From loop clusters and random interlacements to the Gaussian free field

Titus Lupu

Université Paris-Sud, Orsay

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Definition of the random walk "loop soup"

- $\mathcal{G} = (V, E)$ undirected connected graph. V at most countable. Every vertex has finite degree.
- Conductances on edges $(C(e))_{e \in E}$, $C(e) > 0$.
- Killing measure $(\kappa(x))_{x \in V}$, $\kappa(x) \geq 0$.
- $(X_t)_{0 \leq t < \zeta}$ Markov jump process on $\mathcal{G}$ that jumps from x to a neighbour y with rate $C(\{x, y\})$ and to the cemetery point (killing) with rate $\kappa(x)$.
- Assumption: $(X_t)_{0 \leq t < \zeta}$ transient: either $\kappa \not\equiv 0$ or V infinite and C appropriate.
- $(G(x, y))_{x, y \in V}$ Green's function of X.
- $P^t_{x,y}(\cdot)$ bridge probability measures, $p_t(x, y)$ transition probabilities Measure on loops:

$$\mu(\cdot) = \sum_{x \in V} \int_{t > 0} P^t_{x,x}(\cdot) p_t(x, x) \frac{dt}{t}$$

- Possonian ensemble of loops or "loop soup": a Poisson point process $\mathcal{L}_c$ of intensity $c\mu$ where $c > 0$ constant.

Percolation by loops

- Loop clusters: $\gamma, \gamma' \in \mathcal{L}_c$ belong to the same cluster if there is a chain $\gamma_0, \dots, \gamma_n \in \mathcal{L}_c$ such that $\gamma_0 = \gamma$, $\gamma_n = \gamma'$ and γ_i and γ_{i-1} have a common vertex.
- Percolation problem: existence of an unbounded cluster of loops. Natural framework:
  - $\mathbb{Z}^d$, $d \geq 3$, uniform conductances, $\kappa \equiv 0$
  - $\mathbb{Z}^2$, uniform conductances, $\kappa > 0$ uniform
  - discrete half-plane $\mathbb{Z} \times \mathbb{N}$, uniform conductances, instantaneous killing at the boundary $\mathbb{Z} \times \{0\}$
- Previously known: non trivial phase transition (critical $c^* \in (0, +\infty)$), uniqueness of the infinite cluster (Burton-Keane). See:
  - Y. Le Jan, S. Lemaire (2012): Markovian loop clusters on graphs, to appear in *Illinois J. Math.*
  - Y. Chang, A. Sapozhnikov (2014): Phase transition in loop percolation, arXiv:1403.5687v1

Isomorphism with the Gaussian free field

- Occupation field of the "loop soup": $x \in V$

$$\widehat{\mathcal{L}}_c^x := \sum_{\gamma \in \mathcal{L}_c} \int_0^{t_\gamma} 1_{\gamma(s)=x} ds$$

- Gaussian free field $(\phi_x)_{x \in V}$. $\mathbb{E}[\phi_x] = 0$. $\mathbb{E}[\phi_x \phi_y] = G(x, y)$.
- Le Jan's isomorphism (2011):

$$(\widehat{\mathcal{L}}_{\frac{1}{2}}^x)_{x \in V} \overset{(d)}{=} \left(\frac{1}{2}\phi_x^2\right)_{x \in V}$$

Theorem (L. 2014)

There is a coupling between $\mathcal{L}_{\frac{1}{2}}$ and ϕ such that:

- $(\widehat{\mathcal{L}}_{\frac{1}{2}}^x)_{x \in V} = \left(\frac{1}{2}\phi_x^2\right)_{x \in V}$
- *The sign of ϕ is constant on clusters of loops in $\mathcal{L}_{\frac{1}{2}}$*

Brownian motion on metric graphs

- Each edge e of $\mathcal{G}$ are replaced by a continuous interval I_e: topological space $\widetilde{\mathcal{G}}$.
- Metric structure on $\widetilde{\mathcal{G}}$: each interval I_e has length $\frac{1}{2}C(e)^{-1}$. $\widetilde{\mathcal{G}}$ metric graph.
- $B^{\widetilde{\mathcal{G}}}$ Brownian motion on $\widetilde{\mathcal{G}}$. Continuous family of local times $(L_t^z(B^{\widetilde{\mathcal{G}}}))_{z\in\widetilde{\mathcal{G}},t\geq 0}$.
- If $\kappa \not\equiv 0$ then killing measure

$$\tilde{\kappa} = \sum_{x\in V} \kappa(x)\delta_x$$

  on $B^{\widetilde{\mathcal{G}}}$.
- Stopping times $(\tau_l)_{l\geq 0}$:

$$\tau_l := \inf \left\{ t > 0 | \sum_{x\in V} L_t^x(B^{\widetilde{\mathcal{G}}}) > 0 \right\}$$

- $(B^{\widetilde{\mathcal{G}}}_{\tau_l})_{l\geq 0}$ has the same law as the Markov jump process X.

Clusters of loops on metric graphs

- $\tilde{\mu}$ measure on loops on $\widetilde{\mathcal{G}}$ associated to $B^{\widetilde{\mathcal{G}}}$. $\widetilde{\mathcal{L}}_c$ Poisson point process of intensity with intensity $c\tilde{\mu}$.
- Discrete loop soup $\mathcal{L}_c$ can be reconstructed from the metric graph loop soup $\widetilde{\mathcal{L}}_c$:
  - Throw away loops in $\widetilde{\mathcal{L}}_c$ that do not visit V.
  - Take the restriction to V of the loops in $\widetilde{\mathcal{L}}_c$ that visit V
- Occupation field of $\widetilde{\mathcal{L}}_c$: sum of local times of loops.

$$\widehat{\mathcal{L}}_c^z := \sum_{\gamma \in \widetilde{\mathcal{L}}_c} L^z_{t_\gamma}(\gamma)$$

  The restriction of $(\widehat{\mathcal{L}}_c^z)_{z \in \widetilde{\mathcal{G}}}$ to V is the occupation field of $\mathcal{L}_c$.
- $(\widehat{\mathcal{L}}_c^z)_{z \in \widetilde{\mathcal{G}}}$ is a continuous field. The clusters of loops in $\widetilde{\mathcal{L}}_c$ are the connected components of $\{Z \in \widetilde{\mathcal{G}} | \widehat{\mathcal{L}}_c^z > 0\}$.
- The clusters of $\widetilde{\mathcal{L}}_c$ are larger than then clusters of $\mathcal{L}_c$: excursions and loops inside the interior of edges can create connections.

Construction of the coupling

- The discrete Gaussian free field $(\phi_x)_{x\in V}$ can be interpolated to the Gaussian free field on metric graph $(\phi_z)_{z\in\widetilde{\mathcal{G}}}$ by adding independent Brownian bridges.
- Le Jan's isomorphism holds on metric graphs:

$$(\widehat{\mathcal{L}}^z_{\frac{1}{2}})_{z\in\widetilde{\mathcal{G}}} \overset{(d)}{=} \left(\frac{1}{2}\phi_z^2\right)_{z\in\widetilde{\mathcal{G}}}$$

- Given $(|\phi_z|)_{z\in\widetilde{\mathcal{G}}}$ the sign of ϕ is to be chosen independently and uniformly on each connected component of $\{z\in\widetilde{\mathcal{G}}||\phi_z|>0\}$, that is to say on each cluster of $\widetilde{\mathcal{L}}_{\frac{1}{2}}$.
- Thus the sign of ϕ is constant on each cluster of $\widetilde{\mathcal{L}}_{\frac{1}{2}}$ and *a fortiori* on each cluster of $\mathcal{L}_{\frac{1}{2}}$.

Application to percolation

Theorem (L. 2014)

At $c = \frac{1}{2}$ there is no infinite cluster of loops in all the following settings:

- $\mathbb{Z}^d$, $d \geq 3$, *uniform conductances*, $\kappa \equiv 0$
- $\mathbb{Z}^2$, *uniform conductances*, $\kappa > 0$ *uniform*
- *discrete half-plane* $\mathbb{Z} \times \mathbb{N}$, *uniform conductances, instantaneous killing at the boundary* $\mathbb{Z} \times \{0\}$

- In dimension 2 the discrete GFF does not have infinite sign clusters.
- In higher dimension the discrete GFF may have infinite sign clusters (Rodriguez, Sznitman, 2013)
- But in all dimensions the metric graph GFF has only bounded sign clusters.
- $\mathbb{P}(z, z' \text{ in same sign cluser of metric graph GFF}) =$
$$\mathbb{E}[sgn(\phi_z) sgn(\phi_{z'})] = \frac{2}{\pi} Arcsin\left(\frac{G(z,z')}{\sqrt{G(z,z)G(z',z')}}\right) \overset{d(z,z')\to\infty}{\longrightarrow} 0$$

Isomorphism for random interlacements

- $d \geq 3$. I^u random interlacement of level u on $\mathbb{Z}^d$. $\mathcal{V}^u$ vacant set.
- $(L^x(I^u))_{x\in\mathbb{Z}^d}$ occupation field of I^u: add an independent exponential time at x for each visit by random interlacements. Sznitman's isomorphism:

$$\left(L^x(I^u) + \frac{1}{2}\phi_x'^2\right)_{x\in\mathbb{Z}^d} \overset{(d)}{=} \left(\frac{1}{2}(\phi_x - \sqrt{2u})^2\right)_{x\in\mathbb{Z}^d}$$

where ϕ' GFF independent of I^u.

Theorem (L. 2014)

There is a coupling between I^u and the GFF ϕ such that

$$\{x \in \mathbb{Z}^d | \phi_x > \sqrt{2u}\} \subseteq \mathcal{V}^u$$

- $h_* \geq 0$ critical level for the percolation by level sets of GFF. u_* critical level for the vacant set of random interlacements. From the coupling follows that:

$$h_* \leq \sqrt{2u_*}$$

Construction of the coupling for random interlacements

- $\widetilde{\mathbb{Z}}^d$ metric graph associated to $\mathbb{Z}^d$.
- Construct metric graph interlacments by adding Brownian excursions.
- Sznitman's isomorphism holds in metric graph setting.
- The occupation field of metric graph interlacments is strictly positive on the edges and vertices visited by I^u.
- Thus the interlacements are contained in the unbounded connected components of $\{z \in \widetilde{\mathbb{Z}}^d | \phi_z \neq \sqrt{2u}\}$
- $\{z \in \widetilde{\mathbb{Z}}^d | \phi_z > \sqrt{2u}\}$ has only bounded connected components.
- Thus I^u is contained in $\{z \in \widetilde{\mathbb{Z}}^d | \phi_z < \sqrt{2u}\}$.

Thank you for your attention!