Critical density for Activated Random Walk

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Outline

Definition: Activated Random Walk

Two types of particles, A and S,

- *A particles*: continuous time random walk in $\mathbb{Z}^d$, with jumps rate 1, distribution of jumps $P(\cdot)$.
- *S particles*: at rest.

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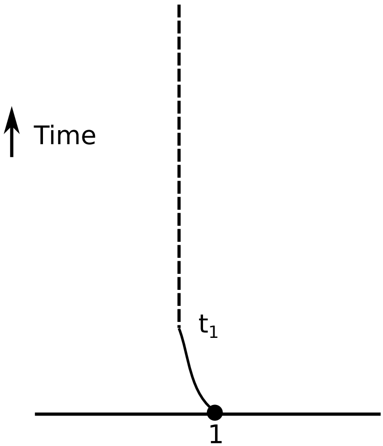
Initial configuration $\eta \in \Sigma = \mathbb{N}^{\mathbb{Z}^d}$. $(\eta(x))_{x\in\mathbb{Z}^d}$ i.i.d. random variables with $\mathbb{E}[\eta(x)] = \mu < \infty$.

Phase Transition

1. *Local Fixation*: a.s. for any finite $V \subset \mathbb{Z}^d$ $\exists t_V$ such that there is no activity in V for all $t > t_V$.

2. *Activity*: there is no local fixation.

About monotonicity



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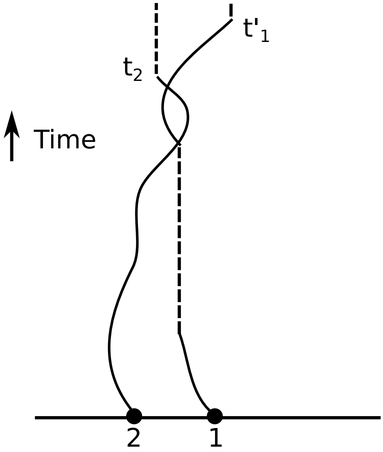

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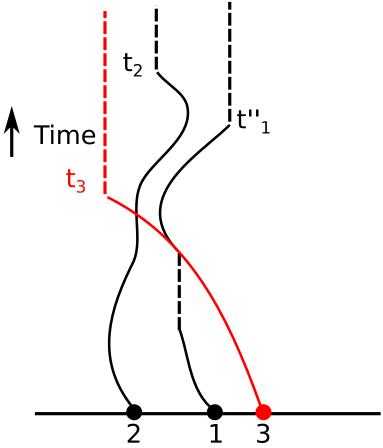

Critical density in $d = 1$

Definition:
$\mu_c = \sup\{\mu \in [0, \infty]$ s.t. ARW starting from $\nu(\mu)$ fixates locally$\}$.

Theorem [Rolla - Sidoravicius (2009)]

- Initial configuration: i.i.d. Poisson random variables with expectation μ.
- Jumps on nearest neighbours.

Then,

a) $\exists! \ \mu_c \in [0, \infty]$

b) If $d = 1$, then $\mu_c \in [\frac{\lambda}{1+\lambda}, 1]$.

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Question: is $\mu_c < 1$?

(Dickmann, Rolla, Sidoravicius - 2010)

Theorem [Taggi (2014)]

- $d = 1$.
- Jumps distribution $P(1) = p$, $P(-1) = 1 - p$, $p \in [0, 1]$.
- Initial configuration: i.i.d. random variables with expectation μ.

Let $\delta(p) = |2p - 1|$. Then $\mu_c \leq \frac{1}{\frac{\delta(p)}{1+\lambda} + 1}$.

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Theorem [Cabezas - Rolla- Sidoravicius (2013)]

- $d = 1$.
- Jumps distribution $P(1) = 1$.
- Initial configuration: i.i.d. random variables with expectation μ.

Then $\mu_c = \frac{\lambda}{1+\lambda}$ and there is no fixation at $\mu = \mu_c$.

Critical density in $d > 1$

Theorem [Shellef (2010), Amir - Gurel Gurevich (2010)]

- Any d, any λ.
- Any bounded jumps distribution.
- Initial configuration: i.i.d random variables with expectation μ.

Then $\mu_c \leq 1$.

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Then $\mu_c \leq 1$.

Theorem [Taggi 2014]

- $d \geq 2$
- Biased ARW
- Initial configuration: i.i.d. random variables, $\eta(x) = 1$ with probability μ and $\eta(0) = 0$ with probability $1 - \mu$.

There exists $K(P(\cdot)) > 0$ such that $\mu_c \leq \frac{1}{\frac{K}{1+\lambda}+1}$.

The case of $\lambda \to \infty$

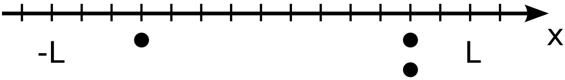
Theorem [Cabezas - Rolla- Sidoravicius (2013); Shellef (2010), Amir - Gurel Gurevich (2010)]

- $\lambda = \infty$, any dimension.
- Any jumps distribution.
- Initial configuration: i.i.d. Poisson random variables with expectation μ.

Then if $\mu_c = 1$ and there is no fixation at $\mu = 1$.

Diaconis-Fulton graphical representation

Jumps distribution of ARW: $P(1) = p$, $P(-1) = 1 - p$.
Stabilization of the set $[-L, L]$.



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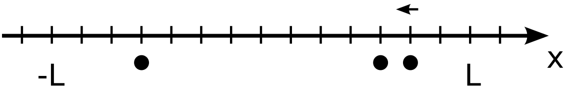

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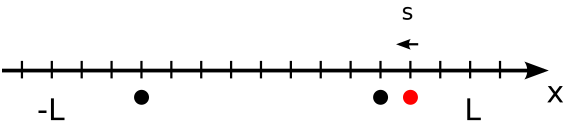

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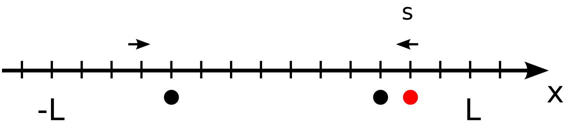

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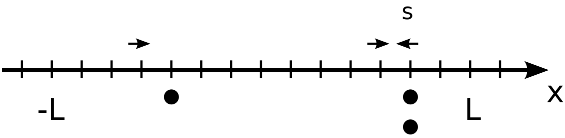

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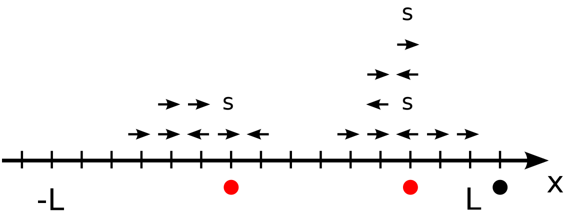

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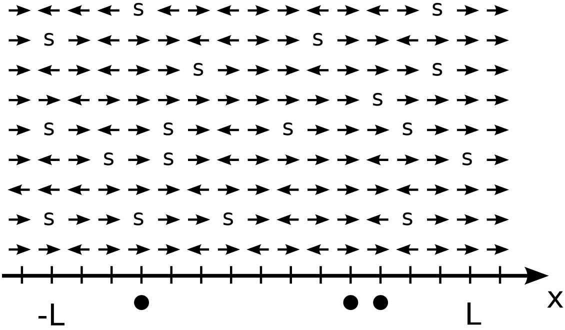
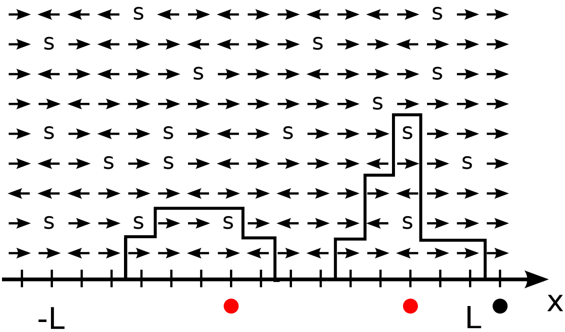


Diaconis-Fulton graphical representation

Definition: let $m_{\eta,V}(x)$ be the number of instructions that must be used at $x \in \mathbb{Z}^d$ in order to stabilize the initial configuration $\eta \in \mathbb{N}^{\mathbb{Z}^d}$ in the (finite) set $V \subset \mathbb{Z}^d$.

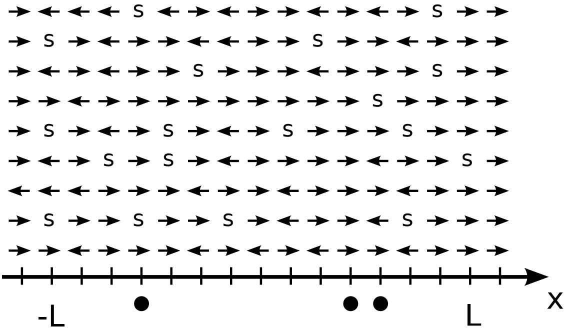


Diaconis-Fulton graphical representation

Space $(\mathbb{N}^{\mathbb{Z}}, \mathcal{I})$ $\quad \mathcal{I} = \{\tau_x^j \mid x \in \mathbb{Z}, j \in \mathbb{N}\}$

Probability measure $\mathcal{P}^\nu$:

$$\mathcal{P}^\nu(\tau_x^j = \text{“}\rightarrow\text{”}) = \frac{p}{1+\lambda}, \quad \mathcal{P}^\nu(\text{“}\leftarrow\text{”}) = \frac{1-p}{1+\lambda}, \quad \mathcal{P}^\nu(\tau_x^j = \text{“s”}) = \frac{\lambda}{1+\lambda}$$



Diaconis-Fulton graphical representation

Lemma [Rolla - Sidoravicius (2009)]

Let ν be a translation-invariant, ergodic distribution with finite density $\nu(\eta(0))$. Then,

$$\mathbb{P}^{\nu}\text{(the system locally fixates)} = \mathcal{P}^{\nu}\big(\lim_{V\uparrow\mathbb{Z}^d} m_{\eta,V}(0) < \infty\big) \in \{0,1\}.$$

Proposition (Monotonicity)

Consider a realization of the instructions, consider

- $\eta \prec \eta'$,
- (finite) $V \subset V' \subset \mathbb{Z}^d$.

Then $\forall x \in \mathbb{Z}^d$, $m_{\eta,V}(x) \le m_{\eta',V'}(x)$.

Thank you for your attention!