Geometric RSK, Whittaker functions and random polymers

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The longest increasing subsequence problem

For a permutation $\sigma \in S_n$, write

$$L_n(\sigma) = \text{ length of longest increasing subsequence in } \sigma$$

E.g. if $\sigma = 154263$ then $L_6(\sigma) = 3$.

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Based on Monte-Carlo simulations, Ulam (1961) conjectured that

$$EL_n = \frac{1}{n!} \sum_{\sigma \in S_n} L_n(\sigma) \sim c\sqrt{n}, \qquad n \to \infty.$$

A classical result from combinatorial geometry (Erdős-Szekeres 1935) implies that $EL_n \geq \sqrt{n-1}/2$.

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Baik, Deift and Johansson (1999): for each $x \in \mathbb{R}$,

$$\frac{1}{n!}|\{\sigma \in S_n : \; n^{-1/6}(L_n(\sigma) - 2\sqrt{n}) \leq x\}| \to F_2(x),$$

where F_2 is the Tracy-Widom (GUE) distribution from random matrix theory (Tracy and Widom 1994 — limiting distribution of largest eigenvalue of high-dimensional random Hermitian matrix)

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How is this possible?

The Robinson-Schensted correspondence

From the representation theory of S_n,

$$n! = \sum_{\lambda \vdash n} d_\lambda^2$$

where d_λ = number of standard tableaux with shape λ.

A standard tableau with shape $(4, 3, 1) \vdash 8$:

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & 8 \\ \cline{1-3} 7 \\ \cline{1-1}\end{array}$$

In other words, S_n has the same cardinality as the set of pairs of standard tableaux of size n with the same shape.

The Robinson-Schensted correspondence

Robinson (38): A bijection between S_n and such pairs

$$\sigma \longleftrightarrow (P, Q)$$

Schensted (61):

$$L_n(\sigma) = \text{ length of longest row of } P \text{ and } Q$$

This yields

$$|\{\sigma \in S_n : L_n(\sigma) \leq k\}| = \sum_{\lambda \vdash n,\ \lambda_1 \leq k} d_\lambda^2.$$

The RSK correspondence

Knuth (70): Extends to a bijection between matrices with nonnegative integer entries and pairs of *semi-standard* tableaux of same shape.

A *semistandard tableau* of shape $\lambda \vdash n$ is a diagram of that shape, filled in with positive integers which are *weakly* increasing along rows and strictly increasing along columns.

A semistandard tableau of shape $(5, 3, 1)$:

1	2	2	5	7
3	3	8		
4				

Cauchy-Littlewood identity

This gives a combinatorial proof of the Cauchy-Littlewood identity

$$\prod_{ij}(1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(x) s_\lambda(y),$$

where s_λ are Schur polynomials, defined by

$$s_\lambda(x) = \sum_{\mathrm{sh}\ P=\lambda} x^P,$$

where $x = (x_1, x_2, \ldots)$ and

$$x^P = x_1^{\sharp 1's\ in\ P} x_2^{\sharp 2's\ in\ P} \ldots.$$

Cauchy-Littlewood identity

Let $(a_{ij}) \mapsto (P, Q)$ under RSK.

Then $C_j = \sum_i a_{ij} = \sharp\, j$'s in P and $R_i = \sum_j a_{ij} = \sharp\, i$'s in Q.

For $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ we have

$$\prod_{ij} (y_i x_j)^{a_{ij}} = \prod_j x_j^{C_j} \prod_i y_i^{R_i} = x^P y^Q.$$

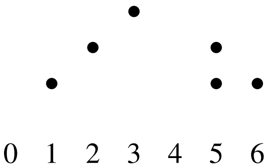
Summing over (a_{ij}) on the left and (P, Q) with sh $P =$ sh Q on the right gives

$$\prod_{ij} (1 - x_i y_j)^{-1} = \sum_\lambda s_\lambda(x) s_\lambda(y).$$

Tableaux and Gelfand-Tsetlin patterns

Semistandard tableaux $\longleftrightarrow$ discrete Gelfand-Tsetlin patterns

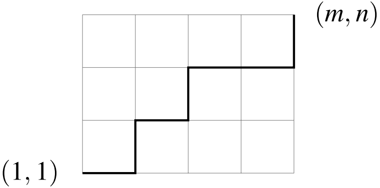
1	1	1	2	2	3
2	2	3	3	3	
3					



The RSK correspondence

If $(a_{ij}) \in \mathbb{N}^{m\times n}$, then length of longest row in corresponding tableaux is

$$M = \max_\pi \sum_{(i,j)\in\pi} a_{ij}$$



Combinatorial interpretation

Consider n queues in series:

$$☺ ☺ ☺| \longrightarrow \quad ☹ ☺| \longrightarrow \quad ☹ ☹ ☺| \longrightarrow \quad \cdots\cdots\cdots \quad ☺| \longrightarrow$$

Data:

$$a_{ij} = \text{ time required to serve } i^{th} \text{ customer at } j^{th} \text{ queue}$$

If we start with all customers in first queue, then M is the time taken for all customers to leave the system (Muth 79).

Combinatorial interpretation

From the RSK correspondence:

If a_{ij} are independent random variables with $P(a_{ij} \geq k) = (p_i q_j)^k$ then

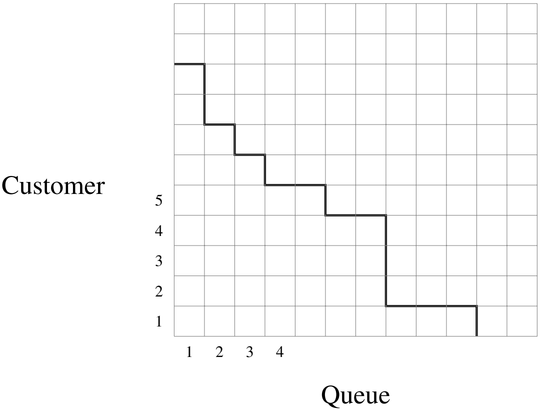
$$P(M \leq k) = \prod_{ij} (1 - p_i q_j) \sum_{\lambda:\ \lambda_1 \leq k} s_\lambda(p) s_\lambda(q).$$

cf. Weber (79): *The interchangeability of $\cdot/M/1$ queues in series.*

Johansson (99): As $n, m \to \infty$, $M \sim$ Tracy-Widom distribution
(and other related asymptotic results)

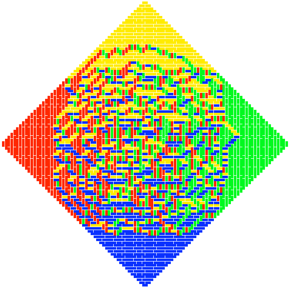
Surface growth and KPZ universality

The queueing system can be thought of as a model for surface growth . . .



Surface growth and KPZ universality

…and belongs to the same *universality class* as:



Random tiling

Burning paper

Bacteria colonies

KPZ = Kardar-Parisi-Zhang (1986)

Geometric RSK correspondence

The RSK mapping can be defined by expressions in the $(\max, +)$-semiring.

Replacing these expressions by their $(+, \times)$ counterparts, A.N. Kirillov (00) introduced a *geometric lifting* of RSK correspondence. It is a bi-rational map

$$T : (\mathbb{R}_{>0})^{n\times n} \to (\mathbb{R}_{>0})^{n\times n}$$

$$X = (x_{ij}) \mapsto (t_{ij}) = T = T(X).$$

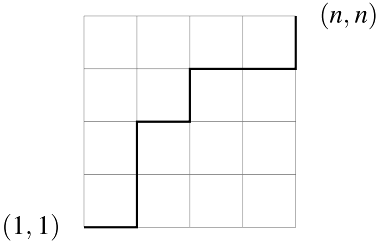
For $n = 2$,

$$\begin{matrix} & x_{21} & \\ x_{11} & & x_{22} \\ & x_{12} & \end{matrix} \quad \mapsto \quad \begin{matrix} & x_{11}x_{21} & \\ x_{12}x_{21}/(x_{12}+x_{21}) & & x_{11}x_{22}(x_{12}+x_{21}) \\ & x_{11}x_{12} & \end{matrix}$$

Geometric RSK correspondence

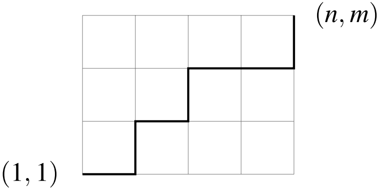
The analogue of the 'longest increasing subsequence' is the matrix element:

$$t_{nn} = \sum_{\phi \in \Pi_{(n,n)}} \prod_{(i,j) \in \phi} x_{ij}$$



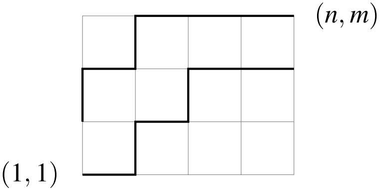
Geometric RSK correspondence

$$t_{nm} = \sum_{\phi \in \Pi_{(n,m)}} \prod_{(i,j) \in \phi} x_{ij}$$



Geometric RSK correspondence

$$t_{n-k+1,m-k+1} \dots t_{nm} = \sum_{\phi \in \Pi^{(k)}_{(n,m)}} \prod_{(i,j) \in \phi} x_{ij}$$



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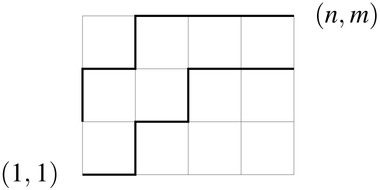
$$T(X)' = T(X')$$



Whittaker functions

A *triangle* P with shape $x \in (\mathbb{R}_{>0})^n$ is an array of positive real numbers:

$$P = \begin{array}{ccccccc} & & & z_{11} & & & \\ & & z_{22} & & z_{21} & & \\ & \iddots & & & & \ddots & \\ z_{nn} & & & \cdots\cdots & & & z_{n1} \end{array}$$

with bottom row $z_{n\cdot} = x$.

Denote by $\Delta(x)$ the set of triangles with shape x.

Whittaker functions

Let

$$P = \begin{array}{ccccccc} & & & z_{11} & & & \\ & & z_{22} & & z_{21} & & \\ & \iddots & & & & \ddots & \\ z_{nn} & & & \cdots & & & z_{n1} \end{array}$$

Define

$$P^\lambda = R_1^{\lambda_1} \left(\frac{R_2}{R_1}\right)^{\lambda_2} \cdots \left(\frac{R_n}{R_{n-1}}\right)^{\lambda_n}, \qquad \lambda \in \mathbb{C}^n, \qquad R_k = \prod_{i=1}^{k} z_{ki}$$

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$$\mathcal{F}(P) = \sum_{a \to b} \frac{z_a}{z_b}$$

$$\begin{tikzcd}[column sep=small] & & z_{11} \arrow[dr] & & \\ & z_{22} \arrow[ur] \arrow[dr] & & z_{21} \arrow[dr] & \\ z_{33} \arrow[ur] & & z_{32} \arrow[ur] & & z_{31} \end{tikzcd}$$

Whittaker functions

For $\lambda \in \mathbb{C}^n$ and $x \in (\mathbb{R}_{>0})^n$, define

$$\Psi_\lambda(x) = \int_{\Delta(x)} P^{-\lambda} e^{-\mathcal{F}(P)} dP,$$

where $dP = \prod_{1 \le i \le k < n} dz_{ki}/z_{ki}$.

For $n = 2$,

$$\Psi_{(\nu/2,-\nu/2)}(x) = 2K_\nu\left(2\pi\sqrt{x_2/x_1}\right).$$

These are called $GL(n)$-Whittaker functions.

They are the analogue of Schur polynomials in the geometric setting.

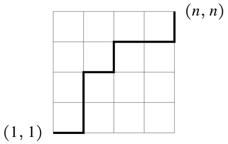
Geometric RSK correspondence

Recall

$$X = (x_{ij}) \mapsto (t_{ij}) = T \quad = \quad \begin{array}{ccccc} & & t_{31} & & \\ & t_{21} & & t_{32} & \\ t_{11} & & t_{22} & & t_{33} \\ & t_{12} & & t_{23} & \\ & & t_{13} & & \end{array}$$

= pair of triangles of same shape $(t_{nn}, \dots, t_{11})$.

$$t_{nn} = \sum_{\phi \in \Pi_{(n,n)}} \prod_{(i,j) \in \phi} x_{ij}$$



Whittaker measures

Let $a, b \in \mathbb{R}^n$ with $a_i + b_j > 0$ and define

$$\mathbb{P}(dX) = \prod_{ij} \Gamma(a_i + b_j)^{-1} x_{ij}^{-a_i - b_j - 1} e^{-1/x_{ij}} dx_{ij}.$$

Theorem (Corwin-O'C-Seppäläinen-Zygouras, '14)

Under $\mathbb{P}$, the law of the shape of the output under geometric RSK is given by the Whittaker measure *on $\mathbb{R}_+^n$ defined by*

$$\mu_{a,b}(dx) = \prod_{ij} \Gamma(a_i + b_j)^{-1} e^{-1/x_n} \Psi_a(x) \Psi_b(x) \prod_{i=1}^{n} \frac{dx_i}{x_i}.$$

Application to random polymers

Corollary

Suppose $a_i > 0$ for each i and $b_j < 0$ for each j. Then

$$\mathbb{E}e^{-st_{nn}} = \int_{\iota\mathbb{R}^m} s^{\sum_{i=1}^n (b_i-\lambda_i)} \prod_{ij} \Gamma(\lambda_i - b_j) \prod_{ij} \frac{\Gamma(a_i+\lambda_j)}{\Gamma(a_i+b_j)} s_n(\lambda) d\lambda,$$

where

$$s_n(\lambda) = \frac{1}{(2\pi\iota)^n n!} \prod_{i\neq j} \Gamma(\lambda_i - \lambda_j)^{-1}.$$

Application to random polymers

If $a_i + b_j = \theta$ for all i, j, this is the log-gamma polymer model introduced by Seppäläinen (2012). Using the above integral formula, Borodin, Corwin and Remenik (2013) have shown that for $\theta < \theta^*$ (for technical reasons)

$$\frac{\log t_{nn} - c(\theta)n}{d(\theta)n^{1/3}} \xrightarrow{dist} F_2.$$

The constant $c(\theta) = -2\Psi(\theta/2)$ and bound on fluctuation exponent $\chi < 1/3$ were established earlier by Seppäläinen (2012).

Combinatorial approach

Recall: $X = (x_{ij}) \mapsto (t_{ij}) = T(X) = (P, Q)$.

Theorem (O'C-Seppäläinen-Zygouras, '14)

- *The map* $(\log x_{ij}) \to (\log t_{ij})$ *has Jacobian* ± 1
- *For* $\nu, \lambda \in \mathbb{C}^n$,
$$\prod_{ij} x_{ij}^{\nu_i + \lambda_j} = P^{\lambda} Q^{\nu}$$
- *The following identity holds:*
$$\sum_{ij} \frac{1}{x_{ij}} = \frac{1}{t_{11}} + \mathcal{F}(P) + \mathcal{F}(Q)$$

This theorem (a) explains the appearance of Whittaker functions and (b) extends to models with symmetry.

Analogue of the Cauchy-Littlewood identity

It follows that

$$\prod_{ij} x_{ij}^{-\nu_i-\lambda_j} e^{-1/x_{ij}} \frac{dx_{ij}}{x_{ij}} = P^{-\lambda} Q^{-\nu} e^{-1/t_{11}-\mathcal{F}(P)-\mathcal{F}(Q)} \prod_{ij} \frac{dt_{ij}}{t_{ij}}.$$

Integrating both sides gives, for $\Re(\nu_i+\lambda_j)>0$:

Corollary (Stade 02)

$$\prod_{ij} \Gamma(\nu_i+\lambda_j) = \int_{\mathbb{R}_+^n} e^{-1/x_n} \Psi_\nu(x) \Psi_\lambda(x) \prod_{i=1}^n \frac{dx_i}{x_i}.$$

This is equivalent to a Whittaker integral identity which was conjectured by Bump (89) and proved by Stade (02). The integral is associated with Archimedean L-factors of automorphic L-functions on $GL(n,\mathbb{R}) \times GL(n,\mathbb{R})$.

Local moves

Proof of second theorem uses new description of the gRSK map T as a composition of a sequence of 'local moves' applied to the input matrix

$$\begin{array}{ccccc} & & x_{31} & & \\ & x_{21} & & x_{32} & \\ x_{11} & & x_{22} & & x_{33} \\ & x_{12} & & x_{23} & \\ & & x_{13} & & \end{array}$$

This description is a re-formulation of Noumi and Yamada's (2004) geometric row insertion algorithm.

Local moves

The basic move is:

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

Local moves

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$$\begin{array}{ccc} & b & \\ \dfrac{bc}{ab+ac} & & bd+cd \\ & c & \end{array}$$

Local moves

This can be applied at any position in the matrix:

$$\begin{array}{ccccc} & & c & & \\ & b & & f & \\ a & & e & & i \\ & d & & h & \\ & & g & & \end{array}$$

Local moves

This can be applied at any position in the matrix:

$$
\begin{array}{ccccc}
 & & c & & \\
 & b & & {\color{red}f} & \\
a & & e & & i \\
 & d & & h & \\
 & & g & &
\end{array}
$$

Local moves

This can be applied at any position in the matrix:

$$
\begin{array}{ccccc}
 & & c & & \\
 & \dfrac{ce}{bc+be} & & cf+ef & \\
a & & e & & i \\
 & d & & h & \\
 & & g & &
\end{array}
$$

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 & b & & f & \\
a & & e & & i \\
 & d & & {\color{red}h} & \\
 & & g & &
\end{array}$$

Local moves

This can be applied at any position in the matrix:

$$
\begin{array}{ccccc}
 & & c & & \\
 & b & & f & \\
a & & e & & i \\
 & \dfrac{eg}{de+dg} & & eh+gh & \\
 & & g & &
\end{array}
$$

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a & & e & & i \\
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 & & dg & &
\end{array}
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Local moves

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 & & g & &
\end{array}$$

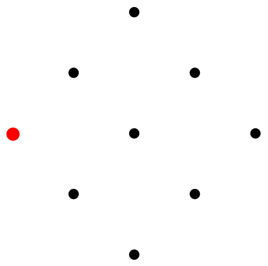
Local moves

Start with:

$$
\begin{array}{ccccc}
 & & x_{31} & & \\
 & x_{21} & & x_{32} & \\
x_{11} & & x_{22} & & x_{33} \\
 & x_{12} & & x_{23} & \\
 & & x_{13} & &
\end{array}
$$

Local moves

Apply the local moves in the following order:



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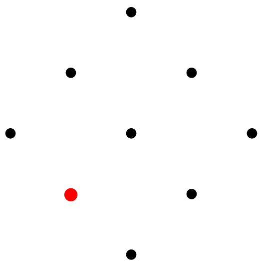

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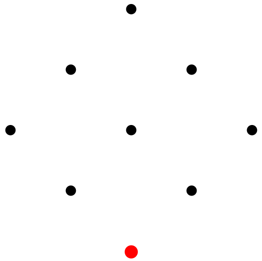

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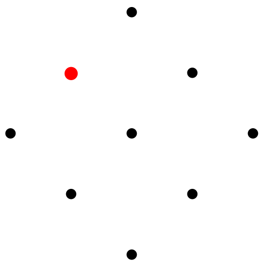

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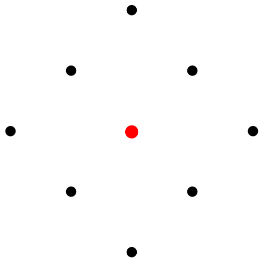

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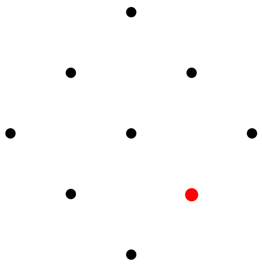

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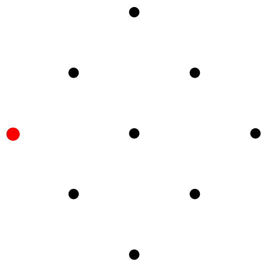

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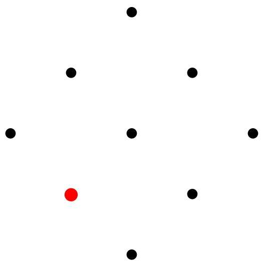

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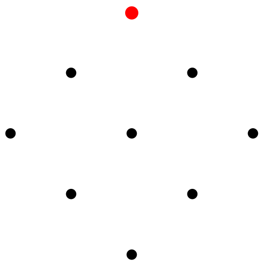

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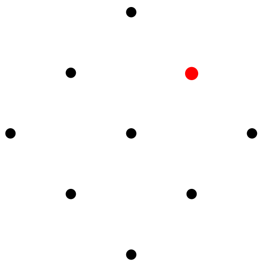

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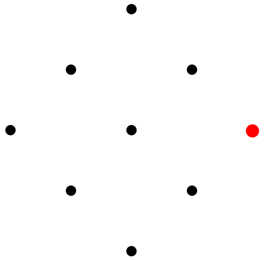

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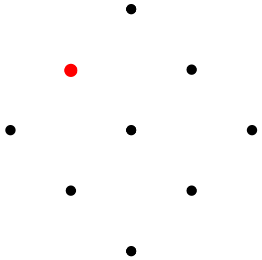

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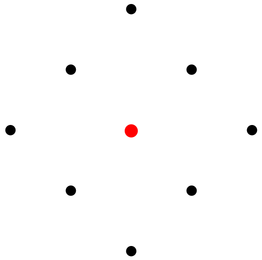

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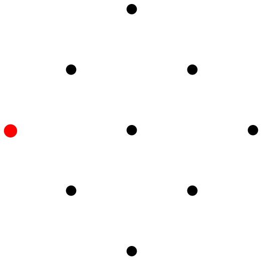

Local moves

To arrive at:

$$\begin{array}{ccccc} & & t_{31} & & \\ & t_{21} & & t_{32} & \\ t_{11} & & t_{22} & & t_{33} \\ & t_{12} & & t_{23} & \\ & & t_{13} & & \end{array}$$

Combinatorial approach

Recall: $X = (x_{ij}) \mapsto (t_{ij}) = T(X) = (P, Q)$.

Theorem (O'C-Seppäläinen-Zygouras, Invent. Math. 14)

- *The map* $(\log x_{ij}) \to (\log t_{ij})$ *has Jacobian* ± 1
- *For* $\nu, \lambda \in \mathbb{C}^n$,
$$\prod_{ij} x_{ij}^{\nu_i + \lambda_j} = P^\lambda Q^\nu$$
- *The following identity holds:*
$$\sum_{ij} \frac{1}{x_{ij}} = \frac{1}{t_{11}} + \mathcal{F}(P) + \mathcal{F}(Q)$$

This theorem (a) explains the appearance of Whittaker functions and (b) extends to models with symmetry.

Symmetric input matrix

Symmetry properties of gRSK:

$$T(X') = T(X)'$$

$$X \mapsto (P, Q) \quad \Longleftrightarrow \quad X' \mapsto (Q, P).$$

$$X = X' \quad \Longleftrightarrow \quad P = Q$$

Theorem (O'C-Seppäläinen-Zygouras 14)

The restriction of T to symmetric matrices is volume-preserving.

Symmetric input matrix

The analogue of the Cauchy-Littlewood identity in this setting is:

Corollary

Suppose $s > 0$ and $\Re\lambda_i > 0$ for each i. Then

$$\int_{(\mathbb{R}_{>0})^n} e^{-sx_1} \Psi^n_{-\lambda}(x) \prod_{i=1}^n \frac{dx_i}{x_i} = s^{-\sum_{i=1}^n \lambda_i} \prod_i \Gamma(\lambda_i) \prod_{i<j} \Gamma(\lambda_i + \lambda_j).$$

This is equivalent to a Whittaker integral identity which was conjectured by Bump-Friedberg (90) and proved by Stade (01).

Symmetric input matrix

Corollary

Let $\alpha_i > 0$ for each i and define

$$\mathbb{P}_\alpha(dX) = Z_\alpha^{-1} \prod_i x_{ii}^{-\alpha_i} \prod_{i<j} x_{ij}^{-\alpha_i-\alpha_j} e^{-\frac{1}{2}\sum_i \frac{1}{x_{ii}} - \sum_{i<j} \frac{1}{x_{ij}}} \prod_{i\le j} \frac{dx_{ij}}{x_{ij}}.$$

Then

$$\mathbb{P}_\alpha(sh\ P \in dx) = c_\alpha^{-1} e^{-1/2x_n} \Psi_\alpha^n(x) \prod_i \frac{dx_i}{x_i},$$

where

$$c_\alpha = \prod_i \Gamma(\alpha_i) \prod_{i<j} \Gamma(\alpha_i + \alpha_j).$$

Application to 'symmetrised' random polymer (reflecting boundary conditions)

Formally, this yields the integral formula:

$$\mathbb{E}_\alpha e^{-st_{nn}} = \int s^{-\sum_i \lambda_i} \prod_i \frac{\Gamma(\lambda_i)}{\Gamma(\alpha_i)} \prod_{i,j} \Gamma(\alpha_i + \lambda_j) \prod_{i<j} \frac{\Gamma(\lambda_i + \lambda_j)}{\Gamma(\alpha_i + \alpha_j)} s_n(\lambda) d\lambda$$

for appropriate vertical contours which stay to the right of zero.