Edge-Reinforced Random Walk, Vertex-Reinforced Jump Process and the second generalised Ray-Knight theorem

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CONTENTS

I) Definition of Vertex Reinforced Jump Process (VRJP)

- $G = (V, E)$ non-oriented locally finite graph
- $(W_e)_{e \in E}$ be positive conductances on edges
- $\varphi = (\varphi_i)_{i \in V}$, $\varphi_i > 0$.
- VRJP $(Y_s)_{s \geq 0}$ continuous-time process: $Y_0 = i_0$ and, if $Y_s = i$, then Y (conditionally) jumps to $j \sim i$ at rate

  $$W_{i,j} L_j(s),$$

  where

  $$L_j(s) = \varphi_j + \int_0^t \mathbb{1}_{Y_u = j} du.$$

- Proposed by Werner ('00) and studied by Davis, Volkov ('02,'04), Collevechio ('09), Basdevant and Singh ('10) on trees.

I) Definition of Vertex Reinforced Jump Process (VRJP)

- Change of time

$$\ell_i := \frac{1}{2}(L_i^2 - \varphi_i^2), \;\; t := \sum_{i \in V} \ell_i$$

defines time-changed VRJP $(Z_t)_{t \geq 0}$.

- Between times t and $t + dt$, X (conditionally) jumps to $j \sim i$ with probability

$$W_{i,j} \sqrt{\frac{\varphi_j^2 + 2\ell_j(t)}{\varphi_i^2 + 2\ell_i(t)}} dt,$$

where

$$\ell_i(t) = \int_0^t \mathbb{1}_{Z_u = i} du.$$

II) LINK with ERRW ($\forall i, \varphi_i = 1$)

- $G = (V, E)$ non-oriented locally finite graph
- $(a_e)_{e \in E}$ weights on edges, $a_e > 0$.
- ERRW on V with initial weights (a_e), starting from $i_0 \in V$, is the discrete-time process (X_n) defined by $X_0 = i_0$ and

$$\mathbb{P}(X_{n+1} = j \,|\, X_k,\, k \leq n) = \mathbb{1}_{j \sim X_n} \frac{Z_n(\{X_n, j\})}{\sum_{i \sim X_n} Z_n(\{X_n, i\})}$$

  where $Z_n(e) = a_e + \sum_{k=0}^{n-1} \mathbb{1}_{\{X_{k-1}, X_k\} = e}$.:
- Coppersmith-Diaconis ('86), Pemantle ('88), Merkl-Rolles ('05-09).

II) LINK with ERRW ($\forall i$, $\varphi_i = 1$)

Theorem (Sabot, T. '11)

- *$W_e \sim Gamma(a_e)$ independent, $e \in E$*
- *Y VRJP with $Gamma(a_e, 1)$ independent conductances.*

Then $(Y_t)_{t \geq 0}$ (at jump times) $\overset{\text{"law"}}{=} (X_n)$.

Two ingredients :

- Rubin construction (Davis '90, Sellke '94) : $(\tilde{X}_t)$ continuous-time version of (X_n) through independent sequences of exponential random variables for each edge.
- Kendall transform ('66): Representation of the timeline at each edge as a Poisson Point Process with Gamma random parameter after change of time.

III) LINK with SuSy hyperbolic sigma model ($\forall i$, $\varphi_i = 1$)

VRJP Mixture of MJPs (G finite, $|V| = N$))

Let $\mathbb{P}_{i_0} :=$ law of (X_t) starting at $i_0 \in V$.

Theorem (Sabot, T. '11)

i) $\forall i \in V$, $\exists U_i := \lim[(\log \ell_i(t))/2 - \sum_{i \in V}(\log \ell_i(t))/2N]$ *s.t.*, *conditionally on* $U = (U_i)_{i \in V}$, *X is a Markov jump process starting from* i_0 *with jump rate from i to j*

$$W_{i,j} e^{U_j - U_i}.$$

In particular

$$\text{VRJP} \overset{\text{(jump times)}}{=} \text{MC with conductances } W_{ij}^U := W_{ij} e^{U_i + U_j}.$$

III) LINK with SuSy hyperbolic sigma model ($\forall i$, $\varphi_i = 1$)

Mixing law of VRJP (G finite, $|V| = N$)

ii) Under $\mathbb{P}_{i_0}$, (U_i) has density on $\mathcal{H}_0 := \{(u_i), \sum u_i = 0\}$

$$\frac{N}{(2\pi)^{(N-1)/2}} e^{u_{i_0}} e^{-H(W,u)} \sqrt{D(W,u)},$$

where, if $\mathcal{T}$ is the set of (non-oriented) spanning trees of G,

$$H(W,u) := 2 \sum_{\{i,j\} \in E} W_{i,j}(\cosh(u_i - u_j) - 1),$$

$$D(W,u) := \sum_{T \in \mathcal{T}} \prod_{\{i,j\} \in T} W_{\{i,j\}} e^{u_i + u_j}.$$

Introduced by Zirnbauer (1991) in quantum field theory as the SuSy hyperbolic sigma model.

III) LINK with SuSy hyperbolic sigma model ($\forall i$, $\varphi_i = 1$)

Recurrence/transience of ERRW/VRJP

Theorem (Recurrence: Sabot-T.'11-'12 (using Disertori and Spencer '10), Angel, Crawford and Kozma '12)

For any graph of bounded degree there exists $\beta_c > 0$ such that, if $W_e < \beta_c$ (resp. $a_e < \beta_c$) for all $e \in E$, the VRJP (resp. ERRW) is recurrent.

Theorem (Transience: Sabot and T. '12 (using Disertori, Spencer and Zirnbauer '10), Disertori, Sabot and T. '13)

On $\mathbb{Z}^d$, $d \geq 3$, there exists $\beta_c > 0$ such that, if $W_e > \beta_c$ (resp. $a_e > \beta_c$) for all $e \in E$, then VRJP (resp. ERRW) is transient a.s.

IV) Ray-Knight and the VRJP

- $\mathbb{P}_{i_0}$ law of MJP $X = (X_t)_{t \geq 0}$ starting at i_0, with local time ℓ.
- $U = E \setminus \{i_0\}$,
- $P^{G,U} = C \exp\{-\mathcal{E}(\varphi, \varphi)/2\} \, \delta_0(\varphi_{i_0}) \prod_{x \in U} d\varphi_x$.
- $\sigma_u = \inf\{t \geq 0; \, \ell_t^{i_0} > u\}, \;\; u \geq 0$.
- $\mathcal{E}(f, f) = \frac{1}{2} \sum_{x,y \in V} W_{x,y}(f(x) - f(y))^2$ Dirichlet form at $f : V \to \mathbb{R}$.

Theorem (Generalized second Ray-Knight theorem)

For any $u > 0$,

$$\left(\ell_{\sigma_u}^x + \frac{1}{2}\varphi_x^2\right)_{x \in V} \quad \textit{under } \mathbb{P}_{i_0} \otimes P^{G,U}, \;\; \textit{has the same law as}$$

$$\left(\frac{1}{2}(\varphi_x + \sqrt{2u})^2\right)_{x \in V} \quad \textit{under } P^{G,U}.$$

IV) Ray-Knight and the VRJP

Let

$$\Phi_i = \sqrt{\varphi_i^2 + 2\ell_i(t)}.$$

Let $\mathbb{P}_{i_0,t}$ (resp. $\mathbb{P}_{i_0,t}^{VRJP}$) be the laws of VRJP (resp. MJP) $(X_t)_{t\geq 0}$ starting at i_0, with conductances $(W_e)_{e\in E}$, up to time t. An elementary calculation yields

$$\frac{d\mathbb{P}_{i_0,t}^{VRJP}}{d\mathbb{P}_{i_0,t}} = \exp\left(\frac{1}{2}\left(\mathcal{E}(\Phi,\Phi) - \mathcal{E}(\varphi,\varphi)\right)\right)\frac{\prod_{j\neq i_0}\varphi_j}{\prod_{j\neq X_t}\Phi_j}$$

- Exponential part is the holding probability, where the fraction is the product of the jump probabilities.
- Implies partial exchangeability easily
- Yields a martingale of the MJP
- Can be used to obtain large deviation estimates

IV) Ray-Knight and the VRJP

- Given $\Phi = (\Phi_i)_{i \in V}$, $\Phi_i > 0$, time-reversed VRJP defined as follows: $\tilde{Y}_0 = i_0$ and, if $\tilde{Y}_s = i$, then $\tilde{Y}$ (conditionally) jumps to $j \sim i$ at rate

$$W_{i,j}\tilde{L}_j(s), \text{ where } \tilde{L}_j(s) = \Phi_j - \int_0^t \mathbb{1}_{\tilde{Y}_u = j} du.$$

- Change of time

$$\tilde{\ell}_i := \frac{1}{2}(\Phi_i^2 - \tilde{L}_i^2), \quad t := \sum_{i \in V} \tilde{\ell}_i$$

defines time-changed VRJP $(\tilde{Z}_t)_{t \geq 0}$.

- Between times t and $t + dt$, X (conditionally) jumps to $j \sim i$ with probability

$$W_{i,j}\sqrt{\frac{\Phi_j^2 - 2\tilde{\ell}_j(t)}{\Phi_i^2 - 2\tilde{\ell}_i(t)}}dt, \text{ where } \tilde{\ell}_i(t) = \int_0^t \mathbb{1}_{\tilde{Z}_u = i} du.$$

IV) Ray-Knight and the VRJP

Let

$$\varphi_i = \sqrt{\Phi_i^2 - 2\tilde{\ell}_i(t)},$$

and assume it is well-defined at time t for all $i \in V$.
Let $\tilde{\mathbb{P}}_{i_0,t}^{VRJP}$ be the law of time-reversed VRJP. Then, similarly,

$$\frac{d\tilde{\mathbb{P}}_{i_0,t}^{VRJP}}{d\mathbb{P}_{i_0,t}} = \exp\left(\frac{1}{2}(\mathcal{E}(\Phi,\Phi) - \mathcal{E}(\varphi,\varphi))\right) \frac{\prod_{j\neq i_0} \Phi_j}{\prod_{j\neq X_t} \varphi_j}$$

IV) Ray-Knight and the VRJP

Let $t = \sigma_u$ and

$$\Phi_i = \sqrt{\varphi_i^2 + 2\ell_i(\sigma_u)},$$
$$\sigma(\varphi) = (\text{sign}(\varphi_i))_{i \in V},$$
$$\sigma = + \iff \forall i \in V, \ \sigma_i = 1.$$

Let us explain why, heuristically,

$$\mathcal{L}\left(\ell \,|\sigma(\varphi) = +\right) = \tilde{\ell},$$

where $\tilde{\ell}$ is distributed under $\tilde{\mathbb{P}}^{VRJP}_{i_0,\sigma_u}$.

IV) Ray-Knight and the VRJP

Indeed, by Ray-Knight Theorem above, and a change of variables,

$$\begin{aligned}\mathbb{P}_{i_0} \otimes P^{G,U}(\Phi + d\Phi) &= \exp(-\frac{1}{2}\mathcal{E}(\Phi,\Phi))d\Phi \qquad (1)\\ &= \exp(-\frac{1}{2}\mathcal{E}(\Phi,\Phi)) \prod_{j\neq i_0} \frac{\varphi_j}{\Phi_j} d\varphi\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d(\mathbb{P}_{i_0} \otimes P^{G,U})}{\mathbb{P}_{i_0} \otimes P^{G,U}(\Phi + d\Phi)} &= \exp\left(\frac{1}{2}(\mathcal{E}(\Phi,\Phi) - \mathcal{E}(\varphi,\varphi))\right) \frac{\prod_{j\neq i_0} \Phi_j}{\prod_{j\neq X_t} \varphi_j} d\mathbb{P}_{i_0,t}\\ &= \left(\frac{d\tilde{\mathbb{P}}^{VRJP}_{i_0,t}}{d\mathbb{P}_{i_0,t}}\right) d\mathbb{P}_{i_0,t} = d\tilde{\mathbb{P}}^{VRJP}_{i_0,t}.\end{aligned}$$

Problem By Ray-Knight Theorem, Φ is the modulus a variable $\tilde{\varphi} + \sqrt{2u}$ with $\mathcal{L}(\tilde{\varphi}) = P^{G,U}$, but

$\sigma = +$ does not imply $\tilde{\varphi} + \sqrt{2u} = +$, i.e. (1) does not hold.

V) The magnetized time-reversed VRJP

- Magnetized time-reversed VRJP $(\check{Y}_s)_{s\geq 0}$, $\check{Y}_0 = i_0$.
- $(\Phi_x)_{x\in V}$ positive reals.
- $\check{L}_i(s) = \Phi_i - \int_0^s \mathbb{1}_{\check{Y}_u = i} du$.
- $< \cdot >_s$ (resp. $F(s)$) the expectation (resp. partition function) of the Ising model with interaction

$$J_{i,j}(s) = W_{i,j}\check{L}_i(s)\check{L}_j(s),$$

  and with boundary condition $\sigma_{i_0} = +1$.
- Conditioned on the past at time s, if $\check{Y}_s = i$, jumps from i to j with a rate

$$W_{i,j}\check{L}_j(s)\frac{< \sigma_j >_s}{< \sigma_i >_s}.$$

- $\check{Y}$ well-defined up to time

$$S = \sup\{s \geq 0,\ \check{L}_i(s) > 0 \text{ for all } i\}.$$

V) The magnetized time-reversed VRJP

Lemma

$$\check{Y}_S = i_0.$$

Let $\check{\mathbb{P}}_{i_0}^{VRJP}$ be the law of $(\check{Y}_s)_{s\geq 0}$ up to the time S, and set

$$\Phi_i = \sqrt{\varphi_i^2 + 2\ell_i(\sigma_u)}.$$

Theorem (Sabot, T. '14)

$$\mathcal{L}\left((\ell,\varphi)|\Phi\right) = \left(\frac{1}{2}(\Phi^2 - \check{L}^2(S)), \sigma\check{L}(S)\right),$$

where

- $\check{L}(S)$ *distributed under* $\check{\mathbb{P}}_{\Phi,i_0}^{VRJP}$
- *conditionally on* $\check{L}(S)$, σ *has law* $< \cdot >_S$.