussian domination inequality (33) implies that for any $\tau : V(\mathbb{T}_L^d)\to\mathbb{R}^n$, the scalar function $\alpha\mapsto Z(\alpha\cdot\tau)$ has a maximum at $\alpha=0$. In particular, its second derivative at $\alpha=0$ is non-positive, yielding that

$$\begin{aligned}\frac{d^2}{d\alpha^2}Z(\alpha\cdot\tau)|_{\alpha=0} &= \frac{d^2}{d\alpha^2}\int_\Omega \exp\left(-\frac{\beta}{2}\sum_{\{u,v\}\in E(\mathbb{T}_L^d)}\|\sigma_u+\alpha\cdot\tau_u-\sigma_v-\alpha\cdot\tau_v\|_2^2\right)d\sigma|_{\alpha=0}\\ &= \int_\Omega\left(\Big(\beta\sum_{\{u,v\}\in E(\mathbb{T}_L^d)}\langle\sigma_u-\sigma_v,\tau_u-\tau_v\rangle\Big)^2-\beta\sum_{\{u,v\}\in E(\mathbb{T}_L^d)}\|\tau_u-\tau_v\|^2\right)\cdot\\ &\qquad\qquad \exp\left(-\frac{\beta}{2}\sum_{\{u,v\}\in E(\mathbb{T}_L^d)}\|\sigma_u-\sigma_v\|_2^2\right)d\sigma\le 0.\end{aligned}$$

Suppose now that σ is randomly sampled from the spin $O(n)$ model on the torus $\mathbb{T}_L^d$. Recalling the density of the spin $O(n)$ model from (1), we may rewrite the last inequality (similarly to the relation (32)) as

$$\mathbb{E}\left(\Big|\beta\sum_{\{u,v\}\in E(\mathbb{T}_L^d)}\langle\sigma_u-\sigma_v,\tau_u-\tau_v\rangle\Big|^2-\beta\sum_{\{u,v\}\in E(\mathbb{T}_L^d)}\|\tau_u-\tau_v\|^2\right)\le 0. \tag{37}$$

This inequality was derived for real-valued τ but remains valid also for complex-valued τ by applying the inequality to the real and imaginary parts. Now recall the *discrete Green identity.* For $r,s : V(\mathbb{T}_L^d)\to\mathbb{R}^n$ we have

$$\sum_{\{u,v\}\in E(\mathbb{T}_L^d)}\langle r_u-r_v,s_u-s_v\rangle=\sum_{u\in V(\mathbb{T}_L^d)}\langle r_u,-(\Delta s)_u\rangle$$

where

$$(\Delta s)_u := \sum_{v:\,\{u,v\}\in E(\mathbb{T}_L^d)}(s_v-s_u).$$

Applying the identity in (37) and rearranging gives

$$\mathbb{E}\left(\Big|\sum_{u\in V(\mathbb{T}_L^d)}\langle\sigma_u,-(\Delta\tau)_u\rangle\Big|^2\right)\le\frac{1}{\beta}\sum_{u\in V(\mathbb{T}_L^d)}\langle\tau_u,-(\Delta\tau)_u\rangle. \tag{38}$$

We now make the following choice for τ. Denote by $\mathbf{1} \in \mathbb{R}^n$ the vector with all 1 coordinates. Let $k \in V((\mathbb{T}_L^d)^*) \setminus \{0\}$ and take

$$\tau_u := e^{i\langle k,u\rangle} \cdot \mathbf{1}.$$

With this choice,

$$(\Delta\tau)_u = \sum_{v\colon \{u,v\}\in E(\mathbb{T}_L^d)} (e^{i\langle k,v\rangle} - e^{i\langle k,u\rangle}) \cdot \mathbf{1} = 2\left(\sum_{j=1}^d (\cos(k_j) - 1)\right) e^{i\langle k,u\rangle} \cdot \mathbf{1}.$$

Substituting back into (38) shows that

$$\mathbb{E}\left(\Big|\sum_{u\in V(\mathbb{T}_L^d)} e^{-i\langle k,u\rangle} \langle \sigma_u, \mathbf{1}\rangle\Big|^2\right) \le \frac{n|V(\mathbb{T}_L^d)|}{2\beta\left(\sum_{j=1}^d (1-\cos(k_j))\right)},$$

which implies the infra-red bound upon expanding the square on the left-hand side. $\square$

Step 4: Long-range order. Observe that for any $a \in \{-2L+1,\dots,2L-1\}^n$,

$$\sum_{k\in V((\mathbb{T}_L^d)^*)} e^{i\langle k,a\rangle} = \prod_{j=1}^n \left(\sum_{k_j=-L+1}^{L} e^{\pi i k_j a_j/L}\right) = \begin{cases} 0 & a \neq 0 \\ |V(\mathbb{T}_L^d)| & a = 0 \end{cases}.$$

Thus, the infra-red bound (Proposition 2.7) implies that

$$\begin{aligned} n|V(\mathbb{T}_L^d)| &= \frac{1}{|V(\mathbb{T}_L^d)|} \sum_{u,v\in V(\mathbb{T}_L^d)} \sum_{k\in V((\mathbb{T}_L^d)^*)} e^{i\langle k,v-u\rangle} \mathbb{E}\left(\langle \sigma_u,\sigma_v\rangle\right) \\ &\le \frac{1}{|V(\mathbb{T}_L^d)|} \sum_{u,v\in V(\mathbb{T}_L^d)} \mathbb{E}\left(\langle \sigma_u,\sigma_v\rangle\right) + \frac{n}{2\beta} \sum_{k\in V((\mathbb{T}_L^d)^*)\setminus\{0\}} \frac{1}{\left(\sum_{j=1}^d (1-\cos(k_j))\right)}. \end{aligned}$$

Rearranging, we finally obtain that

$$\frac{1}{|V(\mathbb{T}_L^d)|^2} \sum_{u,v\in V(\mathbb{T}_L^d)} \mathbb{E}\left(\langle \sigma_u,\sigma_v\rangle\right) \ge 1 - \frac{1}{2\beta|V(\mathbb{T}_L^d)|} \sum_{k\in V((\mathbb{T}_L^d)^*)\setminus\{0\}} \frac{1}{\sum_{j=1}^d (1-\cos(k_j))}. \tag{39}$$

It remains to note that, identifying a Riemann sum on the right-hand side,

$$\lim_{L\to\infty} \frac{1}{|V(\mathbb{T}_L^d)|} \sum_{k\in V((\mathbb{T}_L^d)^*)\setminus\{0\}} \frac{1}{\sum_{j=1}^d (1-\cos(k_j))} = \int_{[-\pi,\pi]^d} \frac{1}{\sum_{j=1}^d (1-\cos(x_j))} dx,$$

and the integral is finite in dimensions $d \ge 3$ as $1-\cos(x_j)$ is of order x_j^2 when x_j is small. Thus the sum on the right-hand side of (39) is uniformly bounded in L for $d \ge 3$, showing that when β is sufficiently large the right-hand side is positive and proving Theorem 2.5.

2.7 Slow decay of correlations in spin $O(2)$ models - heuristics for the Berezinskii-Kosterlitz-Thouless transition and a theorem of Aizenman

In this section we consider the question of proving a *polynomial lower bound* on the decay of correlations in the two-dimensional spin $O(2)$ model. As described in Section 2.2, this was achieved for the XY model at sufficiently low temperatures in the celebrated work of Fröhlich and Spencer on the Berezinskii-Kosterlitz-Thouless transition (*** ref ***). The result is too difficult to present within the scope of our notes and instead we start by giving a heuristic reason for the existence of the transition. The heuristic suggests that a polynomial lower bound on correlations will *always* hold in the spin $O(2)$ model with a potential U of bounded support (as explained below). We then proceed by presenting a theorem of Aizenman [1], following earlier predictions by Patrascioiu and Seiler [20], who made rigorous a version of the last statement.

2.7.1 Heuristic for the Berezinskii-Kosterlitz-Thouless transition and vortices in the XY model

To motivate the result, let us first give a heuristic argument for the Berezinskii-Kosterlitz-Thouless phase transition. Let $h : V(\mathbb{T}_L^2) \to \mathbb{R}$ be a randomly sampled *discrete Gaussian free field*. By this, we mean that $h((0,0)) := 0$ and h is sampled from the probability measure

$$\frac{1}{Z^{\mathrm{DGFF}}_{\mathbb{T}_L^2,\beta}} \exp\left[-\beta \sum_{\{u,v\}\in E(G)} (h_u - h_v)^2\right] \prod_{\substack{v\in V(\mathbb{T}_L^2)\\ v\neq(0,0)}} dm(h_v), \tag{40}$$

with $Z^{\mathrm{DGFF}}_{\mathbb{T}_L^2,\beta}$ a suitable normalization constant and dm standing for the Lebesgue measure on $\mathbb{R}$. As the expression in the exponential is a quadratic form in h, it follows that h has a multi-dimensional Gaussian distribution with zero mean. Moreover, the matrix of this quadratic form is proportional to the graph Laplacian of $\mathbb{T}_L^2$, whence the covariance structure of h is proportional to the Green's function of $\mathbb{T}_L^2$. In particular,

$$\mathrm{Var}(h_x) = \mathrm{Var}(h_x - h_0) \approx \frac{a}{\beta} \log \|x-y\|_1 \tag{41}$$

for large $\|x-y\|_1$, with some specific constant $a > 0$. Now consider the random configuration $\sigma : V(\mathbb{T}_L^2) \to \mathbb{S}^1$, with $\mathbb{S}^1$ identified with the unit circle in the complex plane, obtained by setting

$$\sigma_v := \exp(ih_v). \tag{42}$$

This configuration has some features in common with a sample of the XY model (normalized to have $\sigma_{(0,0)} = 1$). Although its density is not a product of nearest-neighbor terms, one may imagine that the main contribution to it does come from nearest-neighbor interactions, at least for large β when the differences $h_u - h_v$ of nearest neighbors tend to be small. The interaction term $-\beta(h_u - h_v)^2$ in (40) is then rather akin to an interaction term of the form $\frac{\beta}{2}\langle\sigma_u, \sigma_v\rangle$ as in the XY model (as $\langle s, y\rangle$ is the cosine of the difference of arguments between

s and t and one may consider its Taylor expansion around $s = t$). The main advantage in this definition of σ is that it allows a precise calculation of correlation. Indeed, as h_x has a centered Gaussian distribution with variance given by (41), it follows that

$$\rho_{x,(0,0)} := \mathbb{E}(\langle \sigma_x, \sigma_{(0,0)} \rangle) = \mathbb{E}(\cos(h_x)) = e^{-\frac{\mathrm{Var}(h_x)}{2}} \approx \|x-y\|_1^{-\frac{a}{\beta}}, \tag{43}$$

and thus σ exhibits polynomial decay of correlations.

There are many reasons why the analogy between the definition (42) and samples of the XY model should not hold. Of these, the notion of vortices has been highlighted in the literature. Suppose now that $\sigma : V(\mathbb{T}_L^2) \to \mathbb{S}^1$ is an *arbitrary* configuration. Associate to each directed edge (u, v), where $\{u, v\} \in E(\mathbb{T}_L^2)$, the difference $\theta_{(u,v)}$ in the arguments of σ_u and σ_v, with the convention that $\theta_{(u,v)} \in [-\pi, \pi)$. Call a 2×2 'square' in the graph $\mathbb{T}_L^2$ a *plaquette* (these are exactly the simple cycles of length 4 in $\mathbb{T}_L^2$). For a plaquette P, set s_P to be the sum of $\theta_{(u,v)}$ on the edges around the plaquette going in 'clockwise' order, say. We necessarily have that $s_P \in \{-2\pi, 0, 2\pi\}$ and one says that there is a *vortex* at P if $s_P \neq 0$, with charge plus or minus according to the sign of s_P. Vortices form an obstruction to defining a *height function* h for which (42) holds, as one would naturally like the differences of this h to be the $\theta_{(u,v)}$, but then one must have $s_P = 0$ for all plaquettes. Existence of vortices means that h needs to be a multi-valued function, with a non-trivial *monodromy* around plaquettes with $s_P \neq 0$.

Now take σ to be a sample of the XY model on $\mathbb{T}_L^2$ at inverse temperature β. When β is small, the model is disordered as one may deduce from the high-temperature expansion (Section 2.3) and there are vortices of both charges in a somewhat chaotic fashion (a 'plasma' of vortices), making the analogy with the definition (42) rather weak. Indeed, in this case there is exponential decay of correlations violating (43). However, when β is large, it can be shown (e.g., by the so-called chessboard estimate) that large differences $\theta_{(u,v)}$ in the angles are rare, whence vortices are rare too. Thus, one may hope vortices to *bind* together, coming in structures of small diameter of overall neutral charge (the smallest structure is a *dipole*, having one plus and one minus vortex). When this occurs, the height function h can be defined as a single-valued function at most vertices and one may hope that the analogy (42) is of relevance so that, in particular, polynomial decay of correlations holds. This gives a heuristic reason for the Berezinskii-Kosterlitz-Thouless transition (*** ref ***).

2.7.2 Slow decay of correlations for Lipschitz spin $O(2)$ models

The above heuristic suggests the consideration of the spin $O(2)$ model with a potential U of *bounded support*. By this we mean a measurable $U : [-1, 1] \to (-\infty, \infty]$ (allowing here $U(r) = \infty$) which satisfies

$$U(r) = \infty \quad \text{when } r < r_0 \in (-1, 1).$$

This property constrains the corresponding $O(2)$ model so that adjacent spins have difference of arguments at most $\arccos(r_0)$. Such a spin configuration may naturally be called *Lipschitz* (as in a Lipschitz function). If $r_0 \geq 0$, the maximal difference allowed is at most $\frac{\pi}{2}$ which implies that the spin configuration is *free of vortices* with probability one. If indeed vortices are the reason behind the Berezinskii-Kosterlitz-Thouless transition, then one may expect such models to always exhibit polynomial decay of correlations. Patrascioiu and Seiler [20] predicted, based on rigorous mathematical statements and certain yet unproven conjectures, that a phenomenon of this kind should hold. Aizenman [1] then gave a beautiful proof of a version of the above statement, which we now proceed to present.

A function $U : [-1, 1] \to (-\infty, \infty]$ is called *non-increasing* if

$$U(r_1) \geq U(r_2) \text{ when } r_1 \leq r_2. \tag{44}$$

This property implies that the spin $O(2)$ model with potential U is *ferromagnetic* in the sense that configurations in which the differences of angles between adjacent spins are reduced have higher density (see (3)).

Theorem 2.8. *Let $U : [-1, 1] \to (-\infty, \infty]$ be non-increasing and satisfy*

$$U(r) = \infty \quad \text{when } r < \frac{1}{\sqrt{2}}. \tag{45}$$

Suppose that $\sigma : V(\mathbb{T}_L^2) \to \mathbb{S}^1$ is randomly sampled from the two-dimensional spin $O(2)$ model with potential U. Then, for any integer $1 \leq \ell \leq L$,

$$\max_{\substack{x,y \in V(\mathbb{T}_L^2) \\ \|x-y\|_1 \geq \ell}} \rho_{x,y} = \max_{\substack{x,y \in V(\mathbb{T}_L^2) \\ \|x-y\|_1 \geq \ell}} \mathbb{E}(\langle \sigma_x, \sigma_y \rangle) \geq \frac{1}{2\ell^2}. \tag{46}$$

We make a few remarks regarding the statement. First, one would expect that $\rho_{x,y}$ is at least a power of $\|x-y\|_1$ for all $x, y \in V(\mathbb{T}_L^2)$. The bound (46) is a little weaker in that it only shows existence of a pair x, y with this property (the proof actually gives a slightly stronger statement, see *** below), but is still enough to rule out exponential decay of correlations in the sense we saw occurs at high temperatures (see Section 2.3. Second, the bound (46) can be said to hold at all temperatures in that it will continue to hold if we multiply the potential U by any constant. Third, the constraint (45) is stronger than the constraint discussed above which would prohibit vortices ($U(r) = \infty$ if $r < 0$). This stronger assumption is used in the proof and it remains open to understand the behavior with other versions of the constraint. (*** polynomial upper bound too in forthcoming work GMP ***)

We proceed to the proof of Theorem 2.8. Let U be a potential as in the theorem and $\sigma : V(\mathbb{T}_L^2) \to \mathbb{S}^1$ be randomly sampled from the two-dimensional spin $O(2)$ model with potential U.

Step 1: Passing to $\{-1, 1\}$-valued random variables. A main idea in the proof, suggested in the work of Patrascioiu and Seiler [20], is to consider the configuration σ conditioned on the y coordinate of each spin and identify an Ising-type model which is embedded in the configuration. Precisely, let us identify $\mathbb{S}^1$ with the (real) plane $\mathbb{R}^2$ and write the coordinates explicitly,

$$\sigma_v = (\sigma_v^1, \sigma_v^2) \quad \text{at each vertex } v \in V(\mathbb{T}_L^2).$$

We write σ^1 (σ^2) for the function $\sigma_\cdot^1$ ($\sigma_\cdot^2$). Then, for every pair of vertices $x, y \in V(\mathbb{T}_L^2)$,

$$\rho_{x,y} = \mathbb{E}(\langle \sigma_x, \sigma_y \rangle) = \mathbb{E}(\sigma_x^1 \sigma_y^1 + \sigma_x^2 \sigma_y^2) = 2\mathbb{E}(\sigma_x^1 \sigma_y^1),$$

as the distribution of σ is invariant to global rotations ((σ^1, σ^2) has the same distribution as (σ^2, σ^1) from the choice of density (3)). In particular, conditioning on σ^2,

$$\rho_{x,y} = 2\mathbb{E}(\mathbb{E}(\sigma_x^1 \sigma_y^1 \mid \sigma^2)) = 2\mathbb{E}(|\sigma_x^1| \cdot |\sigma_y^1| \cdot \mathbb{E}(\varepsilon_x \varepsilon_y \mid \sigma^2)), \tag{47}$$

where we write $\sigma_v^1 = |\sigma_v^1|\varepsilon_v$ where $\varepsilon_v \in \{-1, 1\}$ and we note that $|\sigma_v^1|$ is determined from σ_v^2 as $\sigma_v \in \mathbb{S}^1$.

Step 2: Non-negativity of conditional correlations. Our goal now is to show that

$$\mathbb{E}(\varepsilon_x \varepsilon_y \mid \sigma^2) \geq 0 \quad \text{for every } x, y \in V(\mathbb{T}_L^2), \text{ almost surely.} \tag{48}$$

To this end, we proceed to develop a high-temperature expansion for the signs ε, conditioned on σ^2, similarly to the derivation in Section 2.3. Observe that for almost every value of σ^2, the density of the signs ε conditioned on σ^2 (with respect to the uniform product measure on $\{-1, 1\}^{V(\mathbb{T}_L^2)}$) is

$$\frac{1}{Z_{\sigma^2}} \exp\left[-\sum_{\{u,v\} \in E(\mathbb{T}_L^2)} U(\langle \sigma_u, \sigma_v \rangle)\right] = \frac{1}{Z_{\sigma^2}} \exp\left[-\sum_{\{u,v\} \in E(\mathbb{T}_L^2)} U(|\sigma_u^1| \cdot |\sigma_v^1|\varepsilon_u \varepsilon_v + \sigma_u^2 \sigma_v^2)\right]$$

where

$$Z_{\sigma^2} = \sum_{\varepsilon \in \{-1,1\}^{V(\mathbb{T}_L^2)}} \exp\left[-\sum_{\{u,v\} \in E(\mathbb{T}_L^2)} U(|\sigma_u^1| \cdot |\sigma_v^1|\varepsilon_u \varepsilon_v + \sigma_u^2 \sigma_v^2)\right].$$

Now observe that our assumption that U is non-increasing implies that

$$U(|\sigma_u^1| \cdot |\sigma_v^1| + \sigma_u^2 \sigma_v^2) \leq U(-|\sigma_u^1| \cdot |\sigma_v^1| + \sigma_u^2 \sigma_v^2) \quad \text{for every } u, v \in V(\mathbb{T}_L^2), \text{ almost surely.}$$

With this in mind, we write

$$f_{\{u,v\}}(\varepsilon) := \exp\big[-U(|\sigma_u^1| \cdot |\sigma_v^1|\varepsilon_u \varepsilon_v + \sigma_u^2 \sigma_v^2) + U(-|\sigma_u^1| \cdot |\sigma_v^1| + \sigma_u^2 \sigma_v^2)\big] - 1 \geq 0, \quad u, v \in V(\mathbb{T}_L^2), \tag{49}$$

and

$$f_0 := \exp\left[-\sum_{\{u,v\} \in E(\mathbb{T}_L^2)} U(-|\sigma_u^1| \cdot |\sigma_v^1| + \sigma_u^2 \sigma_v^2)\right]$$

so that

$$Z_{\sigma^2} = f_0 \sum_{\varepsilon \in \{-1,1\}^{V(\mathbb{T}_L^2)}} \prod_{\{u,v\} \in E(\mathbb{T}_L^2)} (f_{\{u,v\}}(\varepsilon) + 1) = F \sum_{E \subset E(\mathbb{T}_L^2)} \sum_{\varepsilon \in \{-1,1\}^{V(\mathbb{T}_L^2)}} \prod_{\{u,v\} \in E} f_{\{u,v\}}(\varepsilon).$$

Exactly as in Section 2.3, we interpret the last equality as prescribing a probability measure (as $f_{\{u,v\}}(\varepsilon) \geq 0$ from (49)) over pairs (E, ε). Examining the definition (49), we see that in this probability measure, conditioned on E, the signs ε are obtained by independently assigning to the vertices in each connected component of $(V(\mathbb{T}_L^2), E)$ the same spin value, picked uniformly from $\{-1, 1\}$. Thus, for every $x, y \in V(\mathbb{T}_L^2)$, $\mathbb{E}(\varepsilon_x \varepsilon_y \mid \sigma^2)$ equals the probability that x is connected to y in the graph $(V(\mathbb{T}_L^2), E)$. The non-negativity statement (48) is an immediate consequence.

Remark 2.5. *Of course, we may also conclude from* (48) *and* (47) *that the correlations* $\rho_{x,y}$ *themselves are non-negative. In fact, the above derivation may be equally done for the spin* $O(n)$ *model with non-increasing potential* U *on any graph* G *and for any* $n \geq 1$ *(by conditioning on all the coordinates of* σ *but the first) to conclude that such models have non-negative correlations. This is the case, in particular, for the standard spin* $O(n)$ *models. We point out that this need not be the case, however, for completely general potentials. For instance, suppose* σ *is a sample from the standard spin* $O(n)$ *model on* $\mathbb{T}_L^d$ *at inverse temperature* β *and define a new configuration* τ *by setting* $\tau_v = \sigma_v$ *at even vertices* v *and* $\tau_v = -\sigma_v$ *at odd vertices* v*. Here, a vertex is even if the sum of its coordinates is even and otherwise odd. As the correlations of* σ *are non-negative, the correlations* $\rho_{x,y}$ *of* τ*, between an even vertex* x *and an odd vertex* y*, will be non-positive, and in fact negative (this also follows from the above derivation of* (48)*). However, the density of* τ *is the same as in* (1) *with* β *replaced by* $-\beta$ *(*τ *is a sample of the* anti-ferromagnetic *spin* $O(n)$ *model). In other words,* τ *is sampled from the spin* $O(n)$ *model with potential* $U(r) = \beta r$ *(which is indeed increasing).*

Step 3: A lower bound on correlations in terms of connectivity. A key idea in the analysis of Aizenman [1] is the consideration of the following random set of vertices

$$V_0 := \left\{ v \in V(\mathbb{T}_L^2) \colon |\sigma_v^1| \geq \frac{1}{\sqrt{2}} \right\}.$$

Note that this set is measurable with respect to σ^2. Let us consider the relevance of this set to the conditional correlations $\mathbb{E}(\varepsilon_x \varepsilon_y \mid \sigma^2)$ discussed above.

For reasons that will become clear in the next step, we introduce a second adjacency relation on the vertices $V(\mathbb{T}_L^2)$. We say that $u, v \in V(\mathbb{T}_L^2)$ are $\boxtimes$-adjacent if $\{u, v\} \in E(\mathbb{T}_L^2)$ or u, v are next-nearest-neighbors in $\mathbb{T}_L^2$ which differ in both coordinates (they are diagonal neighbors). Now observe that, almost surely,

$$\text{if } u, v \text{ are } \boxtimes\text{-adjacent and both } u, v \in V_0 \text{ then } \varepsilon_u = \varepsilon_v.$$

This is a consequence of the bounded support constraint (45) and it is here that the number $\frac{1}{\sqrt{2}}$ in that constraint is important (as we are allowing next-nearest-neighbors). Together with the non-negativity property (48), it follows that conditionally on σ^2, for every $x, y \in V(\mathbb{T}_L^2)$, almost surely,

$$\mathbb{E}(\varepsilon_x \varepsilon_y \mid \sigma^2) \geq \mathbf{1}(E_{x,y})$$

where we write $\mathbf{1}(E)$ for the indicator function of the event

$$E_{x,y} := \{x \text{ and } y \text{ are connected in the graph on } V_0 \subseteq V(\mathbb{T}_L^2) \text{ with the } \boxtimes\text{-adjacency}\}.$$

Plugging this relation back into the identity (47) for the correlation $\rho_{x,y}$ shows that

$$\rho_{x,y} \geq 2\mathbb{E}(|\sigma_x^1| \cdot |\sigma_y^1| \mathbf{1}(E)) \geq \mathbb{P}(E_{x,y}), \tag{50}$$

where we used that $|\sigma_x^1| \cdot |\sigma_y^1| \geq \frac{1}{2}$ when $x, y \in V_0$. We now proceed to deduce Theorem 2.8 from this lower bound.

Step 4: Duality for vertex crossings. Fix an integer $1 \le \ell \le L$ and define the discrete square $R := \{1, \ldots, \ell\}^2 \subseteq V(\mathbb{T}_L^2)$.

Geometric fact: For any subset $R_0 \subseteq R$, either there is a top-bottom crossing of R with vertices of R_0 and the $\boxtimes$-adjacency or there is a left-right crossing of R with vertices of $R \setminus R_0$ and the standard nearest-neighbor adjacency (that of $\mathbb{T}_L^2$).
The fact is intuitive though finding a simple proof requires some ingenuity. We refer the reader to Timár [23] for this and related statements.

Now consider the two events

$$E := \{\text{there is a top-bottom crossing of } R \text{ with vertices of } V_0 \text{ and the } \boxtimes\text{-adjacency}\},$$
$$F := \{\text{there is a left-right crossing of } R \text{ with vertices of } V(\mathbb{T}_L^2) \setminus V_0 \text{ and the standard adjacency}\}.$$

By rotational-symmetry of the configuration σ (its distribution is invariant under applying a global rotation) we have $\mathbb{P}(F) = \mathbb{P}(\tilde{F})$ where

$$\tilde{F} := \{\text{there is a left-right crossing of } R \text{ with vertices of } V_0 \text{ and the standard adjacency}\}.$$

In particular, as R is a square and since it easier to be connected in the $\boxtimes$-adjacency than in the nearest-neighbor adjacency, we conclude that

$$\mathbb{P}(E) \ge \mathbb{P}(F). \tag{51}$$

Lastly, the geometric fact implies that $\mathbb{P}(E \cup F) = 1$, whence

$$1 = \mathbb{P}(E \cup F) \le \mathbb{P}(E) + \mathbb{P}(F) \overset{(51)}{\le} 2\mathbb{P}(E) \le 2 \sum_{\substack{x=(a,1),\, 1 \le a \le \ell \\ y=(b,\ell),\, 1 \le b \le \ell}} \mathbb{P}(E_{x,y}) \overset{(50)}{\le} 2 \sum_{\substack{x=(a,1),\, 1 \le a \le \ell \\ y=(b,\ell),\, 1 \le b \le \ell}} \rho_{x,y}, \tag{52}$$

from which Theorem 2.8 follows.

3 The Loop $O(n)$ model

3.1 Definitions

Let $\mathbb{H}$ denote the hexagonal lattice. A *loop* is a finite subgraph of $\mathbb{H}$ which is isomorphic to a simple cycle. A *loop configuration* is a spanning subgraph of $\mathbb{H}$ in which every vertex has even degree; see Figure 1. The *non-trivial finite* connected components of a loop configuration are necessarily loops, however, a loop configuration may also contain isolated vertices and infinite simple paths. We shall often identify a loop configuration with its set of edges, disregarding isolated vertices. A *domain* H is a non-empty finite connected induced subgraph of $\mathbb{H}$ whose complement $V(\mathbb{H}) \setminus V(H)$ induces a connected subgraph of $\mathbb{H}$ (in other words, it does not have "holes"). Given a domain H, we denote by $\mathsf{LoopConf}(H)$ the collection of all loop configurations ω that are contained in $\mathbb{H}$. Finally, for a loop configuration ω, we denote by $L(\omega)$ the number of loops in ω and by $o(\omega)$ the number of edges of ω.

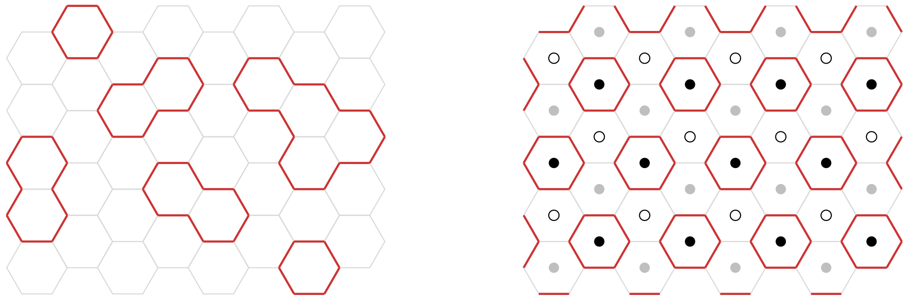


Figure 1: On the left, a loop configuration. On the right, a proper 3-coloring of the triangular lattice $\mathbb{T}$ (the dual of the hexagonal lattice $\mathbb{H}$), inducing a partition of $\mathbb{T}$ into three color classes $\mathbb{T}^0$, $\mathbb{T}^1$, and $\mathbb{T}^2$. The 0-phase ground state ω^0_{gnd} is the (fully-packed) loop configuration consisting of trivial loops around each hexagon in $\mathbb{T}^0$.

Let H be a domain and let n and x be positive real numbers. The loop $O(n)$ measure on H with edge weight x is the probability measure $\mathbb{P}_{H,n,x}$ on $\mathsf{LoopConf}(H)$ defined by

$$\mathbb{P}_{H,n,x}(\omega) := \frac{x^{o(\omega)} n^{L(\omega)}}{Z^{\text{loop}}_{H,n,x}},$$

where $Z^{\text{loop}}_{H,n,x}$, the partition function, is given by

$$Z^{\text{loop}}_{H,n,x} := \sum_{\omega \in \mathsf{LoopConf}(H)} x^{o(\omega)} n^{L(\omega)}.$$

The $x = \infty$ Model. We also consider the limit of the loop $O(n)$ model as the edge weight x tends to infinity. This means restricting the model to 'optimally packed loop configurations', i.e., loop configurations having the maximum possible number of edges.

Let H be a domain and let $n > 0$. The loop $O(n)$ measure on H with edge weight $x = \infty$ is the probability measure on $\mathsf{LoopConf}(H)$ defined by

$$\mathbb{P}_{H,n,\infty}(\omega) := \lim_{x\to\infty} \mathbb{P}_{H,n,x}(\omega) = \begin{cases} \frac{n^{L(\omega)}}{Z_{H,n,\infty}} & \text{if } o(\omega) = o_H \\ 0 & \text{otherwise} \end{cases},$$

where $o_H := \max\{o(\omega) : \omega \in \mathsf{LoopConf}(H)\}$ and $Z_{H,n,\infty}$ is the unique constant making $\mathbb{P}_{H,n,\infty}$ a probability measure. We note that if a loop configuration $\omega \in \mathsf{LoopConf}(H)$ is *fully-packed*, i.e., every vertex in $V(H)$ has degree 2, then ω is optimally packed, i.e., $o(\omega) = o_H$. In particular, if such a configuration exists for the domain H then the measure $\mathbb{P}_{H,n,\infty}$ is supported on fully-packed loop configurations.

3.2 Relation to the spin $O(n)$ model

We note that the loop $O(n)$ model is defined for any *real* $n > 0$ whereas the spin $O(n)$ model is only defined for positive *integer* n (the loop $O(n)$ model may be defined also with $n = 0$ by taking the limit $n \to 0$, giving rise to a self-avoiding walk model). For integer n, there is a connection between the loop and the spin $O(n)$ models on a domain $H \subset \mathbb{H}$. Rewriting the partition function $Z^{\text{spin}}_{H,n,\beta}$ given by (2) using the approximation $e^t \approx 1 + t$ gives

$$\begin{aligned}
Z^{\text{spin}}_{H,n,\beta} &= \int_{\Omega} \prod_{\{u,v\}\in E(H)} \exp\left[\beta\langle \sigma_u, \sigma_v\rangle\right] \, d\sigma \\
&\approx \int_{\Omega} \prod_{\{u,v\}\in E(H)} (1 + \beta\langle \sigma_u, \sigma_v\rangle) \, d\sigma = \sum_{\omega\subset E(H)} (\beta/n)^{o(\omega)} \int_{\Omega} \prod_{\{u,v\}\in E(\omega)} \langle \sqrt{n}\cdot\sigma_u, \sqrt{n}\cdot\sigma_v\rangle \, d\sigma, \\
&\qquad\qquad\qquad\qquad\qquad\qquad = \sum_{\omega\in\mathsf{LoopConf}(H)} (\beta/n)^{o(\omega)} n^{L(\omega)},
\end{aligned}$$

where the last equality follows by splitting the integral into a product of integrals on each connected component of ω and then using the following calculation.

Exercise. Let $E \subset E(\mathbb{H})$ be finite and connected. Show that

$$\int_{\Omega} \prod_{\{u,v\}\in E} \langle \sqrt{n}\cdot\sigma_u, \sqrt{n}\cdot\sigma_v\rangle \, d\sigma = \begin{cases} n & \text{if } E \text{ is a loop} \\ 0 & \text{otherwise} \end{cases}.$$

(see [8, Appendix A] for the calculation)

Hence, substituting x for β/n, we obtain

$$Z^{\text{spin}}_{H,n,nx} \approx Z^{\text{loop}}_{H,n,x}.$$

In the same manner, the correlation $\rho_{u,v}$ for $u, v \in V(H)$ in the spin $O(n)$ model at inverse temperature $\beta = nx$ may be approximated as follows.

$$\rho_{u,v} = \frac{\displaystyle\int_{\Omega} \langle \sigma_u, \sigma_v\rangle \prod_{\{w,z\}\in E(H)} \exp\left[\beta\langle \sigma_w, \sigma_z\rangle\right]}{Z^{\text{spin}}_{H,n,\beta}} \approx n \cdot \frac{\displaystyle\sum_{\lambda\in\mathsf{LoopConf}(H,u,v)} x^{o(\lambda)} n^{L'(\lambda)} J(\lambda)}{Z^{\text{loop}}_{H,n,x}}, \tag{53}$$

where $\mathsf{LoopConf}(H, u, v)$ is the set of spanning subgraphs of H in which the degrees of u and v are odd and the degrees of all other vertices are even. Here, for $\lambda \in \mathsf{LoopConf}(H, u, v)$, $o(\lambda)$ is the number of edges of λ, $L'(\lambda)$ is the number of loops in λ after removing an arbitrary simple path in λ between u and v, and $J(\lambda) := \frac{3n}{n+2}$ if there are three disjoint paths in λ between u and v and $J(\lambda) := 1$ otherwise (in which case, there is a unique simple path in λ between u and v).

Exercise. Use the approximation $e^t \approx 1 + t$ to obtain the asserted representation in (53) (see [8, Appendix A] for the calculation).

Unfortunately, the above approximation is not justified for any $x > 0$. Nevertheless, (53) provides a heuristic connection between the spin and the loop $O(n)$ models and suggests that both these models reside in the same universality class. For this reason, it is natural to ask whether the prediction about the absence of phase transition is valid for the loop $O(n)$ model.

Question: Does the quantity on the right-hand side of (53) decay exponentially fast in the distance between u and v, uniformly in the domain H, whenever $n > 2$ and $x > 0$?

This question is partially answered in [8], where it is showed that for all sufficiently large n and any $x > 0$, the quantity on the right-hand side of (53) decays exponentially fast for a large class of domains H. The result is a consequence of a more detailed understanding of the loop $O(n)$ model with large n which we elaborate on in Section 3.5.

(*** Add section on exact representations of spin $O(n)$ model ***)

3.3 Conjectured phase diagram and equivalent models

It is predicted (*** ref ***) that the loop $O(n)$ model exhibits critical behavior when the parameter $n \in [0, 2]$. In this regime, the model should have a critical value $x_c(n)$ with the formula

$$x_c(n) := \frac{1}{\sqrt{2 + \sqrt{2 - n}}}.$$

The prediction is that for $x < x_c$ the model is *sub-critical* in the sense that the probability that a loop passing through a given point has length longer than t decays *exponentially* in t. When $x \geq x_c$, the model should be *critical*, with the same probability decaying only *polynomially* in t and with the model exhibiting a conformally invariant scaling limit. Furthermore, there should be two critical regimes: when $x = x_c$ and $x > x_c$, each characterized by its own conformally invariant scaling limit (the same one for all $x > x_c$ and a different one for $x = x_c$). Kager and Nienhuis (*** ref ***) predict that in both cases, the loops should scale in a proper limit to random *Schramm Löwner evolution* (SLE) curves with parameter κ satisfying

$$n = -2 \cos\left(\frac{4\pi}{\kappa}\right),$$

where, however, we take the solution of the above equation to satisfy $\kappa \in [\frac{8}{3}, 4]$ when $x = x_c$ and $\kappa \in [4, 8]$ when $x > x_c$. When the parameter n satisfies $n > 2$ it is predicted that the model is always subcritical in the sense of exponential decay of loop lengths described above (*** see also later section ***). These predictions have been mathematically validated only in very special cases. The Ising model case, when $n = 1$, $x = x_c(1) = \frac{1}{\sqrt{3}}$ is known to be critical with its loops scaling to SLE(3) (*** ref, DCS10? ***). In the percolation case, $n = x = 1$, Smirnov proved that the loops scale to SLE(6) (*** ref ***). In the self-avoiding walk case, $n = 0$ (*** explain this value? ***), it was proved by Duminil-Copin and Smirnov that the critical value of x (the inverse of the connective constant of the hexagonal lattice) is $x_c(0) = \frac{1}{\sqrt{2+\sqrt{2}}}$ (*** see later section? ***), though conformal invariance and convergence to SLE have not been established. Furthermore, it was shown in the $n = 0$ case that for $x > x_c(0)$ the self-avoiding path is space filling (*** ref ***). Lastly, it has been shown (*** ref ***) that for large values of n, there is exponential decay of loop lengths for all values of x (*** see later section? ***).

Like in the spin model, special cases of the loop $O(n)$ model have names of their own:

- When $n = 0$, one formally obtains the self-avoiding walk (SAW).

- When $n = 1$, the model is equivalent to the Ising model on the triangular lattice (the loops represent the interfaces between spins of different value).
- When $n = 1$ and $x = 1$, the model is equivalent to the Ising model on the triangular lattice at infinite temperature, which in turn is the same as critical site percolation on the triangular lattice.
- When $n = 1$ and $x = \infty$, the model is equivalent to the anit-ferromagnetic Ising model on the triangular lattice at zero temperature, and is also called the dimer model.
- When $n \geq 2$ is an integer, the model is a marginal of a discrete random Lipschitz function on the triangular lattice. When $n = 2$ this function takes integer values and when $n \geq 3$ it takes values in the n-regular tree. See Section 3.3.1 for more details.
- When $n = \infty$ (formally, $n \to \infty$ and $nx^6 \to c > 0$), the model becomes the hard-hexagon model.

(*** Also dilute Potts model when $n = \sqrt{q}$, q integer, ref Nienhuis (see Kager and Nienhuis) ***)

3.3.1 Relation to Lipschitz functions

When the parameter n is a positive integer, the loop $O(n)$ model admits a height function representation, which is a special case of a graph homomorphism. Let T'_n be the n-regular tree (so that $T'_1 = \{+,-\}$ and $T'_2 = \mathbb{Z}$) rooted at an arbitrary vertex ρ, and let T_n be the graph obtained from T'_n by adding a loop at every vertex. Consider a graph homomorphism φ from the triangular lattice $\mathbb{T}$ (the dual of the hexagonal lattice) to T_n, and note that the interfaces arising from the level lines of φ define a loop configuration ω_φ. Given a domain H, we restrict our attention to those homomorphism φ satisfying that $\varphi(t) = \rho$ for all hexagons $t \in \mathbb{T}$ which are not entirely contained in H (where t is seen as a face of $\mathbb{H}$). Now, if one samples such a random homomorphism φ with probability proportional to $x^{\text{number of edges in } \omega_\varphi}$, then it is straightforward to check that ω_φ is distributed according to $\mathbb{P}_{H,n,x}$. In this sense, φ is a height function representation of the model. In particular, for $n = 1$ this representation is an Ising model (either ferromagnetic or antiferromagnetic, depending on x), and for $n = 2$ it is a restricted Solid-On-Solid model (a Lipschitz function).

3.4 Self-avoiding walk and the connective constant

When $n \to 0$ and under vacant boundary, the probability of any non-empty loop configuration tends to zero. Thus, under vacant boundary conditions, the $n = 0$ model is trivial. However, as can be done for the spin $O(n)$, here too one may impose different *boundary conditions* on the model, where the states of certain edges are pre-specified. Taking boundary conditions for which precisely two edges e_1 and e_2 on the boundary of the domain H are present, one may force a self-avoiding path between these two edges within the domain in addition to possible loops. Thus, under such boundary conditions, in the limit as $n \to 0$, one obtains a random self-avoiding walk. The probability of such a given self-avoiding walk γ is proportional to $x^{\mathrm{length}(\gamma)}$. The partition function, $Z^{\mathrm{saw}}_{x,e_1,e_2}$, is given by

$$Z^{\mathrm{saw}}_{x,H,e_1,e_2} := \sum_{\substack{\gamma: e_1 \to e_2 \\ \gamma \subset H}} x^{\mathrm{length}(\gamma)} = \sum_{k=0}^{\infty} s_{k,H,e_1,e_2} x^k,$$

where s_{k,H,e_1,e_2} is the number of self-avoiding walks of length k from e_1 to e_2 in H.

We consider the related partition function, Z^{saw}_x, of all self-avoiding walks starting at the origin, given by

$$Z^{\mathrm{saw}}_x := \sum_{\gamma:\ \gamma_0 = 0} x^{\mathrm{length}(\gamma)} = \sum_{k=0}^{\infty} s_k x^k,$$

where s_k is the number of self-avoiding walks of length k starting at a fixed vertex. The series defining Z^{saw}_x has a radius of convergence $x_c \in [0, \infty]$ so that $Z^{\mathrm{saw}}_x < \infty$ when $x < x_c$ and $Z^{\mathrm{saw}}_x = \infty$ when $x > x_c$. This is the critical point of the model. The critical value x_c is directly related to the exponential rate of growth of s_k.

An important and simple observation is that s_k is sub-multiplicative. That is,

$$s_{k+m} \leq s_k s_m.$$

It follows that the limit

$$\mu := \lim_{k \to \infty} s_k^{1/k}$$

exists and is finite. The number μ, called the *connective constant* of the hexagonal lattice, clearly relates to the critical value via $\mu = 1/x_c$.

Exercise. Show that μ is well-defined and that $\mu = \inf_k s_k^{1/k}$.

Exercise. Show that $2^{k/2} \leq s_k \leq 3 \cdot 2^{k-1}$ and deduce that $\sqrt{2} \leq \mu \leq 2$.

Recently, Hugo Duminil-Copin and Stanislav Smirnov [9] showed the following.

Theorem 3.1. *The connective constant of the hexagonal lattice is*

$$\mu = \sqrt{2 + \sqrt{2}}.$$

We do not give the proof in these notes and refer the interested reader to [9].

3.5 Large n

It is believed that the loop $O(n)$ model, although only an approximation of the spin $O(n)$ model, resides in the same universality class as the spin $O(n)$ model. Thus, as in the case of the spin $O(n)$ model, it has been conjectured that the loop $O(n)$ model exhibits exponential decay of correlations when $n > 2$. In a recent work, Hugo Duminil-Copin, Ron Peled, Wojciech Samotij and Yinon Spinka [8] established this for large n, showing that long loops are exponentially unlikely to occur, uniformly in the edge weight x. This result is the content of the following theorem. First, we require some definitions (see Figure 1 for their illustration).

We consider the triangular lattice $\mathbb{T} := (0,2)\mathbb{Z} + (\sqrt{3},1)\mathbb{Z}$, and view the hexagonal lattice $\mathbb{H}$ as its dual lattice, obtained by placing a vertex at the center of every face (triangle) of $\mathbb{T}$, so that each edge e of $\mathbb{H}$ corresponds to the unique edge e^* of $\mathbb{T}$ which intersects e. Since vertices of $\mathbb{T}$ are identified with faces of $\mathbb{H}$, they will be called *hexagons* instead of vertices. We also say that a vertex or an edge of $\mathbb{H}$ *borders* a hexagon if it borders the corresponding face of $\mathbb{H}$.

Fix a proper 3-coloring of the triangular lattice $\mathbb{T}$ (there is a unique such coloring up to permutations of the colors), and let $\mathbb{T}^0$, $\mathbb{T}^1$ and $\mathbb{T}^2$ denote the color classes of this coloring. The 0-phase ground state ω^0_{gnd} is defined to be the (fully-packed) loop configuration consisting of trivial loops (loops of length 6) around each hexagon in $\mathbb{T}^0$. A domain $H \subset \mathbb{H}$ is said to be *of type 0* if no edge on its boundary belongs to ω^0_{gnd}, or equivalently, if every edge bordering a hexagon in $\mathbb{T}^0$ has either both or neither of its endpoints in $V(H)$. Finally, we say that a loop surrounds a vertex u of $\mathbb{H}$ if any infinite simple path in $\mathbb{H}$ starting at u intersects a vertex of this loop. In particular, if a loop passes through a vertex then it surrounds it as well.

Theorem 3.2. *There exist $n_0, c > 0$ such that for any $n \geq n_0$, any $x \in (0,\infty]$ and any domain H of type 0 the following holds. Suppose ω is sampled from the loop $O(n)$ model in domain H with edge-weight x. Then, for any vertex $u \in V(H)$ and any integer $k > 6$,*

$$\mathbb{P}(\textit{there exists a loop of length } k \textit{ surrounding } u) \leq n^{-ck}.$$

The reasons behind this exponential decay are quite different when x is small or large. While there is no transition to slow decay of loop lengths as x increases, there is a different kind of transition in terms of the structure of the random loop configuration and, in particular, in how the loops pack in the domain. When x is small, the model is dilute and disordered, whereas, when x is large, the model is dense and ordered; these behaviors are depicted in Figure 2. The proof for small x is very similar in nature to the high-temperature case of the spin $O(n)$ model, as described in Section 2.3, while the proof for large x is more intricate.

Given a loop configuration ω, we say that u and v are *loop-connected* if there exists a path between u and v consisting only of vertices which belong to loops in ω, and we say that u and v are *ground-connected* if there exists a path between u and v consisting only of vertices which belong to loops in $\omega \cap \omega^0_{\mathrm{gnd}}$.

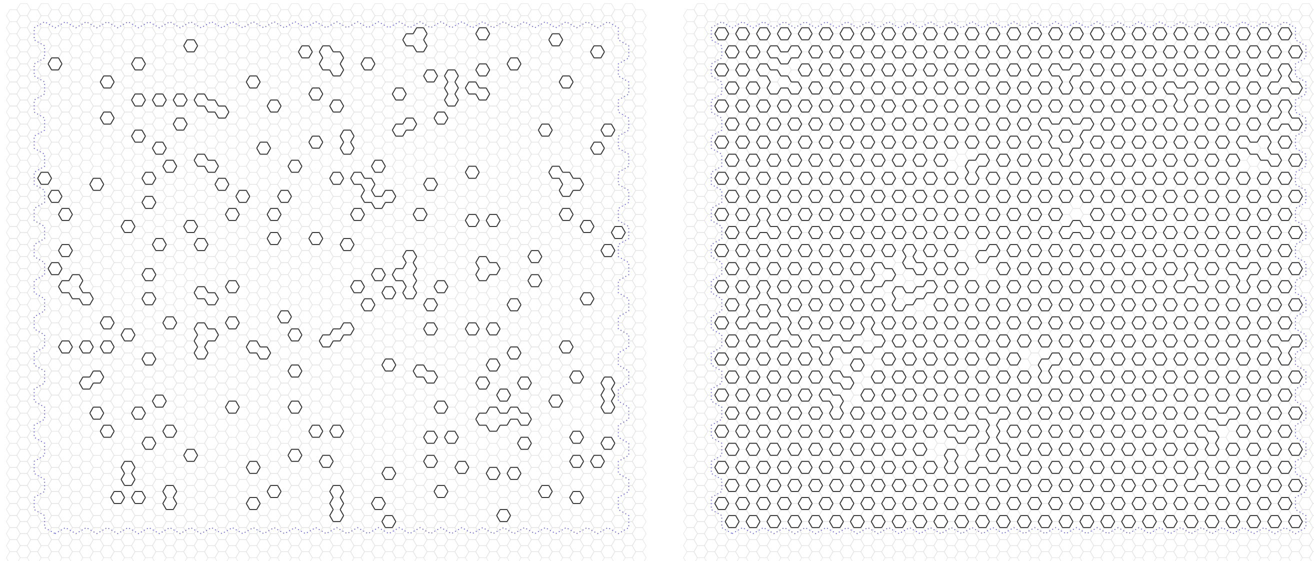
Theorem 3.3. *There exists $C, c > 0$ such that for any $n \geq C$, any $x \in (0,\infty]$, any domain H of type 0 and any $u \in V(H)$ the following holds. Suppose ω is sampled from the loop $O(n)$ model in domain H with edge-weight x. Then on the one hand,*

$$\mathbb{P}(u \textit{ is loop-connected to distance } k) \leq C(nx^6)^{ck},$$

and on the other hand,

$$\mathbb{P}(u \textit{ is ground-connected to } \partial V(H)) \geq 1 - C(n \min\{x^6, 1\})^{-c}.$$

Thus, when n is large, the theorem establishes a change in behavior as nx^6 transitions from small to large values. In particular, when nx^6 is large, the hexagons in $\mathbb{T}^0$ are very likely



(a) $n = 8$ and $x = 0.5$. When x is small, the limiting measure is unique for domains with vacant boundary conditions, and the model is in a dilute, disordered phase.

(b) $n = 8$ and $x = 2$. When n and x are large, the model is in an ordered phase where typical configurations are small perturbations of the ground state.

Figure 2: Two samples of random loop configurations with large n. Configurations are on a 60×45 domain of type 0 and are sampled via Glauber dynamics for 100 million iterations started from the empty configuration.

to be surrounded by trivial loops. In fact, as the proof shows, in this regime of parameters, the model is in a dense, ordered phase which is a small perturbation of the 0-phase ground state ω^0_{gnd}. On the other hand, when nx^6 is small, every hexagon is unlikely to be surrounded by a loop.

3.5.1 Proof of results for large n

In these notes, we give an extended overview of the proof of Theorem 3.2 and Theorem 3.3, omitting many of the details. The techniques of the proof are combinatorial in nature and rely on a general principle captured by the following simple lemma.

Lemma 3.4. *Let $p, q > 0$ and let E and F be two events in a discrete probability space. If there exists a map $\mathsf{T}\colon E \to F$ such that $\mathbb{P}(\mathsf{T}(e)) \geq p \cdot \mathbb{P}(e)$ for every $e \in E$, and $|\mathsf{T}^{-1}(f)| \leq q$ for every $f \in F$, then*

$$\mathbb{P}(E) \leq \frac{q}{p} \cdot \mathbb{P}(F).$$

Proof. We have

$$p \cdot \mathbb{P}(E) \leq \sum_{e\in E} \mathbb{P}(\mathsf{T}(e)) = \sum_{e\in E}\sum_{f\in F} \mathbb{P}(f)\mathbf{1}_{\{\mathsf{T}(e)=f\}} = \sum_{f\in F} |\mathsf{T}^{-1}(f)| \cdot \mathbb{P}(f) \leq q \cdot \mathbb{P}(F). \qquad \square$$

The results for small x are obtained via a fairly standard, and short, Peierls argument, by applying the above lemma to a map which removes loops (see Lemma 3.8 below). Thus, the main focus here lies in the study of the loop $O(n)$ model for large x.

In the large x regime, the main idea is to identify the region having an atypical structure (which is called the breakup) and apply the above lemma to a suitably defined 'repair map'. This map takes a configuration ω sampled in a domain of type 0 and having a large breakup and returns a 'repaired' configuration in which the breakup is significantly reduced (see Figure 4). In order to use Lemma 3.4, it is important that the number of preimages of a given loop configuration is exponentially smaller than the probability gain. This yields the main lemma, Lemma 3.7, from which the results for large x are later deduced.

Basic definitions. A *circuit* is a simple closed path in $\mathbb{T}$ of length at least 3. We may view a circuit γ as a sequence of hexagons $(\gamma_0, \ldots, \gamma_m)$ with $\gamma_0 = \gamma_m$. Define γ^* to be the set of edges $\{\gamma_i, \gamma_{i+1}\}^* \in E(\mathbb{H})$ for $0 \le i < m$. We now state two standard geometric facts regarding circuits and domains, which may be seen as a discrete version of the Jordan curve theorem. Proofs of these facts can be found in [8, Appendix B].

Fact 3.5. *If γ is a circuit then the removal of γ^* splits $\mathbb{H}$ into exactly two connected components, one of which is infinite, denoted by $\mathrm{Ext}(\gamma)$, and one of which is finite, denoted by $\mathrm{Int}(\gamma)$. Moreover, each of these are induced subgraphs of $\mathbb{H}$.*

Fact 3.6. *Circuits are in one-to-one correspondence with domains via $\gamma \leftrightarrow \mathrm{Int}(\gamma)$.*

Hence, every domain H may be written as $H = \mathrm{Int}(\gamma)$ for some circuit γ. Note also that H is of type 0 if and only if $\gamma \subset \mathbb{T} \setminus \mathbb{T}^0$. We denote the vertex sets and edge sets of $\mathrm{Int}(\gamma), \mathrm{Ext}(\gamma)$ by $\mathrm{Int}^{\mathrm{V}}(\gamma), \mathrm{Ext}^{\mathrm{V}}(\gamma)$ and $\mathrm{Int}^{\mathrm{E}}(\gamma), \mathrm{Ext}^{\mathrm{E}}(\gamma)$, respectively. Note that $\{\mathrm{Int}^{\mathrm{V}}(\gamma), \mathrm{Ext}^{\mathrm{V}}(\gamma)\}$ is a partition of $V(\mathbb{H})$ and that $\{\mathrm{Int}^{\mathrm{E}}(\gamma), \mathrm{Ext}^{\mathrm{E}}(\gamma), \gamma^*\}$ is a partition of $E(\mathbb{H})$. We also define $\mathrm{Int}^{\mathrm{hex}}(\gamma)$ to be the set of faces of $\mathrm{Int}(\gamma)$, i.e., the set of hexagons $z \in \mathbb{T}$ having all their six bordering vertices in $\mathrm{Int}^{\mathrm{V}}(\gamma)$. Since $\mathrm{Int}(\gamma)$ is induced, this is equivalent to having all six bordering edges in $\mathrm{Int}^{\mathrm{E}}(\gamma)$.

Definition 3.1 (c-flower, c-garden, c-cluster, vacant circuit; see Figure 3)**.** *Let $\mathsf{c} \in \{0,1,2\}$ and let ω be a loop configuration. A hexagon $z \in \mathbb{T}^{\mathsf{c}}$ is a* c-flower of ω *if it is surrounded by a trivial loop in ω. A subset $E \subset E(\mathbb{H})$ is a* c-garden of ω *if there exists a circuit $\sigma \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$ such that $E = \mathrm{Int}^{\mathrm{E}}(\sigma) \cup \sigma^*$ and every $z \in \mathbb{T}^{\mathsf{c}} \cap \partial \mathrm{Int}^{\mathrm{hex}}(\sigma)$ is a c-flower of ω. In this case, we denote $\sigma(E) := \sigma$. A garden of ω is a c-garden of ω for some $\mathsf{c} \in \{0,1,2\}$. A subset $C \subset E(\mathbb{H})$ is a* c-cluster *of ω if it is a c-garden of ω and it is not contained in any other garden of ω. A cluster of ω is a c-cluster of ω for some $\mathsf{c} \in \{0,1,2\}$. A circuit σ is* vacant *in ω if $\omega \cap \sigma^* = \emptyset$.*

We stress the fact that a garden/cluster is a *subset of the edges of* $\mathbb{H}$. We remark that distinct clusters of ω are edge disjoint and that, moreover, distinct c-clusters (for some c) are slightly separated from one another. Here and below, when A is a subset of vertices of a graph G, we use ∂A to denote the *(vertex) boundary* of A, i.e.,

$$\partial A := \big\{ u \in A \ : \ \{u,v\} \in E(G) \text{ for some } v \notin A \big\}.$$

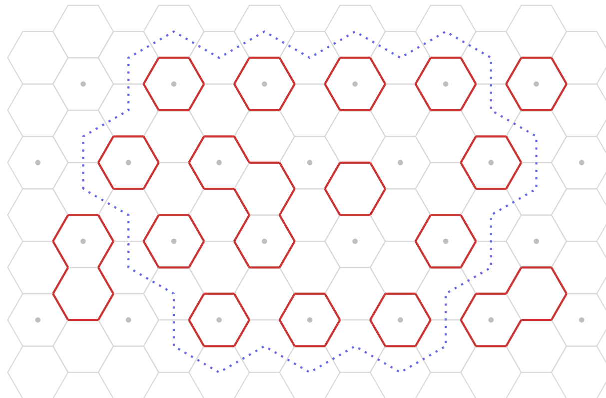


Figure 3: A garden. The dashed line denotes a vacant circuit $\sigma \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$, where $\mathsf{c} \in \{0,1,2\}$. The edges inside σ, along with the edges crossing σ, then comprise a c-garden of ω, since every hexagon in $\mathbb{T}^{\mathsf{c}} \cap \partial \mathrm{Int}^{\mathrm{hex}}(\sigma)$ is surrounded by a trivial loop.

Statement of the main lemma. For a loop configuration ω and a vacant circuit γ in ω, denote by $V(\omega,\gamma)$ the set of vertices $v \in \mathrm{Int}^{\mathrm{V}}(\gamma)$ such that the three edges of $\mathbb{H}$ incident to v are not all contained in the same cluster of $\omega \cap \mathrm{Int}^{\mathrm{E}}(\gamma)$. One may check that a vertex $v \in \mathrm{Int}^{\mathrm{V}}(\gamma)$ satisfies $v \in V(\omega,\gamma)$ if and only if v is incident to an edge which is not in any cluster or each of its incident edges lies in a different cluster. The set $V(\omega,\gamma)$ specifies the deviation in ω from the 0-phase ground state along the interior boundary of γ. The main lemma shows that having a large deviation is exponentially unlikely.

Lemma 3.7. *There exists $c > 0$ such that for any $n > 0$, any $x \in (0,\infty]$ and any circuit $\gamma \subset \mathbb{T} \setminus \mathbb{T}^0$ the following holds. Suppose ω is sampled from the loop $O(n)$ model in domain* $\mathrm{Int}(\gamma)$ *with edge-weight x. Then, for any positive integer k,*

$$\mathbb{P}\big(\partial \mathrm{Int}^{\mathrm{V}}(\gamma) \subset V(\omega,\gamma) \text{ and } |V(\omega,\gamma)| \geq k\big) \leq (cn \cdot \min\{x^6, 1\})^{-k/15}.$$

Definition of the repair map. Fix a circuit $\gamma \subset \mathbb{T} \setminus \mathbb{T}^0$ and set $H := \mathrm{Int}(\gamma)$. Consider a loop configuration ω such that γ is vacant in ω. The idea of the repair map is to modify ω as follows (see Figure 4 for an illustration):

- Edges in 1-clusters inside γ are shifted down "into the 0-phase".
- Edges in 2-clusters inside γ are shifted up "into the 0-phase".
- Edges in 0-clusters inside γ are left untouched.
- The remaining edges which are not inside (the shifted) clusters, but are in the interior of γ (these edges will be called *bad*), are overwritten to "match" the 0-phase ground state, ω^0_{gnd}.

In order to formalize this idea, we need a few definitions. A *shift* is a graph automorphism of $\mathbb{T}$ which maps every hexagon to one of its neighbors. We henceforth fix a shift $\uparrow$ which maps $\mathbb{T}^0$ to $\mathbb{T}^1$ (and hence, maps $\mathbb{T}^1$ to $\mathbb{T}^2$ and $\mathbb{T}^2$ to $\mathbb{T}^0$), and denote its inverse by $\downarrow$. A shift naturally induces mappings on the set of vertices and the set of edges of $\mathbb{H}$. We shall use the same symbols, $\uparrow$ and $\downarrow$, to denote these mappings. Recall that $\mathbb{T}$ has a coordinate system given by $(0,2)\mathbb{Z}+(\sqrt{3},1)\mathbb{Z}$ and that $(\mathbb{T}^0,\mathbb{T}^1,\mathbb{T}^2)$ are the color classes of an arbitrary proper 3-coloring of $\mathbb{T}$. In the figures we make the choice that $(0,0)\in\mathbb{T}^0$ and $(0,2)\in\mathbb{T}^1$ so that $\uparrow$ is the map $(a,b)\mapsto(a,b+2)$.

For a loop configuration $\omega\in\mathsf{LoopConf}(H)$ and $\mathsf{c}\in\{0,1,2\}$, let $E^{\mathsf{c}}(\omega)\subset E(\mathbb{H})$ be the union of all c-clusters of ω, and define

$$E^{\mathrm{bad}}(\omega) := (\mathrm{Int}^{\mathrm{E}}(\gamma)\cup\gamma^*)\setminus\big(E^0(\omega)\cup E^1(\omega)^{\downarrow}\cup E^2(\omega)^{\uparrow}\big), \tag{54}$$

$$\overline{E}(\omega) := (\mathrm{Int}^{\mathrm{E}}(\gamma)\cup\gamma^*)\setminus\big(E^0(\omega)\cup E^1(\omega)\cup E^2(\omega)\big). \tag{55}$$

One may check that $\{E^0(\omega),E^1(\omega),E^2(\omega),\overline{E}(\omega)\}$ is a partition of $\mathrm{Int}^{\mathrm{E}}(\gamma)\cup\gamma^*$ so that $\omega\cap E^0(\omega)$, $\omega\cap E^1(\omega)$, $\omega\cap E^2(\omega)$ and $\omega\cap\overline{E}(\omega)$ are pairwise disjoint loop configurations. Finally, we define the *repair map*

$$R\colon \mathsf{LoopConf}(H)\to\mathsf{LoopConf}(H)$$

by

$$R(\omega) := \big(\omega\cap E^0(\omega)\big)\cup\big(\omega\cap E^1(\omega)\big)^{\downarrow}\cup\big(\omega\cap E^2(\omega)\big)^{\uparrow}\cup\big(\omega^0_{\mathrm{gnd}}\cap E^{\mathrm{bad}}(\omega)\big).$$

The fact that the mapping is well-defined, i.e., that $R(\omega)$ is indeed in $\mathsf{LoopConf}(H)$, is not completely straightforward. However, it is indeed well-defined and, moreover,

$$\omega\cap E^0(\omega),\quad (\omega\cap E^1(\omega))^{\downarrow}\cup(\omega\cap E^2(\omega))^{\uparrow}\quad\text{and}\quad \omega^0_{\mathrm{gnd}}\cap E^{\mathrm{bad}}(\omega)$$

are pairwise disjoint loop configurations in $\mathsf{LoopConf}(H)$.

Proof of the main lemma. Let V be such that $\partial\mathrm{Int}^{\mathrm{V}}(\gamma)\subset V\subset\mathrm{Int}^{\mathrm{V}}(\gamma)$. We first bound the probability of the event

$$E_V := \{\omega\in\mathsf{LoopConf}(H)\ :\ V(\omega,\gamma)=V\}.$$

To do so, we wish to apply Lemma 3.4 to the repair map. To this end, we must estimate the gain in probability (parameter p in Lemma 3.4) and the number of preimages of a given configuration (parameter q in Lemma 3.4). Let $n\geq 1$ and $x\in(0,\infty]$ satisfy $nx^6\geq 1$. Then

$$\frac{\mathbb{P}_{H,n,x}(R(\omega))}{\mathbb{P}_{H,n,x}(\omega)}\geq (n\cdot\min\{x^6,1\})^{|V|/15}\qquad\text{for }\omega\in E_V, \tag{56}$$

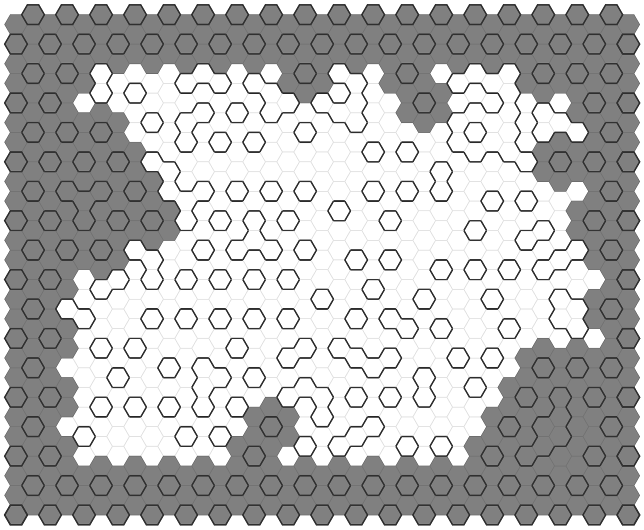
$$|E_V\cap R^{-1}(\omega')|\leq (2\sqrt{2})^{|V|}\qquad\text{for }\omega'\in\mathsf{LoopConf}(H). \tag{57}$$

The proof of (56) is based on a precise understanding of the change in the number of edges $\Delta o := o(R(\omega))-o(\omega)$ and in the number of loops $\Delta L := L(R(\omega))-L(\omega)$. Indeed, one may show (see Figure 4) that

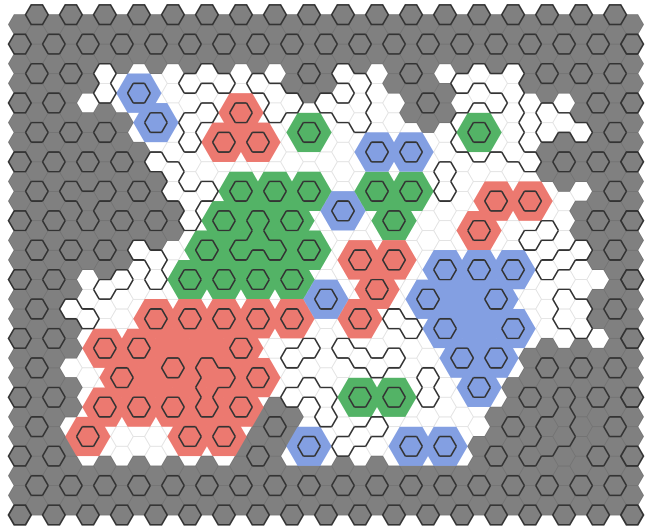
$$\Delta o = |V|-|\omega\cap\overline{E}(\omega)|\qquad\text{and}\qquad \Delta L = |V|/6 - L(\omega\cap\overline{E}(\omega)).$$

Using this, one deduces that

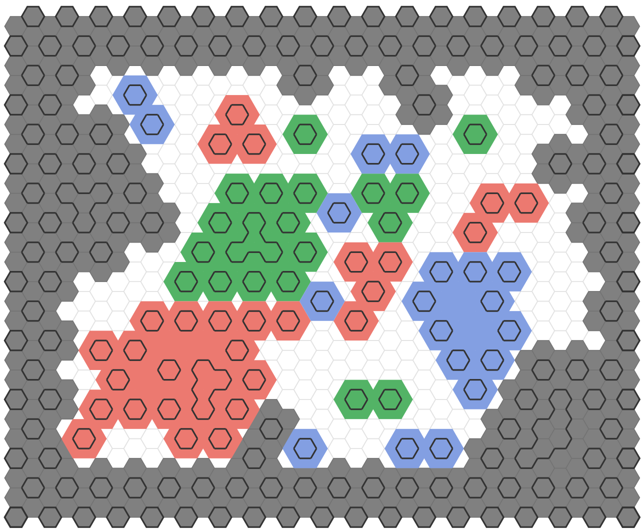
$$0\leq\Delta o\leq|V|\qquad\text{and}\qquad \Delta L\geq \tfrac{|V|}{15}+\tfrac{|\Delta o|}{10},$$



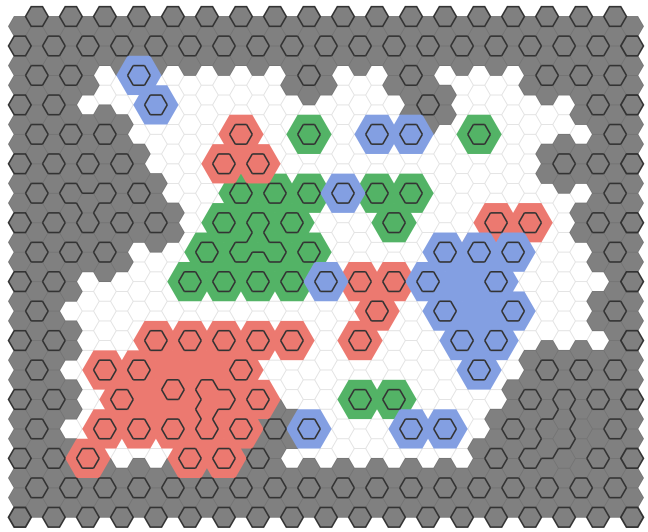
(a) The breakup is found by exploring 0-flowers from the boundary.



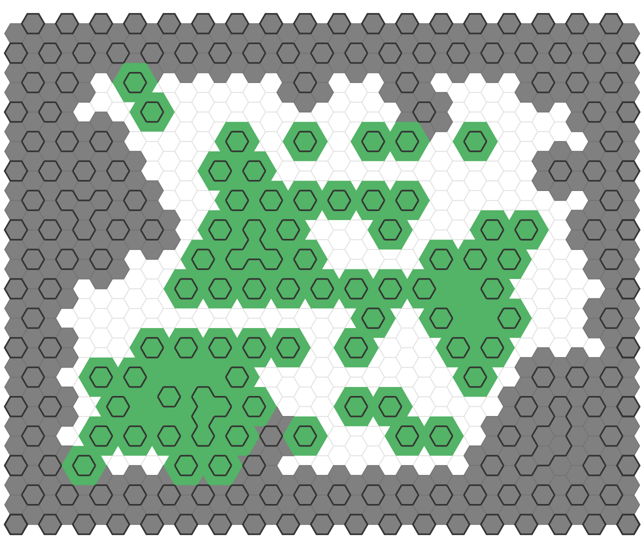
(b) The clusters are found within the breakup (with 0/1/2-clusters shown in green/red/blue).



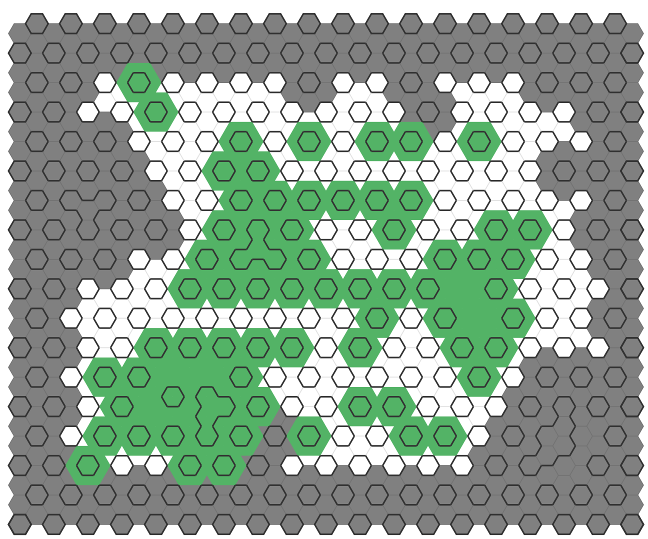
(c) Bad edges are discarded.



(d) The clusters are shifted into the 0-phase.



(e) The empty area outside the shifted clusters is now compatible with the 0-phase ground state.



(f) Trivial loops are packed in the empty area outside the shifted clusters.

Figure 4: An illustration of finding the breakup and applying the repair map in it. The initial loop configuration is modified step-by-step, resulting in a loop configuration with many more loops and at least as many edges.

from which (56) easily follows.

The proof of (57) relies on the fact that the only loss of information incurred by the repair map is in the bad edges (see Figure 4c). More precisely, the mapping $\omega \mapsto (R(\omega), \omega \cap E(V))$ is injective on E_V. Thus, the size of $E_V \cap R^{-1}(\omega')$ is at most the number of subsets of $E(V)$. Since $|E(V)| \leq 3|V|/2$, (57) follows.

Now, using (56) and (57), Lemma 3.4 implies that

$$\mathbb{P}(E_V) \leq (2\sqrt{2})^{|V|} \cdot (n \cdot \min\{x^6, 1\})^{-|V|/15}.$$

To complete the proof, we must sum over the possible choices for V. For this, we use a connectivity property of $V(\omega, \gamma)$. Let $\mathbb{H}^\times$ be the graph obtained from $\mathbb{H}$ by adding an edge between each pair of opposite vertices of every hexagon, so that $\mathbb{H}^\times$ is a 6-regular non-planar graph. One may show that $V(\omega, \gamma)$ is connected in $\mathbb{H}^\times$ whenever $\partial \mathrm{Int}^{\mathrm{V}}(\gamma) \subset V(\omega, \gamma)$. Thus, recalling Lemma 2.2, when $n \cdot \min\{x^6, 1\}$ is sufficiently large, we have

$$\begin{aligned}
\mathbb{P}\big(\partial \mathrm{Int}^{\mathrm{V}}(\gamma) \subset V(\omega,\gamma) \text{ and } |V(\omega,\gamma)| \geq k\big) &\leq \sum_{\substack{V\colon\, |V| \geq k \\ V \text{ connected in } \mathbb{H}^\times \\ \partial \mathrm{Int}^{\mathrm{V}}(\gamma) \subset V \subset \mathrm{Int}^{\mathrm{V}}(\gamma)}} \mathbb{P}(E_V) \\
&\leq \sum_{\ell=k}^{\infty} C^\ell \cdot (2\sqrt{2})^\ell \cdot (n \cdot \min\{x^6, 1\})^{-\ell/15} \\
&\leq (cn \cdot \min\{x^6, 1\})^{-k/15}.
\end{aligned}$$

Proofs of main theorems. The proofs of the main theorems for large x mostly rely on the main lemma, Lemma 3.7. The results for small x follow via a Peierls argument, the basis of which is given by the following lemma which gives an upper bound on the probability that a given collection of loops appears in a random loop configuration.

Lemma 3.8. *For any domain H, any $n, x > 0$ and any $A \in \mathsf{LoopConf}(H)$, we have*

$$\mathbb{P}_{H,n,x}(A \subset \omega) \leq n^{L(A)} x^{o(A)}.$$

Proof. Consider the map

$$\mathsf{T}\colon \{\omega \in \mathsf{LoopConf}(H) : A \subset \omega\} \to \mathsf{LoopConf}(H)$$

defined by

$$\mathsf{T}(\omega) := \omega \setminus A.$$

Clearly, T is well-defined and injective. Moreover, since $L(\mathsf{T}(\omega)) = L(\omega) - L(A)$ and $o(\mathsf{T}(\omega)) = o(\omega) - o(A)$, we have

$$\mathbb{P}_{H,n,x}(\mathsf{T}(\omega)) = \mathbb{P}_{H,n,x}(\omega) \cdot n^{-L(A)} x^{-o(A)}.$$

Hence, the statement follows from Lemma 3.4. $\square$

Recall the notion of a loop surrounding a vertex given prior to Theorem 3.2.

Corollary 3.9. *Let $n, x > 0$ and let H be a domain. Then, for any vertex $u \in V(H)$ and any positive integer k, we have*

$$\mathbb{P}_{H,n,x}(\text{there exists a loop of length } k \text{ surrounding } u) \leq kn(2x)^k.$$

Moreover, for any $u_1, \dots, u_m \in V(H)$ and $k_1, \dots, k_m \geq 1$ with $k = k_1 + \cdots + k_m$, we have

$$\mathbb{P}_{H,n,x}(\forall i \text{ there exists a distinct loop of length } k_i \text{ passing through } u_i) \leq n^m(2x)^k,$$

Proof. Denote by a_k the number of simple paths of length k in $\mathbb{H}$ starting at a given vertex. Clearly, $a_k \leq 3 \cdot 2^{k-1}$. It is then easy to see that the number of loops of length k surrounding u is at most $ka_{k-1} \leq k2^k$. Thus, the result follows by the union bound and Lemma 3.8.

The moreover part follows similarly. □

The main lemma, Lemma 3.7, shows that for a given circuit γ (with a type) it is unlikely that the set $V(\omega, \gamma)$ is large. The set $V(\omega, \gamma)$ specifies deviations from the ground states which are 'visible' from γ, i.e., deviations which are not 'hidden' inside clusters. In Theorem 3.2, we claim that it is unlikely to see long loops surrounding a given vertex. Any such long loop constitutes a deviation from all ground states. Thus, the theorem would follow from the main lemma (in the main case, when x is large) if the long loop was captured in $V(\omega, \gamma)$. The next lemma (whose proof we omit) bridges the gap between the main lemma and the theorem, by showing that even when a deviation is not captured by $V(\omega, \gamma)$, there is necessarily a smaller circuit σ which captures it in $V(\omega, \sigma)$.

Lemma 3.10. *Let ω be a loop configuration, let $\gamma \subset \mathbb{T} \setminus \mathbb{T}^0$ be a vacant circuit in ω and let $L \subset \mathrm{Int}(\gamma)$ be a non-trivial loop. Then there exists $\mathsf{c} \in \{0, 1, 2\}$ and a circuit $\sigma \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$ such that $\mathrm{Int}(\sigma) \subset \mathrm{Int}(\gamma)$, σ is vacant in ω and $V(L) \cup \partial\mathrm{Int}^{\mathrm{V}}(\sigma) \subset V(\omega, \sigma)$.*

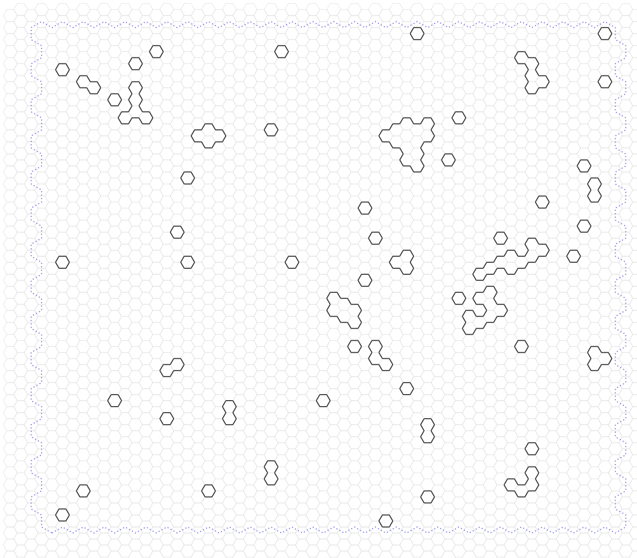
Proof of Theorem 3.2. Suppose that n_0 is a sufficiently large constant, let $n \geq n_0$ and let $x \in (0, \infty]$ be arbitrary. Let H be a domain of type 0 and let $u \in V(H)$. We shall estimate the probability that, in a random loop configuration drawn from $\mathbb{P}_{H,n,x}$, the vertex u is surrounded by a non-trivial loop of length k. We consider two cases, depending on the relative values of n and x.

Suppose first that $nx^6 < n^{1/50}$. Since $n \geq n_0$, we may assume that $2x \leq n^{-4/25}$ and that $kn^{-k/120} \leq 1$ for all $k > 0$. By Corollary 3.9, for every $k \geq 7$,

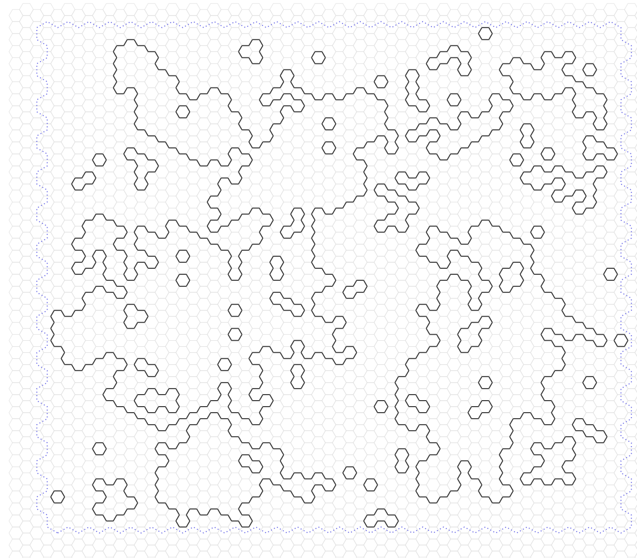
$$\begin{aligned}\mathbb{P}_{H,n,x}(\text{there exists a loop of length } k \text{ surrounding } u) &\leq kn(2x)^k \leq kn^{1-4k/25} \\ &\leq kn^{-k/60} \leq n^{-k/120}.\end{aligned}$$

Suppose now that $nx^6 \geq n^{1/50}$. Since $n \geq n_0$, we may assume that $n \cdot \min\{x^6, 1\}$ is sufficiently large for our arguments to hold. Let $L \subset H$ be a non-trivial loop of length k surrounding u. Note that, if $\omega \in \mathsf{LoopConf}(H)$ has $L \subset \omega$ then, by Lemma 3.10, for some $\mathsf{c} \in \{0, 1, 2\}$, there exists a circuit $\sigma \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$ such that $\mathrm{Int}(\sigma) \subset H$, σ is vacant in ω and $V(L) \cup \partial\mathrm{Int}^{\mathrm{V}}(\sigma) \subset V(\omega, \sigma)$. Using the fact that H is of type 0, the domain Markov property and Lemma 3.7 imply that for every fixed circuit $\sigma \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$ with $\mathrm{Int}(\sigma) \subset H$,

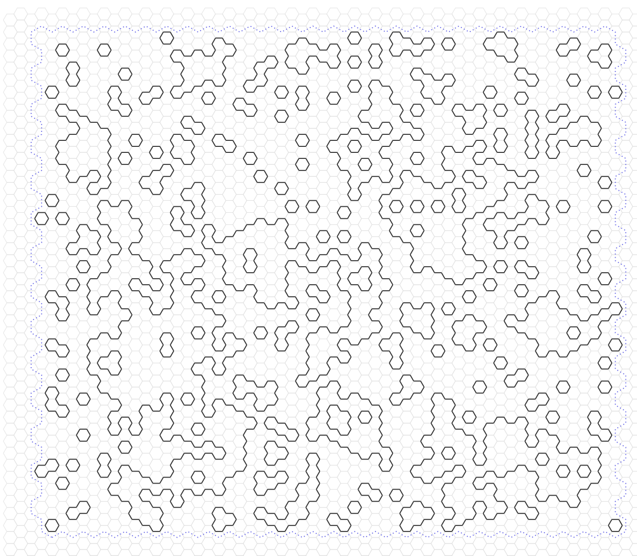
$$\mathbb{P}_{H,n,x}(\sigma \text{ vacant and } V(L) \cup \partial\mathrm{Int}^{\mathrm{V}}(\sigma) \subset V(\omega, \sigma)) \leq (cn \cdot \min\{x^6, 1\})^{-|V(L) \cup \partial\mathrm{Int}^{\mathrm{V}}(\sigma)|/15}.$$



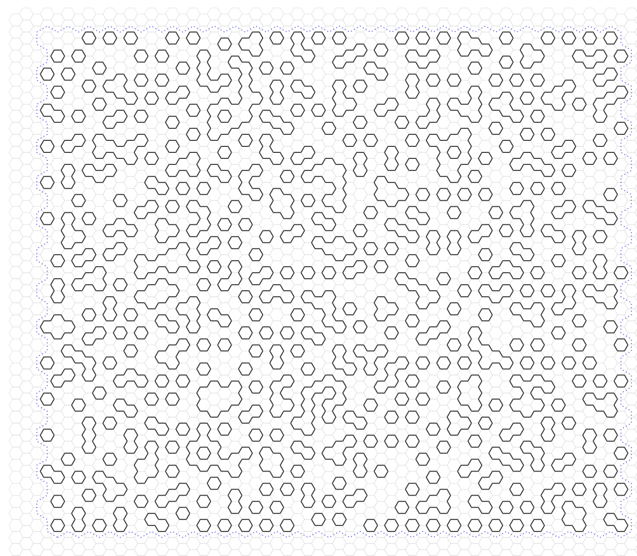
(a) $n = 0.8$ and $x = 0.55$.



(b) $n = 0.8$ and $x = 0.6$.



(c) $n = 2$ and $x = 1/\sqrt{2} \approx 0.707$.



(d) $n = 8$ and $x = 1$.

Figure 5: A few samples of random loop configurations. Configurations are on a 60×45 domain of type 0 and are sampled via Glauber dynamics for 100 million iterations started from the empty configuration. The conjectured phase transition point for $n = 0.8$ is $x_c = 1/\sqrt{2 + \sqrt{2 - 0.8}} \approx 0.568$ and for $n = 2$ is $x_c = 1/\sqrt{2} \approx 0.707$.

Thus, denoting by $\mathcal{G}(u)$ the set of circuits σ contained in $\mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$ for some $\mathsf{c} \in \{0,1,2\}$ and having $u \in \mathrm{Int}^{\mathrm{V}}(\sigma)$, we obtain

$$\begin{aligned}\mathbb{P}_{H,n,x}(L \subset \omega) &\le \sum_{\sigma \in \mathcal{G}(u)} (cn \cdot \min\{x^6, 1\})^{-|V(L) \cup \partial \mathrm{Int}^{\mathrm{V}}(\sigma)|/15} \\ &\le \sum_{\ell=1}^{\infty} D^{\ell} (cn \cdot \min\{x^6,1\})^{-\max\{\ell,k\}/15} \\ &\le (c'n \cdot \min\{x^6,1\})^{-k/15},\end{aligned}$$

where we used the facts that the length of a circuit σ such that $|\partial \mathrm{Int}^{\mathrm{V}}(\sigma)| = \ell$ is at most 3ℓ, that the number of circuits σ of length at most 3ℓ with $u \in \mathrm{Int}^{\mathrm{V}}(\sigma)$ is bounded by D^{ℓ} for some sufficiently large constant D, and in the last inequality we used the assumption that $n \cdot \min\{x^6, 1\}$ is sufficiently large. Since the number of loops of length k surrounding a given vertex is smaller than $k2^k$, our assumptions that $nx^6 \ge n^{1/50}$ and $n \ge n_0$ yield

$$\mathbb{P}_{H,n,x}(\text{there exists a loop of length } k \text{ surrounding } u) \le k2^k (c'n^{1/50})^{-k/15} \le n^{-k/800}.$$

Proof of Theorem 3.3. Let $n > 0$, let $x \in (0, \infty]$ and let H be a domain of type 0.

Suppose first that nx^6 is sufficiently small. Observe that if u is loop-connected to distance k, then there exist $m \ge 1$, $\ell_1, \dots, \ell_m \ge 6$ and $u_0, u_1, \dots, u_m \in V(H)$ such that $\ell := \ell_1 + \dots + \ell_m \ge k$, $u_0 = u$ and, for all $1 \le i \le m$, $\mathrm{dist}(u_i, u_{i-1}) \le \ell_i$ and u_i belongs to a distinct loop of length ℓ_i. Thus, summing over the possible choices and applying Corollary 3.9, we obtain

$$\begin{aligned}\mathbb{P}(u \text{ is loop-connected to distance } k) &\le \sum_{\substack{\ell \ge k \\ \ell \ge m \ge 1}} \sum_{\substack{\ell_1,\dots,\ell_m \\ u_1,\dots,u_m}} \mathbb{P}(\forall i \ u_i \text{ belongs to a distinct loop of length } \ell_i) \\ &\le \sum_{\ell \ge k} \ell \cdot 2^{\ell} \cdot 3^{\ell} \cdot n^m (2x)^{\ell} \le \sum_{\ell \ge k} (Cnx^6)^{\ell/6} \le C(Cnx^6)^{k/6}.\end{aligned}$$

Suppose now that both n and nx^6 are sufficiently large. Let ω be a loop configuration in $\mathsf{LoopConf}(H)$ and assume that u is not ground-connected to the boundary $\partial V(H)$. Let $A(\omega)$ be the set of vertices of $\mathbb{H}$ belonging to loops in $\omega \cap \omega^0_{\mathrm{gnd}}$ and let $B(\omega)$ be the unique infinite connected component of $A(\omega) \cup (V(\mathbb{H}) \setminus V(H))$. Define $\mathcal{C}$ (the breakup) to be the connected component of $\mathbb{H} \setminus B(\omega)$ containing u (note that $u \notin B(\omega)$ by assumption).

One may check that the subgraph induced by $\mathcal{C}$ is a domain of type 0, and that the enclosing circuit Γ (i.e., the circuit satisfying $\mathcal{C} = \mathrm{Int}^{\mathrm{V}}(\Gamma)$ which exists by Fact 3.6) is vacant in ω and is contained in $\mathbb{T} \setminus \mathbb{T}^0$. Furthermore, $\partial \mathrm{Int}^{\mathrm{V}}(\Gamma) \subset V(\omega, \Gamma)$. This follows as Γ is vacant in ω and, by the definition of $B(\omega)$, no vertex of $\partial \mathrm{Int}^{\mathrm{V}}(\Gamma)$ belongs to a trivial loop surrounding a hexagon in $\mathbb{T}^0$.

Thus, denoting by $\mathcal{G}$ the set of circuits $\gamma \subset \mathbb{T} \setminus \mathbb{T}^0$ having $u \in \mathrm{Int}^{\mathrm{V}}(\gamma)$, Lemma 3.7 implies that

$$
\begin{aligned}
\mathbb{P}(u \text{ is not ground-connected to } \partial V(H)) \\
&\le \mathbb{P}(\text{there exists a breakup}) \\
&= \sum_{\gamma\in\mathcal{G}} \mathbb{P}(\Gamma = \gamma) \\
&\le \sum_{\gamma\in\mathcal{G}} \mathbb{P}(\gamma \text{ vacant and } \partial\mathrm{Int}^{\mathrm{V}}(\gamma) \subset V(\omega,\gamma)) \\
&\le \sum_{\gamma\in\mathcal{G}} (cn\cdot\min\{x^6,1\})^{-|\partial\mathrm{Int}^{\mathrm{V}}(\gamma)|/15} \\
&\le \sum_{k\ge1} D^k (cn\cdot\min\{x^6,1\})^{-k/15} \le C(cn\cdot\min\{x^6,1\})^{-c}.
\end{aligned}
$$

In the final inequality, we used the facts that the length of a circuit γ such that $|\partial\mathrm{Int}^{\mathrm{V}}(\gamma)| = k$ is at most $3k$, and that the number of circuits of length at most $3k$ surrounding u is bounded by D^k for some sufficiently large constant D.

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