On Level-Set Percolation for the Gaussian Free Field in High Dimensions

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The Model

- Gaussian Free Field on $\mathbb{Z}^d$, $d \geq 3$:

  $\varphi = (\varphi_x)_{x \in \mathbb{Z}^d}$: centered Gaussian field with

$$\mathbb{E}[\varphi_x \varphi_y] = g(x, y) \sim c|x - y|^{2-d}.$$

- Level sets:

$$\mathcal{L}_{\varphi}^{\geq h} = \{x \in \mathbb{Z}^d;\ \varphi_x \geq h\}, \quad \text{for } h \in \mathbb{R}.$$

Q: percolation of $\mathcal{L}_{\varphi}^{\geq h}$, as h varies ?

Critical parameters

Define

$$h_*(d) = \inf\left\{h \in \mathbb{R}\,;\, \mathbb{P}[0 \text{ lies in an } \infty \text{ cluster of } \mathcal{L}_\varphi^{\geq h}] = 0\right\}$$

(convention $\inf \emptyset = \infty$).

Then: $\mathbb{P}$-a.s.,

- $\forall h > h_*$, $\mathcal{L}_\varphi^{\geq h}$ contains only finite clusters
- $\forall h < h_*$, $\mathcal{L}_\varphi^{\geq h}$ contains a *unique* infinite conn. comp.

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Moreover, let

$$h_{**}(d) = \inf \left\{ h \in \mathbb{R} \,;\, \lim_{L \to \infty} \mathbb{P}[B(0, L) \overset{\geq h}{\longleftrightarrow} S(0, 2L)] = 0 \right\}.$$

Non-trivial phase transition

Theorem (Lebowitz et al. '87, R., Sznitman '12)

$$0 \le h_*(d) \le h_{**}(d) < \infty, \qquad \textit{for all } d \ge 3.$$

In fact, for $h > h_*$, and as $L \to \infty$

$$\mathbb{P}[0 \overset{\ge h}{\longleftrightarrow} S(0,L)] \text{ decays } \begin{cases} \text{(str.) exponentially in } L, & \text{for } h > h_{**} \\ \text{at most polynomially in } L, & \text{for } h \in (h_*, h_{**}) \end{cases}$$

and more! (Popov-Ráth, Popov-Teixeira)

High dimensions - baby version

Theorem (R., Sznitman '12)

For all sufficiently large d,

$$h_*(d) > 0, \qquad (\textit{in fact} \stackrel{d\to\infty}{\longrightarrow} \infty).$$

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Proof: perturbative argument. For $\ell \ll d$,

$$\begin{array}{ccccc} g_{|_{\mathbb{Z}^\ell \times \mathbb{Z}^\ell}}(x,y) & = & \sigma^2 \cdot \delta(x,y) & + & g'(x,y) \\ \updownarrow & & \updownarrow & & \updownarrow \\ \varphi & = & \psi & + & \xi \end{array} \quad \text{with } g' \text{ "small"}$$

Choose ℓ s.t. $p_c^{\text{site}}(\mathbb{Z}^\ell) < \frac{1}{2}$ (ok if $\ell = 3$). When d is large enough,

$$\Longrightarrow \mathcal{L}_\varphi^{\geq \varepsilon} \supset \mathcal{L}_\psi^{\geq 2\varepsilon} \cap \mathcal{L}_\xi^{\geq -\varepsilon} \text{ percolates for some } \varepsilon > 0.$$

High dimensions - leading asymptotics

Theorem (Drewitz, R. '13)

$$\mathbb{P}[\varphi_0 \geq h_*] = \frac{1}{d^{1+o(1)}}, \quad \textit{as } d \to \infty$$

(N.B.: for Bernoulli percolation, $p_c(\mathbb{Z}^d) \sim \frac{1}{2d}$).

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In fact, we show

$$\limsup_{d\to\infty} h_{**}(d)/\sqrt{2\log d} = 1$$

$$\liminf_{d\to\infty} h_*(d)/\sqrt{2\log d} = 1.$$

It follows that

$$h_* \sim h_{**}, \text{ as } d \to \infty.$$

Proof requires good understanding of "local" connectivity.
Key: suitable orthogonal expansion of GFF.

Thank you!