Electric network for non-reversible Markov chains

joint with Áron Folly

Márton Balázs

University of Bristol

School and Workshop on Random Interacting Systems
Bath, June 2014.

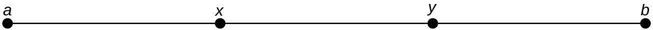
Reversible chains and resistors

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

$$h_x = \sum_y P_{xy} h_y, \qquad h_a = 1, \quad h_b = 0.$$



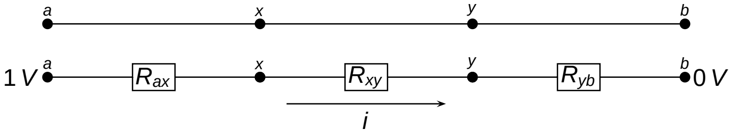
Reversible chains and resistors

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

$$h_x = \sum_y P_{xy} h_y, \qquad h_a = 1, \quad h_b = 0.$$



Electric resistor network: the voltage u is harmonic ($C = 1/R$):

$$u_x = \sum_y \frac{C_{xy}}{\sum_z C_{xz}} \cdot u_y; \qquad u_a = 1, \quad u_b = 0.$$

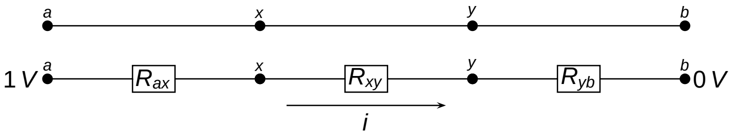
Reversible chains and resistors

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

$$h_x = \sum_y P_{xy} h_y, \qquad h_a = 1, \quad h_b = 0.$$



Electric resistor network: the voltage u is harmonic ($C = 1/R$):

$$u_x = \sum_y \frac{C_{xy}}{\sum_z C_{xz}} \cdot u_y; \qquad u_a = 1, \quad u_b = 0.$$

Reversible chains and resistors

Thus,

$$P_{xy} = \frac{C_{xy}}{\sum_z C_{xz}}$$

Reversible chains and resistors

Thus,

$$P_{xy} = \frac{C_{xy}}{\sum_z C_{xz}} =: \frac{C_{xy}}{C_x}.$$

Stationary distribtuion:

$$\rightsquigarrow \; C_x = \mu_x.$$

Reversible chains and resistors

Thus,

$$P_{xy} = \frac{C_{xy}}{\sum_z C_{xz}} =: \frac{C_{xy}}{C_x}.$$

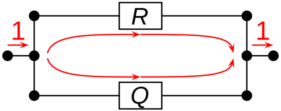
Stationary distribtuion:

$$\rightsquigarrow \quad C_x = \mu_x.$$

Notice $\mu_x P_{xy} = C_{xy} = C_{yx} = \mu_y P_{yx}$, so the chain is reversible.

Thomson, Dirichlet principles

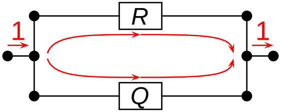
Thomson principle:



The physical unit current is the unit flow that minimizes the sum of the ohmic power losses $\sum i^2 R$.

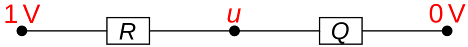
Thomson, Dirichlet principles

Thomson principle:



The physical unit current is the unit flow that minimizes the sum of the ohmic power losses $\sum i^2 R$.

Dirichlet principle:



The physical voltage is the function that minimizes the ohmic power losses $\sum (\nabla u)^2 / R$.

Monotonicity, transience, recurrence

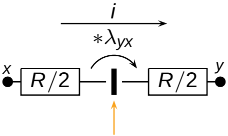
The monotonicity property:
Between any disjoint sets of vertices, the effective resistance is a non-decreasing function of the individual resistances.

Monotonicity, transience, recurrence

The monotonicity property:
Between any disjoint sets of vertices, the effective resistance is a non-decreasing function of the individual resistances.

$\rightsquigarrow$ can be used to prove transience-recurrence by reducing the graph to something manageable in terms of resistor networks.

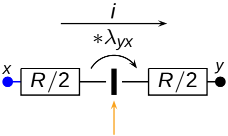
The part



Voltage amplifier: keeps the current, multiplies the potential.

$$(u_x - i \cdot \frac{R}{2}) \cdot \lambda_{yx} - i \cdot \frac{R}{2} = u_y$$

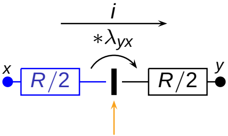
The part



Voltage amplifier: keeps the current, multiplies the potential.

$$(u_x - i \cdot \frac{R}{2}) \cdot \lambda_{yx} - i \cdot \frac{R}{2} = u_y$$

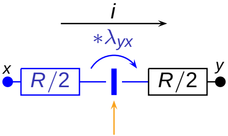
The part



Voltage amplifier: keeps the current, multiplies the potential.

$$\left(u_x - i\cdot\frac{R}{2}\right)\cdot\lambda_{yx} - i\cdot\frac{R}{2} = u_y$$

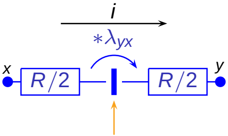
The part



Voltage amplifier: keeps the current, multiplies the potential.

$$(u_x - i \cdot \frac{R}{2}) \cdot \lambda_{yx} - i \cdot \frac{R}{2} = u_y$$

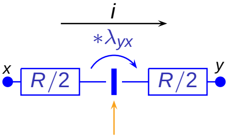
The part



Voltage amplifier: keeps the current, multiplies the potential.

$$(u_x - i \cdot \frac{R}{2}) \cdot \lambda_{yx} - i \cdot \frac{R}{2} = u_y$$

The part

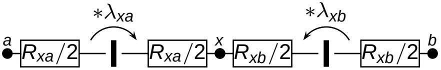


Voltage amplifier: keeps the current, multiplies the potential.

$$(u_x - i \cdot \frac{R}{2}) \cdot \lambda_{yx} - i \cdot \frac{R}{2} = u_y$$

It is clear that $\lambda_{yx} = \dfrac{1}{\lambda_{xy}}$.

Harmonicity



$$u_x = \sum_y \frac{D_{xy}\gamma_{xy}}{D_x} \cdot u_y$$

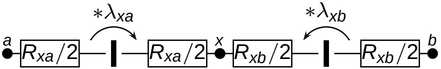
with

$$\gamma_{xy} = \sqrt{\lambda_{xy}} = \frac{1}{\gamma_{yx}},$$

$$D_{xy} = \frac{2\gamma_{xy}C_{xy}}{(\lambda_{xy}+1)} = D_{yx},$$

$$D_x = \sum_z D_{xz}\gamma_{zx}.$$

Harmonicity



$$u_x = \sum_y \frac{D_{xy}\gamma_{xy}}{D_x} \cdot u_y$$

with

$$\gamma_{xy} = \sqrt{\lambda_{xy}} = \frac{1}{\gamma_{yx}},$$

$$D_{xy} = \frac{2\gamma_{xy} C_{xy}}{(\lambda_{xy} + 1)} = D_{yx},$$

$$D_x = \sum_z D_{xz}\gamma_{zx}.$$

$\gamma_{xy} = \frac{1}{\gamma_{yx}}$ $\quad u_x = \sum_y \frac{D_{xy}\gamma_{xy}}{D_x} \cdot u_y$ $\quad D_{xy} = D_{yx}$

From network to chain

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

$$h_x = \sum_y P_{xy} h_y, \qquad h_a = 1, \quad h_b = 0.$$

$$u_x = \sum_y \frac{D_{xy}\gamma_{xy}}{D_x} \cdot u_y, \qquad u_a = 1, \quad u_b = 0.$$

$\gamma_{xy} = \frac{1}{\gamma_{yx}}$ $\qquad u_x = \sum_y \frac{D_{xy}\gamma_{xy}}{D_x} \cdot u_y$ $\qquad D_{xy} = D_{yx}$

From network to chain

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

$$h_x = \sum_y P_{xy} h_y, \qquad h_a = 1, \quad h_b = 0.$$

$$u_x = \sum_y \frac{D_{xy}\gamma_{xy}}{D_x} \cdot u_y, \qquad u_a = 1, \quad u_b = 0.$$

$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x}.$$

$\gamma_{xy} = \frac{1}{\gamma_{yx}}$ $\qquad u_x = \sum_y \frac{D_{xy}\gamma_{xy}}{D_x} \cdot u_y$ $\qquad D_{xy} = D_{yx}$

From network to chain

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

$$h_x = \sum_y P_{xy} h_y, \qquad h_a = 1, \quad h_b = 0.$$

$$u_x = \sum_y \frac{D_{xy}\gamma_{xy}}{D_x} \cdot u_y, \qquad u_a = 1, \quad u_b = 0.$$

$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x}.$$

$\gamma_{xy} = \frac{1}{\gamma_{yx}}$ $\qquad P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x}$ $\qquad D_{xy} = D_{yx}$

From chain to network

Stationary distribution: $\rightsquigarrow$ $D_x = \mu_x$.

$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x} = \frac{D_{xy}\gamma_{xy}}{\mu_x}$$

$$\mu_x P_{xy} \cdot \mu_y P_{yx} = D_{xy}^2;$$

$$\frac{\mu_x P_{xy}}{\mu_y P_{yx}} = \gamma_{xy}^2 = \lambda_{xy}.$$

$\gamma_{xy} = \frac{1}{\gamma_{yx}}$ $\qquad P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x}$ $\qquad D_{xy} = D_{yx}$

From chain to network

Stationary distribution: $\rightsquigarrow$ $D_x = \mu_x$.

$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x} = \frac{D_{xy}\gamma_{xy}}{\mu_x}$$

$$\mu_x P_{xy} \cdot \mu_y P_{yx} = D_{xy}^2;$$

$$\frac{\mu_x P_{xy}}{\mu_y P_{yx}} = \gamma_{xy}^2 = \lambda_{xy}.$$

Reversed chain: Replace P_{xy} by $\hat{P}_{xy} = P_{yx} \cdot \frac{\mu_y}{\mu_x}$.

$\rightsquigarrow$ D_{xy} stays, λ_{xy} reverses to λ_{yx}.

$\gamma_{xy} = \frac{1}{\gamma_{yx}}$ $\qquad$ $P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x}$ $\qquad$ $D_{xy} = D_{yx}$

Markovian network

$$u_x = \sum_z P_{xz} u_z ; \qquad \sum_z P_{xz} = 1$$

$u_x \equiv$ const. is a solution of the network with no external sources. This is now nontrivial.

Effective resistance

Suppose u_a, u_b given, the solution is $\{u_x\}_{x\in\Omega}$ and $\{i_{xy}\}_{x\sim y\in\Omega}$. Current

$$i_a = \sum_{x\sim a} i_{ax}$$

flows in the network at a.

Effective resistance

Suppose u_a, u_b given, the solution is $\{u_x\}_{x\in\Omega}$ and $\{i_{xy}\}_{x\sim y\in\Omega}$. Current

$$i_a = \sum_{x\sim a} i_{ax}$$

flows in the network at a.

$\rightsquigarrow$ Then

$$\frac{i_a}{u_a - u_b} = \text{const.} =: C^{\text{eff}}_{ab} = \frac{1}{R^{\text{eff}}_{ab}}.$$

What works

... the analogy with $\mathbf{P}\{\tau_a < \tau_b\}$.

What works

... the analogy with $\mathbf{P}\{\tau_a < \tau_b\}$.

$$\mathrm{cap}(A, B) = C^{\mathrm{eff}}_{AB} = \frac{1}{R^{\mathrm{eff}}_{AB}} \qquad \text{for all sets } A, B$$

What works

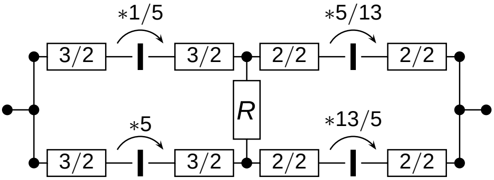
... the analogy with $\mathbf{P}\{\tau_a < \tau_b\}$.

$$\mathrm{cap}(A, B) = C^{\mathrm{eff}}_{AB} = \frac{1}{R^{\mathrm{eff}}_{AB}} \qquad \text{for all sets } A, B$$

Theorem

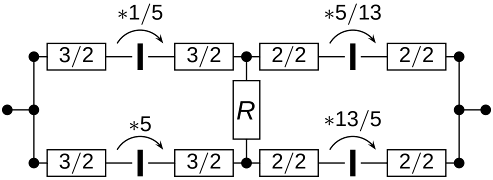
Commute time $= R^{\mathrm{eff}} \cdot$ *all conductances.*

Nonmonotonicity



$$R^{\text{eff}} = \frac{27}{14} + \frac{1296}{1225R + 2268}.$$

Nonmonotonicity



$$R^{\text{eff}} = \frac{27}{14} + \frac{1296}{1225R + 2268}. \qquad ☹$$

Dirichlet principle

Classical case:

Dirichlet principle

Classical case:

$$(i_u)_{xy} = C_{xy} \cdot \big(u(x) - u(y)\big),$$

$$E_{\mathrm{Ohm}}(i_u) = \sum_{x\sim y} (i_u)^2_{xy} \cdot R_{xy}.$$

Dirichlet principle

Classical case:

$$R_{ab}^{\text{eff}} = \qquad E_{\text{Ohm}}(i_u),$$

$$(i_u)_{xy} = C_{xy} \cdot \big(u(x) - u(y)\big),$$

$$E_{\text{Ohm}}(i_u) = \sum_{x \sim y} (i_u)_{xy}^2 \cdot R_{xy}.$$

Dirichlet principle

Classical case:

$$R_{ab}^{\text{eff}} = \min_{u:u(a)=1,\, u(b)=0} E_{\text{Ohm}}(i_u),$$

$$(i_u)_{xy} = C_{xy} \cdot (u(x) - u(y)),$$

$$E_{\text{Ohm}}(i_u) = \sum_{x \sim y} (i_u)_{xy}^2 \cdot R_{xy}.$$

Dirichlet principle

Classical case:

$$R^{\text{eff}}_{ab} = \min_{u:u(a)=1,\,u(b)=0} E_{\text{Ohm}}(i_u),$$

$$(i_u)_{xy} = C_{xy} \cdot \big(u(x) - u(y)\big),$$

$$E_{\text{Ohm}}(i_u) = \sum_{x\sim y} (i_u)^2_{xy} \cdot R_{xy}.$$

Irreversible case (A. Gaudillière, C. Landim / M. Slowik):

$$(i^*_u)_{xy} = D_{xy} \cdot \big(\gamma_{xy} u(x) - \gamma_{yx} u(y)\big),$$

$$E_{\text{Ohm}}(i^*_u - \Psi) = \sum_{x\sim y} \big(i^*_u - \Psi_{xy}\big)^2 \cdot R_{xy}.$$

Dirichlet principle

Classical case:

$$R_{ab}^{\text{eff}} = \min_{u:u(a)=1,\,u(b)=0} E_{\text{Ohm}}(i_u),$$

$$(i_u)_{xy} = C_{xy} \cdot \big(u(x) - u(y)\big),$$

$$E_{\text{Ohm}}(i_u) = \sum_{x\sim y} (i_u)_{xy}^2 \cdot R_{xy}.$$

Irreversible case (A. Gaudillière, C. Landim / M. Slowik):

$$R_{ab}^{\text{eff}} = \qquad E_{\text{Ohm}}(i_u^* - \Psi),$$

$$(i_u^*)_{xy} = D_{xy} \cdot \big(\gamma_{xy} u(x) - \gamma_{yx} u(y)\big),$$

$$E_{\text{Ohm}}(i_u^* - \Psi) = \sum_{x\sim y} \big(i_u^* - \Psi_{xy}\big)^2 \cdot R_{xy}.$$

Dirichlet principle

Classical case:

$$R_{ab}^{\text{eff}} = \min_{u:u(a)=1,\, u(b)=0} E_{\text{Ohm}}(i_u),$$

$$(i_u)_{xy} = C_{xy} \cdot \big(u(x) - u(y)\big),$$

$$E_{\text{Ohm}}(i_u) = \sum_{x\sim y} (i_u)_{xy}^2 \cdot R_{xy}.$$

Irreversible case (A. Gaudillière, C. Landim / M. Slowik):

$$R_{ab}^{\text{eff}} = \min_{\Psi:\text{ flow}} E_{\text{Ohm}}(i_u^* - \Psi),$$

$$(i_u^*)_{xy} = D_{xy} \cdot \big(\gamma_{xy} u(x) - \gamma_{yx} u(y)\big),$$

$$E_{\text{Ohm}}(i_u^* - \Psi) = \sum_{x\sim y} \big(i_u^* - \Psi_{xy}\big)^2 \cdot R_{xy}.$$

Dirichlet principle

Classical case:

$$R_{ab}^{\text{eff}} = \min_{u: u(a)=1,\, u(b)=0} E_{\text{Ohm}}(i_u),$$

$$(i_u)_{xy} = C_{xy} \cdot \big(u(x) - u(y)\big),$$

$$E_{\text{Ohm}}(i_u) = \sum_{x\sim y} (i_u)_{xy}^2 \cdot R_{xy}.$$

Irreversible case (A. Gaudillière, C. Landim / M. Slowik):

$$R_{ab}^{\text{eff}} = \min_{u: u(a)=1,\, u(b)=0} \min_{\Psi:\ \text{flow}} E_{\text{Ohm}}(i_u^* - \Psi),$$

$$(i_u^*)_{xy} = D_{xy} \cdot \big(\gamma_{xy} u(x) - \gamma_{yx} u(y)\big),$$

$$E_{\text{Ohm}}(i_u^* - \Psi) = \sum_{x\sim y} \big(i_u^* - \Psi_{xy}\big)^2 \cdot R_{xy}.$$

Dirichlet principle

Classical case:

$$R^{\text{eff}}_{ab} = \min_{u:u(a)=1,\,u(b)=0} E_{\text{Ohm}}(i_u),$$

$$(i_u)_{xy} = C_{xy} \cdot \big(u(x) - u(y)\big),$$

$$E_{\text{Ohm}}(i_u) = \sum_{x\sim y} (i_u)^2_{xy} \cdot R_{xy}.$$

Irreversible case (A. Gaudillière, C. Landim / M. Slowik):

$$R^{\text{eff}}_{ab} = \min_{u:u(a)=1,\,u(b)=0} \ \min_{\Psi:\ \text{flow}} E_{\text{Ohm}}(i^*_u - \Psi),$$

$$(i^*_u)_{xy} = D_{xy} \cdot \big(\gamma_{xy} u(x) - \gamma_{yx} u(y)\big),$$

$$E_{\text{Ohm}}(i^*_u - \Psi) = \sum_{x\sim y} \big(i^*_u - \Psi_{xy}\big)^2 \cdot R_{xy}.$$

Thank you.