Scaling limit of a layer of unstable phase

Yvan Velenik

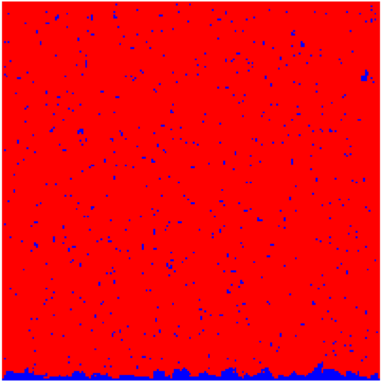
based on joint work with **Dima Ioffe** and **Senya Shlosman**

UNIVERSITÉ DE GENÈVE

Structure of the talk

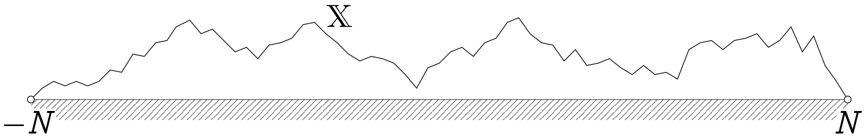
Critical prewetting in the Ising model

- 2d Ising model in a square box
- Boundary conditions: $+++-$
- Magnetic field: $h > 0$



As $h \downarrow 0$, the thickness of the layer of unstable $-$ phase increases as $h^{-1/3+o(1)}$ (as long as $N \gg h^{-2/3}$) [V., PTRF 2004]

$1+1$-dimensional effective model



Probability of a nonnegative trajectory $\mathbb{X} = (X_{-N}, \dots, X_N)$:

$$\mathbb{P}_{N,+,\lambda}(\mathbb{X}) = \frac{1}{Z_{N,+,\lambda}} \exp\Big\{-\lambda \sum_{i=-N}^{N} X_i\Big\}\, \mathrm{p}_{\mathrm{RW}}(\mathbb{X}),$$

where $\lambda > 0$.

Motivating example

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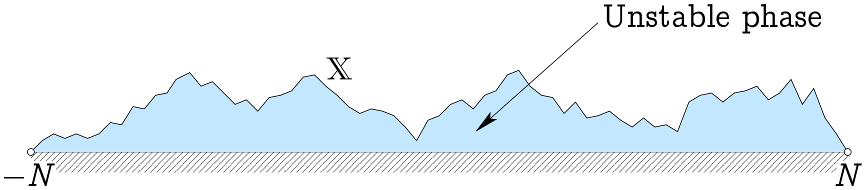

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Behavior as $\lambda \downarrow 0$:

- **Free energy** $\sim \lambda^{2/3}$
- **Thickness** $\sim \lambda^{-1/3}$
- **Correlation length** $\sim \lambda^{-2/3}$

[Hryniv and V., PTRF 2004] (see also [Abraham& Smith, JSP 1986])

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Useful to consider a more general situation...

1. The underlying random walk

$(p_x)_{x\in\mathbb{Z}}$: trans. probab. of an aperiodic, irreducible RW on $\mathbb{Z}$ such that

$$\sum_x p_x = 0\,, \qquad \sum_x e^{tx} p_x < \infty \text{ for small } t$$

Let $\sigma^2 = \sum_x x^2 p_x$ and, for $\mathbb{X} = (X_{-N}, \ldots, X_N)$,

$$\mathsf{p}_{\mathrm{RW}}(\mathbb{X}) = \prod_{i=-N}^{N-1} p_{X_{i+1}-X_i}$$

2. The potentials $(V_\lambda)_{\lambda>0}$

Let $V_\lambda : \mathbb{N} \to \mathbb{R}_+$ be such that

$$V_\lambda(0) = 0\,, \qquad V_\lambda \text{ increasing}, \qquad \lim_{x\to\infty} V_\lambda(x) = +\infty$$

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Additional assumptions (on $(V_\lambda)_{\lambda>0}$):

$$\lim_{\lambda\downarrow 0} H_\lambda = +\infty\,, \qquad \lim_{\lambda\downarrow 0} H_\lambda^2 V_\lambda(rH_\lambda) = q(r)\,,$$

with $q \in \mathsf{C}^2(\mathbb{R}_+)$ such that $\lim_{r\to\infty} q(r) = +\infty$.

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Example: $V_\lambda(x) = \lambda x^\alpha$, $\alpha > 0$, $H_\lambda = \lambda^{-1/(2+\alpha)}$, $q(r) = r^\alpha$

3. The effective model

Probability of a nonnegative trajectory $\mathbb{X} = (X_{-N}, \dots, X_N)$ with $X_{-N} = \mathsf{u}$, $X_N = \mathsf{v}$:

$$\mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}(\mathbb{X}) = \frac{1}{Z^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}} \exp\Big\{-\sum_{i=-N}^{N} V_\lambda(X_i)\Big\}\, \mathsf{p}_{\mathrm{RW}}(\mathbb{X})$$

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Goal: Determine the scaling limit of $x_\lambda(t) = H_\lambda^{-1} X_{[H_\lambda^2 t]}$ as $\lambda \downarrow 0$

Singular Sturm-Liouville problem on $\mathbb{R}_+$

$$\mathsf{L} = \mathsf{L}_{\sigma,q} = \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} - q(r)$$

with zero boundary condition: $\varphi(0) = 0$.

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Well-known: $\exists$ orthonormal basis $\{\varphi_i\}_{i\geq 0}$ of simple eigenfunctions in $\mathbb{L}_2(\mathbb{R}_+)$ with eigenvalues

$$0 > -\mathsf{e}_0 > -\mathsf{e}_1 > -\mathsf{e}_2 > \dots, \qquad \lim_{i\to\infty} \mathsf{e}_i = +\infty$$

$\forall i \geq 0$, φ_i is smooth and possesses exactly i zeroes in $(0, \infty)$

In particular φ_0 can be taken positive.

Ferrari-Spohn diffusions on $(0, \infty)$

Generators:

$$\mathsf{G}_{\sigma,q}\psi = \frac{1}{\varphi_0}(\mathsf{L} + \mathsf{e}_0\mathsf{I})(\psi\varphi_0) = \frac{\sigma^2}{2}\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} + \sigma^2\frac{\varphi_0'}{\varphi_0}\frac{\mathrm{d}\psi}{\mathrm{d}r}$$

Limiting objects

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The corresponding diffusions are ergodic and reversible w.r.t.

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Stationary path measure: $\mathbb{P}_{\sigma,q}$

Main result

- **Scaled process:** $x_\lambda(t) = \frac{1}{H_\lambda} X_{[H_\lambda^2 t]}$ (with linear interpolation)

Theorem (Ioffe, Shlosman, V., 2014)

Let $(\lambda_N)_{N\geq 1}$ *be such that* $\lambda_N \downarrow 0$ *and* $H^2_{\lambda_N}/N \to 0$. *Then,*

$$\textit{Law of } x_{\lambda_N} \textit{ under } \mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda_N} \overset{N\to\infty}{\Longrightarrow} \mathbb{P}_{\sigma,q},$$

uniformly in $0 \leq \mathsf{u},\mathsf{v} \leq C H_\lambda$.

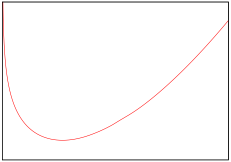
Main result

Example: $V_\lambda(x) = \lambda x$
In this case:

$$H_\lambda = \lambda^{-1/3}, \qquad \varphi_0(r) = \mathrm{Ai}(\chi r - \omega_1), \qquad e_0 = \frac{\omega_1}{\chi},$$

where $-\omega_1$ is the first zero of the Airy function Ai and $\chi = \sqrt[3]{2/\sigma^2}$.

Scaling limit: diffusion in log-Airy potential



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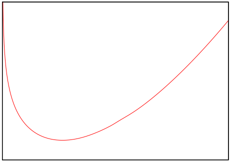

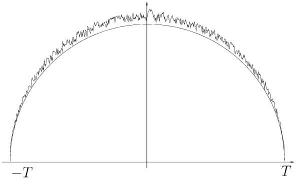

The corresponding diffusion was already derived in [Ferrari&Spohn, AoP 2005] in the context of a Brownian bridge conditioned to stay above a circular barrier.

Transfer operator

(To simplify, I assume here that $p_x = p_{-x}$)

$$\forall x, y \in \mathbb{N}, \qquad \widetilde{\mathsf{T}}_\lambda(x,y) = p_{y-x}\, e^{-\frac{1}{2}(V_\lambda(\mathsf{x})+V_\lambda(\mathsf{y}))}$$

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Krein-Rutman $\Longrightarrow$ $\widetilde{\mathsf{T}}_\lambda$ possesses a leading e.f. $\phi_\lambda > 0$ of e.v. E_λ

Normalized version: $\mathsf{T}_\lambda = \frac{1}{E_\lambda} \widetilde{\mathsf{T}}_\lambda$

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Ground-state chain:

$$\pi_\lambda(\mathsf{x},\mathsf{y}) = \frac{1}{\phi_\lambda(\mathsf{x})}\mathsf{T}_\lambda(\mathsf{x},\mathsf{y})\phi_\lambda(\mathsf{y})$$

$\rightsquigarrow$ Pos. recurrent Markov chain, with inv. meas. $\mu_\lambda(\mathsf{x}) = c_\lambda \phi_\lambda(\mathsf{x})^2$
Stationary path-measure: $\mathbb{P}_\lambda$

Convergence of fdds

Let $\mathbb{N}_\lambda = H_\lambda^{-1}\mathbb{N}$.

$$\mathbb{E}_\lambda\{u_0(x_\lambda(0))u_1(x_\lambda(t))\} = \sum_{\mathsf{r},\mathsf{s}\in\mathbb{N}_\lambda} \mu_\lambda(H_\lambda \mathsf{r})\, \pi_\lambda^{[H_\lambda^2 t]}(H_\lambda \mathsf{r}, H_\lambda \mathsf{s})\, u_0(\mathsf{r})u_1(\mathsf{s})$$

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Assume one can show that, as $\lambda \downarrow 0$,

$$H_\lambda c_\lambda \to 1, \qquad \phi_\lambda(H_\lambda \mathsf{r}) \to \varphi_0(\mathsf{r}), \qquad \mathsf{T}_\lambda^{[H_\lambda^2 t]}[f_\lambda](H_\lambda \mathsf{r}) \to \mathsf{T}^t[f](\mathsf{r})$$

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$$\sum_{\mathsf{r}\in\mathbb{N}_\lambda} \mu_\lambda(H_\lambda\mathsf{r})\cdots = c_\lambda \sum_{\mathsf{r}\in\mathbb{N}_\lambda} \phi_\lambda(H_\lambda\mathsf{r})^2\cdots \to \int \mathrm{d}\mathsf{r}\,\varphi_0(\mathsf{r})^2\cdots = \int \mu_0(\mathrm{d}\mathsf{r})\cdots$$

and

$$\sum_{\mathsf{s}\in\mathbb{N}_\lambda} \pi_\lambda^{[H_\lambda^2 t]}[u_1(H_\lambda^{-1}\cdot)](H_\lambda\mathsf{r}) = \frac{1}{\phi_\lambda(H_\lambda\mathsf{r})}\mathsf{T}_\lambda^{[H_\lambda^2 t]}[\phi_\lambda u_1(H_\lambda^{-1}\cdot)](H_\lambda\mathsf{r})$$

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$$\to \frac{1}{\varphi_0(\mathsf{r})} \mathsf{T}^t[\varphi_0 u_1](\mathsf{r}) = e^{G_{\sigma,q} t}[u_1](\mathsf{r})$$

Thus,

$$\mathbb{E}_\lambda\{u_0(x_\lambda(0))u_1(x_\lambda(t))\} \to \mathbb{E}_{\sigma,q}\{u_0(x(0))u_1(x(t))\}$$

Three main probabilistic inputs

1. The free energy is of order H_λ^{-2}
Setting $\mathrm{e}_\lambda = -H_\lambda^2 \log E_\lambda$, this implies that

$$0 < \liminf_{\lambda \downarrow 0} \mathrm{e}_\lambda \leq \limsup_{\lambda \downarrow 0} \mathrm{e}_\lambda < \infty.$$

$\rightsquigarrow$ compactness of $(e_\lambda)_{\lambda > 0}$

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2. Tail estimate:

$$\mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda_N}(X_0 > K H_\lambda) \leq \exp\{-\nu K (\sqrt{q(K)} \wedge H_\lambda)\}$$

uniformly in $K > 0$ and $\lambda \leq \lambda_0$.

$\rightsquigarrow$ tightness of $(x_\lambda, \mathbb{P}_\lambda)$ and compactness of $(\phi_\lambda(H_\lambda \cdot))_{\lambda > 0}$

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3. Approximation by stationary distribution
Suppose $N \gg H_{\lambda_N}^2$ and $\mathsf{u}_N, \mathsf{v}_N \leq c H_{\lambda_N}$. Then, for any local event A,

$$\lim_{N \to \infty} |\mathbb{P}_{N,+,\lambda_N}^{\mathsf{u},\mathsf{v}}(A) - \mathbb{P}_{\lambda_N}(A)| = 0.$$

$\rightsquigarrow$ sufficient to prove convergence of fdd under $\mathbb{P}_{\lambda_N}$

Convergence of semigroup

Let $\mathsf{L}_\lambda f(\mathsf{r}) = \frac{\mathsf{T}_\lambda - \mathsf{I}}{H_\lambda^{-2}} f(\mathsf{r})$

"Fact": $\lim_{\lambda\downarrow 0} \mathsf{T}_\lambda^{\lfloor H_\lambda^2 t \rfloor} f_\lambda = e^{(\mathsf{L}+e\mathsf{I})t} f$ follows from $\lim_{\lambda\downarrow 0} \mathsf{L}_\lambda u_\lambda = (\mathsf{L} + e\mathsf{I})u$

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- Computation:

$$E_\lambda \mathsf{e}^{V_\lambda(H_\lambda \mathsf{r})} \mathsf{L}_\lambda u_\lambda(\mathsf{r}) =$$

$$\frac{1}{h_\lambda^2} \sum_{\mathsf{s}\in\mathbb{N}_\lambda} p_\lambda(H_\lambda \mathsf{s} - H_\lambda \mathsf{r}) \Big(\mathsf{e}^{\frac{V_\lambda(H_\lambda \mathsf{r}) - V_\lambda(H_\lambda \mathsf{s})}{2}} u(\mathsf{s}) - u(\mathsf{r}) \Big)$$

$$+ \frac{1 - E_\lambda \mathsf{e}^{V_\lambda(H_\lambda \mathsf{r})}}{h_\lambda^2} u(\mathsf{r})$$

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- Computation: Assume that $\lim_{\lambda\downarrow 0} \mathsf{e}_\lambda = \mathsf{e}$

$$\underbrace{E_\lambda \mathsf{e}^{V_\lambda(H_\lambda \mathsf{r})}}_{\to 1} \mathsf{L}_\lambda u_\lambda(\mathsf{r}) =$$

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$$+ \underbrace{\frac{1 - E_\lambda \mathsf{e}^{V_\lambda(H_\lambda \mathsf{r})}}{h_\lambda^2} u(\mathsf{r})}_{\to (\mathsf{e} - q(\mathsf{r}))u(\mathsf{r})}$$

One then easily deduce:

Proposition

$$e_0 = \lim_{\lambda \downarrow 0} e_\lambda, \qquad \varphi_0(r) = \lim_{\lambda \downarrow 0} \phi_\lambda(H_\lambda r)$$

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$$\mathsf{T}_\lambda \phi_\lambda = \phi_\lambda \implies e^{(\mathsf{L}+\mathsf{e}\mathsf{I})t}\varphi = \varphi$$

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$$\mathsf{e}_0 = \lim_{\lambda\downarrow 0} \mathsf{e}_\lambda, \qquad \varphi_0(\mathsf{r}) = \lim_{\lambda\downarrow 0} \phi_\lambda(H_\lambda \mathsf{r})$$

Indeed, w.l.o.g., consider a subsequence $(\lambda_k)_{k>0}$ such that $\mathsf{e}_{\lambda_k} \to \mathsf{e}$ and $\phi_{\lambda_k}(H_{\lambda_k}\cdot) \to \varphi(\cdot)$

$$\mathsf{T}_\lambda \phi_\lambda = \phi_\lambda \implies e^{(\mathsf{L}+\mathsf{e}\mathsf{I})t}\varphi = \varphi$$

$\implies$ φ is a non-negative (normalized) eigenfunction of L with eigenvalue $-\mathsf{e}$.

Identification of the limit

One then easily deduce:

Proposition

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$$\mathsf{T}_\lambda \phi_\lambda = \phi_\lambda \implies e^{(\mathsf{L}+\mathsf{e}\mathsf{I})t}\varphi = \varphi$$

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$\implies$ $\varphi = \varphi_0$ and $\mathsf{e} = \mathsf{e}_0$

Thanks
for your attention!