Ising critical exponents on random graphs

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Joint work with
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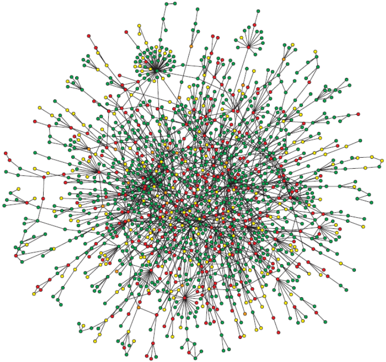
Introduction

There are many *complex real-world networks,* e.g., social, biological, technological, . . .

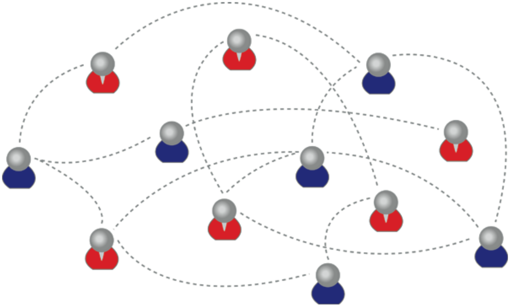


Many have *scale-free* behavior.

Processes on networks: opinion formation, virus spreading. . .

Introduction

What are effects of *structure* of complex networks on *behavior* of processes on the network?



Here, the effect on the *phase transition* in the *Ising model,* a toy model for *opinion formation.*

Configuration model

Configuration model

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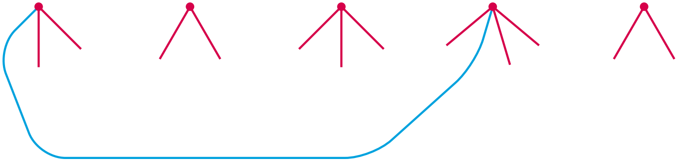
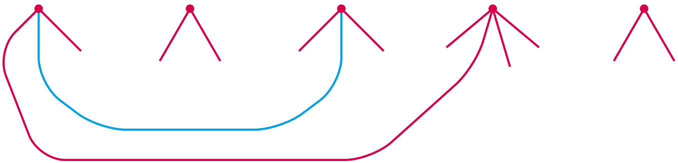
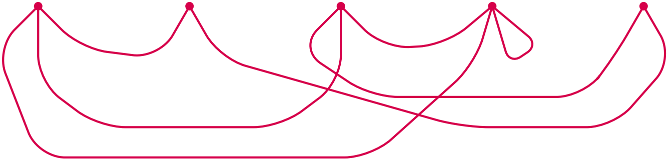


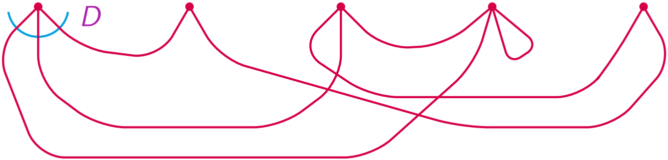
Configuration model



Configuration model



Configuration model

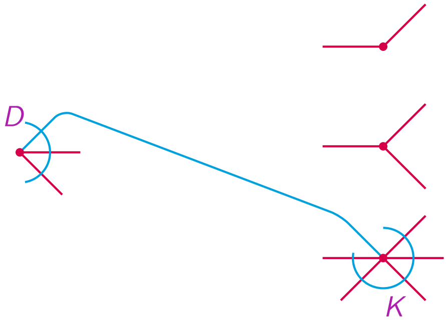


Special attention to *power-law degree sequences,* i.e.,

$$ck^{-\tau} \leq \mathbb{P}[D = k] \leq Ck^{-\tau}, \qquad \tau > 2.$$

Locally tree-like structure

Neighborhood of random vertex



Branching process with offspring D in first generation and K in further generations.

Ising model

For spin configuration $\sigma \in \{-1, +1\}^n$,

$$\mu(\sigma) = \frac{1}{Z_n(\beta, B)} \exp\left\{\beta \sum_{(i,j)\in E_n} \sigma_i \sigma_j + B \sum_{i=1}^{n} \sigma_i\right\},$$

where

$\beta \geq 0$	inverse temperature
B	external magnetic field
$Z_n(\beta, B)$	normalization factor (partition function)

Magnetization

Magnetization

$$M(\beta, B) \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \langle \sigma_i \rangle_\mu,$$

Spontaneous magnetization

$$M(\beta, 0^+) \equiv \lim_{B\searrow 0} M(\beta, B).$$

Critical temperature

$$\beta_c \equiv \inf\{\beta : M(\beta, 0^+) > 0\}.$$

Critical temperature

Theorem (Lyons, '89, DGvdH, '14)

$$\beta_c = \operatorname{atanh}(1/\mathbb{E}[K])$$

Note that, for $\tau \in (2,3)$, we have $\mathbb{E}[K] = \infty$, so that $\beta_c = 0$.

We study *critical exponents* for $\tau > 3$.

Critical exponents

$$M(\beta, 0^+) \asymp (\beta - \beta_c)^{\boldsymbol{\beta}}, \qquad \text{for } \beta \searrow \beta_c;$$

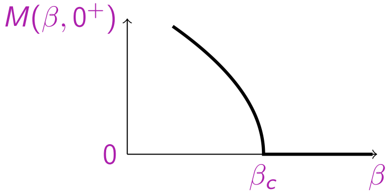
$$M(\beta_c, B) \asymp B^{1/\boldsymbol{\delta}}, \qquad \text{for } B \searrow 0.$$

Theorem (DGvdH, '14)

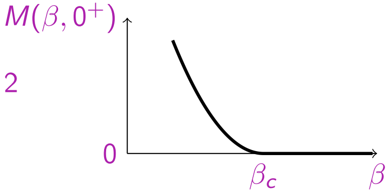
	$\mathbb{E}[K^3] < \infty$	$\tau \in (3, 5)$
$\boldsymbol{\beta}$	$1/2$	$1/(\tau - 3)$
$\boldsymbol{\delta}$	3	$\tau - 2$

Smoothness of transition

For $\mathbb{E}[K^3] < \infty$, $\boldsymbol{\beta} = \frac{1}{2}$

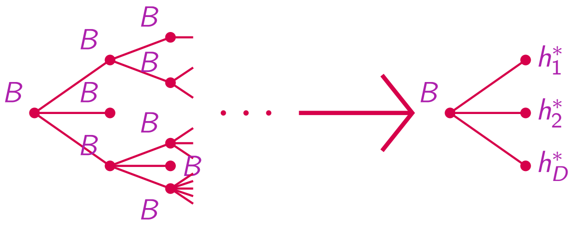


For $\tau = 3.5$, $\boldsymbol{\beta} = \frac{1}{3.5-3} = 2$



Tree recursion

Root *magnetization* on a tree:



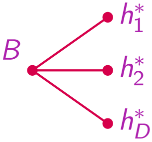
Effective field h^* is *unique* solution to recursion

$$h^{(t+1)} \stackrel{d}{=} B + \sum_{i=1}^{K_t} \xi(h_i^{(t)}),$$

where,

$$\xi(h) = \operatorname{atanh}(\tanh(\beta) \tanh(h)).$$

Magnetization



The *magnatization* equals

$$M(\beta, B) = \mathbb{E}\left[\tanh\left(B + \sum_{i=1}^{D} \xi(h_i)\right)\right]$$
$$\approx B + \mathbb{E}[D]\mathbb{E}[\xi(h)].$$

Hence, same scaling for $M(\beta, B)$ and $\mathbb{E}[\xi(h)]$.

Taylor expansion

Taylor expansion of $\mathbb{E}[\xi(h)]$:

$$\begin{aligned}\mathbb{E}[\xi(h)] &\approx \tanh(\beta)\mathbb{E}[h] - C\mathbb{E}[h^3] \\ &= \tanh(\beta)\left(B + \mathbb{E}[K]\mathbb{E}[\xi(h)]\right) - C\mathbb{E}\left[\left(B + \sum_{i=1}^{K}\xi(h_i)\right)^3\right].\end{aligned}$$

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Only allowed for $\mathbb{E}[K^3] < \infty$. In that case

$$\mathbb{E}[\xi(h)] \approx \tanh(\beta)B + \tanh(\beta)\mathbb{E}[K]\mathbb{E}[\xi(h)] - C\mathbb{E}[\xi(h)]^3.$$

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For $\tau \in (3,5)$, split analysis for small and large K to obtain

$$\mathbb{E}[\xi(h)] \approx \tanh(\beta)B + \tanh(\beta)\mathbb{E}[K]\mathbb{E}[\xi(h)] - C\mathbb{E}[\xi(h)]^{\tau-2}.$$

Critical exponents

Theorem (DGvdH, '14)

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