

# Planar lattices do not recover from forest fires

Ioan Manolescu

Joint work with Demeter Kiss and Vladas Sidoravicius



**UNIVERSITÉ  
DE GENÈVE**

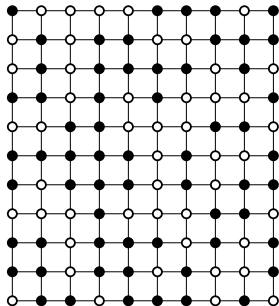
23 June 2014

# What is self-destructive percolation?

Let  $p, \delta \in [0, 1]$ .

Two percolation configurations:

- $\omega$  - intensity  $p$  (measure  $\mathbb{P}_p$ ).

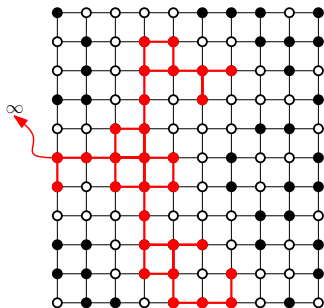


# What is self-destructive percolation?

Let  $p, \delta \in [0, 1]$ .

Two percolation configurations:

- $\omega$  - intensity  $p$  (measure  $\mathbb{P}_p$ ).



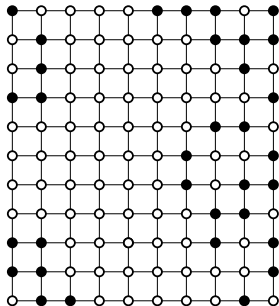
# What is self-destructive percolation?

Let  $p, \delta \in [0, 1]$ .

Two percolation configurations:

- $\omega$  - intensity  $p$  (measure  $\mathbb{P}_p$ ).

$$\omega \xrightarrow{\text{close } \infty\text{-cluster}} \bar{\omega}$$



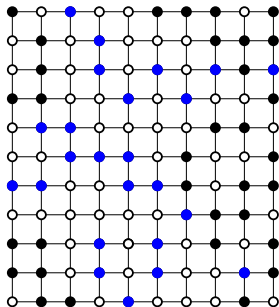
# What is self-destructive percolation?

Let  $p, \delta \in [0, 1]$ .

Two percolation configurations:

- $\omega$  - intensity  $p$  (measure  $\mathbb{P}_p$ ).
- $\sigma$  - intensity  $\delta$  (small).

$$\omega \xrightarrow{\text{close } \infty\text{-cluster}} \bar{\omega} \xrightarrow{\text{enhancement}} \bar{\omega}^\delta = \bar{\omega} \vee \sigma.$$



# What is self-destructive percolation?

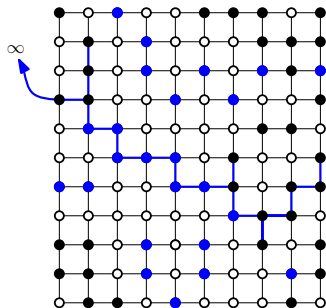
Let  $p, \delta \in [0, 1]$ .

Two percolation configurations:

- $\omega$  - intensity  $p$  (measure  $\mathbb{P}_p$ ).
- $\sigma$  - intensity  $\delta$  (small).

$$\omega \xrightarrow{\text{close } \infty\text{-cluster}} \bar{\omega} \xrightarrow{\text{enhancement}} \bar{\omega}^\delta = \bar{\omega} \vee \sigma.$$

$$\delta_c(p) = \sup\{\delta : \mathbb{P}_{p,\delta}(0 \xleftrightarrow{\bar{\omega}^\delta} \infty) = 0\}.$$



# What is self-destructive percolation?

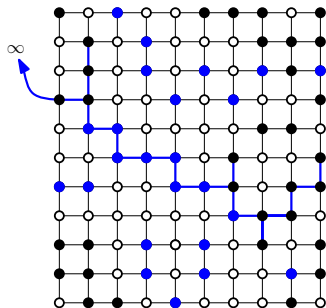
Let  $p, \delta \in [0, 1]$ .

Two percolation configurations:

- $\omega$  - intensity  $p$  (measure  $\mathbb{P}_p$ ).
- $\sigma$  - intensity  $\delta$  (small).

$$\omega \xrightarrow{\text{close } \infty\text{-cluster}} \bar{\omega} \xrightarrow{\text{enhancement}} \bar{\omega}^\delta = \bar{\omega} \vee \sigma.$$

$$\delta_c(p) = \sup\{\delta : \mathbb{P}_{p,\delta}(0 \xleftrightarrow{\bar{\omega}^\delta} \infty) = 0\}.$$



**Question:**  $\delta_c(p) \rightarrow 0$  as  $p \searrow p_c$ ?

# What is self-destructive percolation?

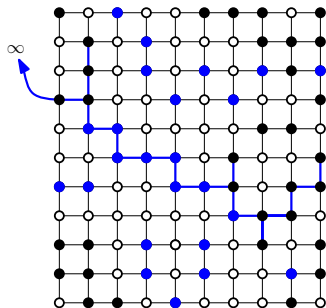
Let  $p, \delta \in [0, 1]$ .

Two percolation configurations:

- $\omega$  - intensity  $p$  (measure  $\mathbb{P}_p$ ).
- $\sigma$  - intensity  $\delta$  (small).

$$\omega \xrightarrow{\text{close } \infty\text{-cluster}} \bar{\omega} \xrightarrow{\text{enhancement}} \bar{\omega}^\delta = \bar{\omega} \vee \sigma.$$

$$\delta_c(p) = \sup\{\delta : \mathbb{P}_{p,\delta}(0 \xleftrightarrow{\bar{\omega}^\delta} \infty) = 0\}.$$



**Theorem** [Kiss, M., Sidoravicius] : There exists  $\delta > 0$  such that, for all  $p > p_c$ ,

$$\mathbb{P}_{p,\delta}(\text{infinite cluster in } \bar{\omega}^\delta) = 0.$$

In particular  $\lim_{p \rightarrow p_c} \delta_c(p) > 0$



# Critical percolation facts

There exists  $p_c \in [0, 1]$  such that

- $p < p_c \Rightarrow \mathbb{P}_p(\text{infinite cluster}) = 0,$
- $p > p_c \Rightarrow \mathbb{P}_p(\text{infinite cluster}) = 1.$

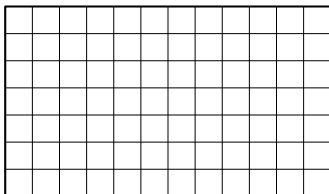
# Critical percolation facts

There exists  $p_c \in [0, 1]$  such that

- $p < p_c \Rightarrow \mathbb{P}_p(\text{infinite cluster}) = 0,$
- $p > p_c \Rightarrow \mathbb{P}_p(\text{infinite cluster}) = 1.$

At  $p_c..$

$$\forall n, \mathbf{P}_{p_c} \left[ \begin{array}{c} \text{[Diagram: A square of width } 2n \text{ and height } n \text{ with a red path from left to right]} \\ 2n \\ n \end{array} \right] \geq \epsilon$$



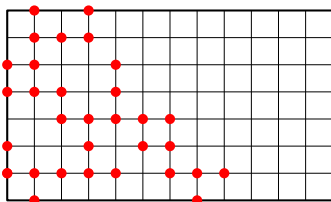
# Critical percolation facts

There exists  $p_c \in [0, 1]$  such that

- $p < p_c \Rightarrow \mathbb{P}_p(\text{infinite cluster}) = 0,$
- $p > p_c \Rightarrow \mathbb{P}_p(\text{infinite cluster}) = 1.$

At  $p_c..$

$$\forall n, \mathbf{P}_{p_c} \left[ \text{[Diagram]} \right] \geq \epsilon$$



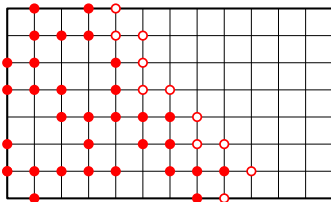
# Critical percolation facts

There exists  $p_c \in [0, 1]$  such that

- $p < p_c \Rightarrow \mathbb{P}_p(\text{infinite cluster}) = 0,$
- $p > p_c \Rightarrow \mathbb{P}_p(\text{infinite cluster}) = 1.$

At  $p_c..$

$$\forall n, \mathbf{P}_{p_c} \left[ \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{c} n \\ 2n \end{array} \right] \geq \epsilon$$



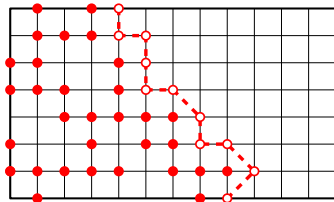
# Critical percolation facts

There exists  $p_c \in [0, 1]$  such that

- $p < p_c \Rightarrow \mathbb{P}_p(\text{infinite cluster}) = 0,$
- $p > p_c \Rightarrow \mathbb{P}_p(\text{infinite cluster}) = 1.$

At  $p_c..$

$$\forall n, \mathbf{P}_{p_c} \left[ \text{[Diagram of a square with a red curve]} \right] \geq \epsilon$$



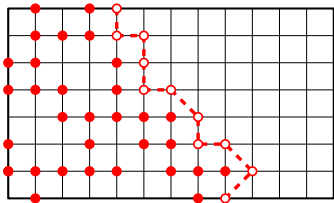
# Critical percolation facts

There exists  $p_c \in [0, 1]$  such that

- $p < p_c \Rightarrow \mathbb{P}_p(\text{infinite cluster}) = 0,$
- $p > p_c \Rightarrow \mathbb{P}_p(\text{infinite cluster}) = 1.$

At  $p_c..$

$$\forall n, \mathbf{P}_{p_c} \left[ \text{[Diagram: Solid red line crossing a box of size } 2n \times 2n \text{ from left to right]} \right] \geq \epsilon \quad \mathbf{P}_{p_c} \left[ \text{[Diagram: Dashed red line crossing a box of size } 2n \times 2n \text{ from top to bottom]} \right] \geq \epsilon$$



# Critical percolation facts

There exists  $p_c \in [0, 1]$  such that

- $p < p_c \Rightarrow \mathbb{P}_p(\text{infinite cluster}) = 0,$
- $p > p_c \Rightarrow \mathbb{P}_p(\text{infinite cluster}) = 1.$

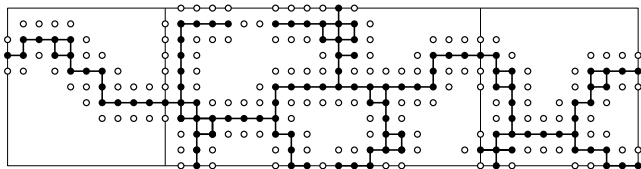
At  $p_c$ ..

$$\forall n, \mathbf{P}_{p_c} \left[ \text{Diagram 1} \right] \geq \epsilon$$

$$\mathbf{P}_{p_c} \left[ \text{Diagram 2} \right] \geq \epsilon$$

$$\mathbf{P}_{p_c} \left[ \text{Diagram 3} \right] \leq n^{-\alpha_1}$$

$$\mathbf{P}_{p_c} \left[ \text{Diagram 4} \right] \leq n^{-(2+\lambda)}$$



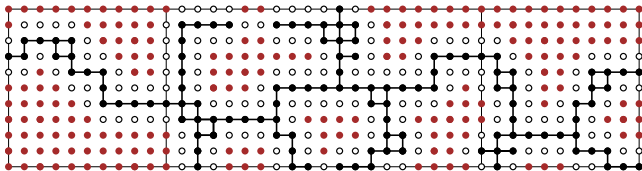
$\omega$  containing crossing

## Proposition

For  $\delta > 0$  small enough, as  $n \rightarrow \infty$ ,

$$\mathbb{P}_{p_C, \delta} \left( \begin{array}{c} \text{[Diagram of } \omega \text{ in a } 4n \times n \text{ rectangle]} \\ \text{and} \\ \text{[Diagram of } \tilde{\omega}^\delta \text{ in a } 4n \times n \text{ rectangle]} \end{array} \right) \rightarrow 0$$





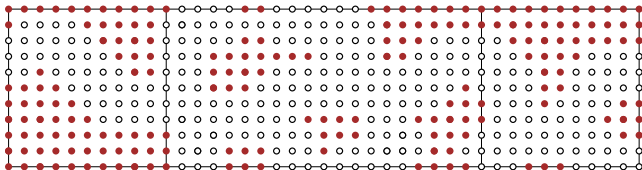
$\omega$  containing crossing  $\xrightarrow{\text{delete crossing cluster}} \tilde{\omega}$

## Proposition

For  $\delta > 0$  small enough, as  $n \rightarrow \infty$ ,

$$\mathbb{P}_{p_c, \delta} \left( \begin{array}{c} \left[ \begin{array}{|c|c|c|} \hline \omega \\ \hline \end{array} \right] \text{ and } \left[ \begin{array}{|c|c|c|} \hline \tilde{\omega}^\delta \\ \hline \end{array} \right] \end{array} \right) \rightarrow 0$$

$\left[ \begin{array}{|c|c|c|} \hline \omega \\ \hline \end{array} \right]$ 
and
 $\left[ \begin{array}{|c|c|c|} \hline \tilde{\omega}^\delta \\ \hline \end{array} \right]$

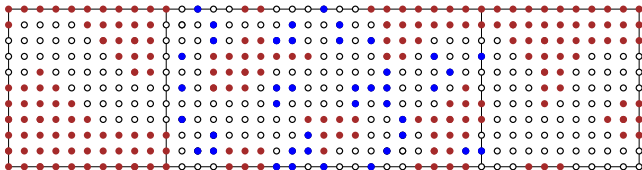


$\omega$  containing crossing  $\xrightarrow{\text{delete crossing cluster}}$   $\tilde{\omega}$

## Proposition

For  $\delta > 0$  small enough, as  $n \rightarrow \infty$ ,

$$\mathbb{P}_{p_c, \delta} \left( \left( \begin{array}{c} \text{Diagram 1: A rectangle of height } n \text{ and width } 6n. \text{ The width is divided into three segments of length } n. \text{ A black curve } \omega \text{ crosses the top and bottom edges.} \\ \text{Diagram 2: A rectangle of height } n \text{ and width } 6n. \text{ The width is divided into three segments of length } n. \text{ A blue curve } \tilde{\omega}^\delta \text{ is contained within the rectangle.} \end{array} \right) \rightarrow 0 \right)$$

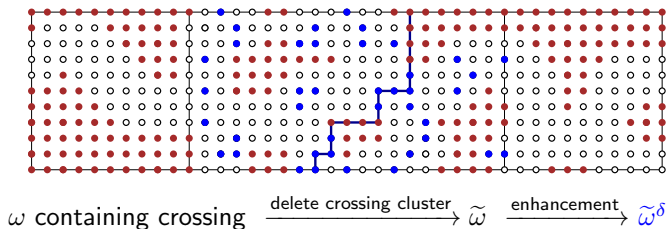


$$\omega \text{ containing crossing} \xrightarrow{\text{delete crossing cluster}} \tilde{\omega} \xrightarrow{\text{enhancement}} \tilde{\omega}^\delta$$

## Proposition

For  $\delta > 0$  small enough, as  $n \rightarrow \infty$ ,

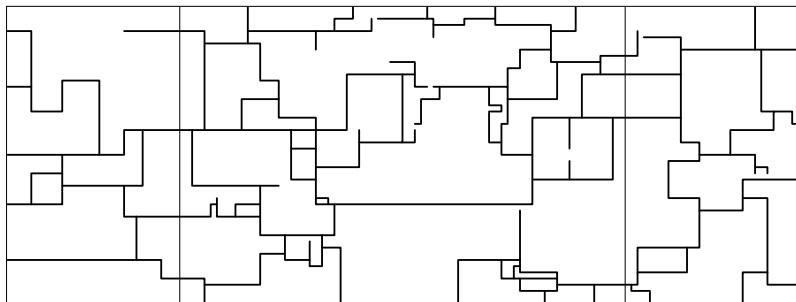
$$\mathbb{P}_{p_c, \delta} \left( \left( \begin{array}{c} \text{Diagram 1: A rectangle of height } n \text{ and width } 4n \text{ divided into three vertical sections of width } n. \text{ A black curve } \omega \text{ crosses the rectangle from top to bottom.} \\ \text{Diagram 2: A rectangle of height } n \text{ and width } 4n \text{ divided into three vertical sections of width } n. \text{ A blue curve } \tilde{\omega}^\delta \text{ crosses the rectangle from top to bottom.} \end{array} \right) \right) \rightarrow 0$$

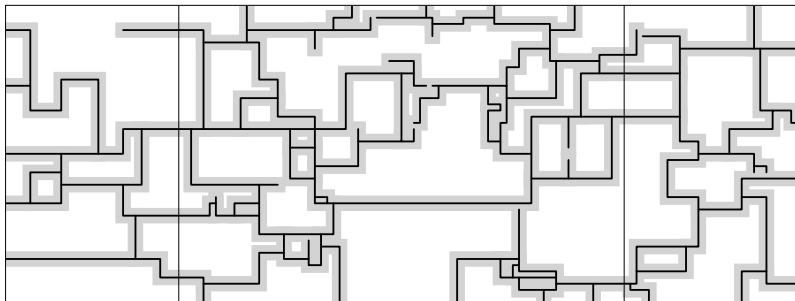


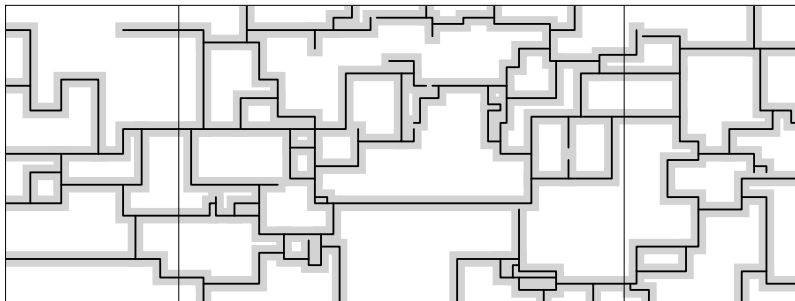
## Proposition

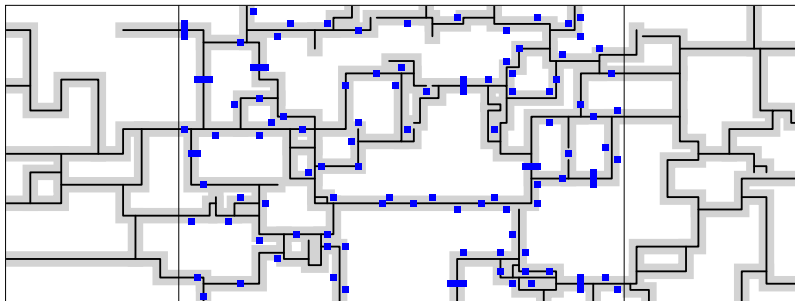
For  $\delta > 0$  small enough, as  $n \rightarrow \infty$ ,

$$\mathbb{P}_{p_c, \delta} \left( \left( \begin{array}{c} \text{Diagram 1: A rectangle with width } 6n \text{ and height } n. \text{ The bottom edge is divided into three segments of length } n, 4n, \text{ and } n. \text{ A wavy path } \omega \text{ crosses the rectangle.} \\ \text{Diagram 2: A rectangle with width } 6n \text{ and height } n. \text{ The bottom edge is divided into three segments of length } n, 4n, \text{ and } n. \text{ A blue path } \tilde{\omega}^\delta \text{ is shown, which is smoother than } \omega. \end{array} \right) \rightarrow 0 \right)$$

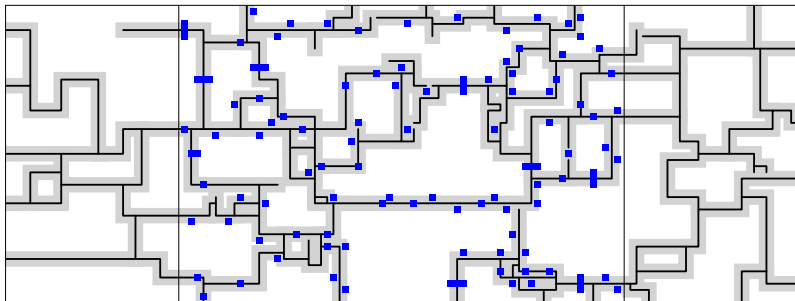




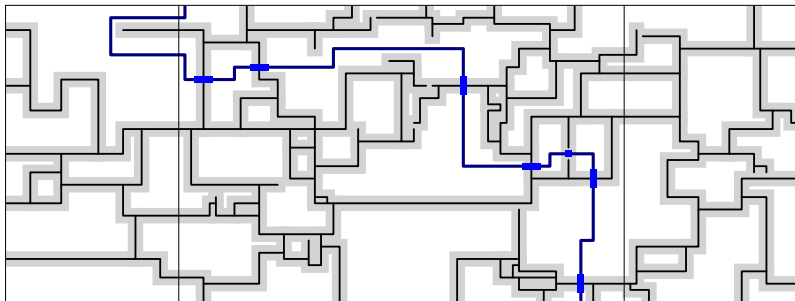








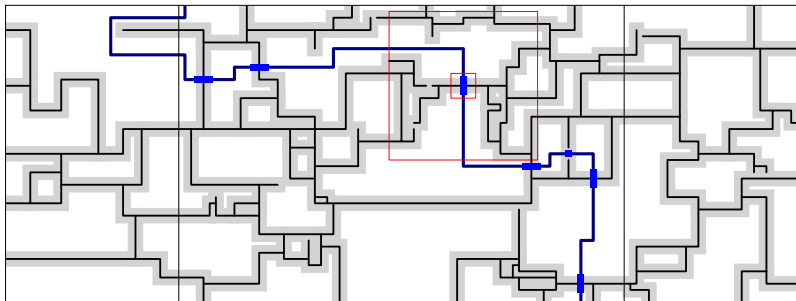
$\gamma$  - vertical crossing with minimal number of enhanced points.



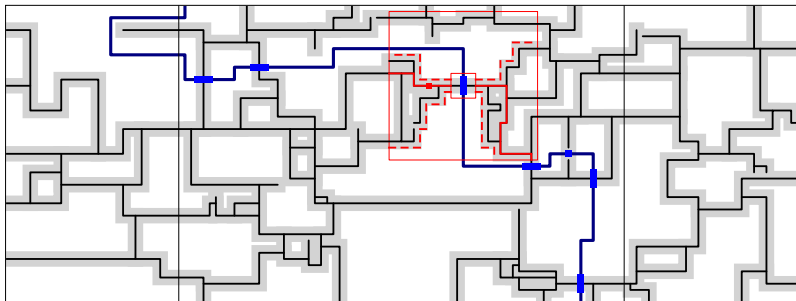
$\gamma$  - vertical crossing with minimal number of enhanced points.

$\mathcal{X} = \{\text{enhanced points used by } \gamma\}$ . If no crossing  $\mathcal{X} = \emptyset$ .

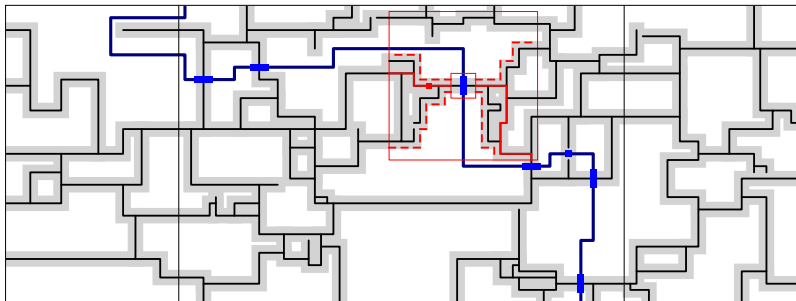
$$\mathbb{P}_{\rho_c, \delta}(\text{vertical crossing in } \tilde{\omega}^\delta) = \sum_{\mathcal{X} \neq \emptyset} \mathbb{P}_{\rho_c, \delta}(\mathcal{X} = X).$$



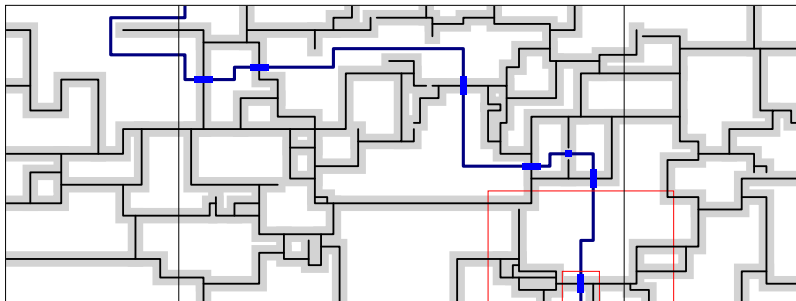
Annulus surrounding passage points but not containing passage points:  
 6 arms or 4 half-plane arms in  $\omega$  (possibly with one defect).



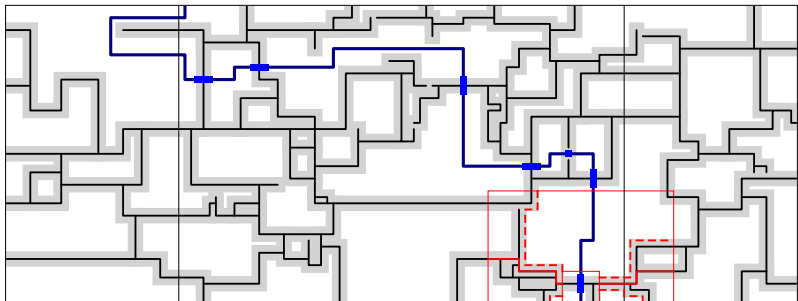
Annulus surrounding passage points but not containing passage points:  
 6 arms or 4 half-plane arms in  $\omega$  (possibly with one defect).



Annulus surrounding passage points but not containing passage points:  
 6 arms or 4 half-plane arms in  $\omega$  (possibly with one defect).

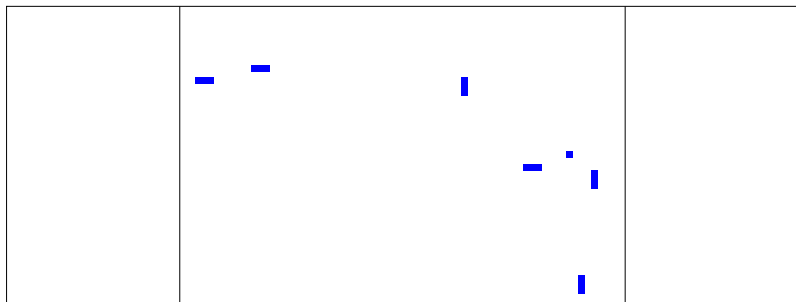


Annulus surrounding passage points but not containing passage points:  
 6 arms or 4 half-plane arms in  $\omega$  (possibly with one defect).



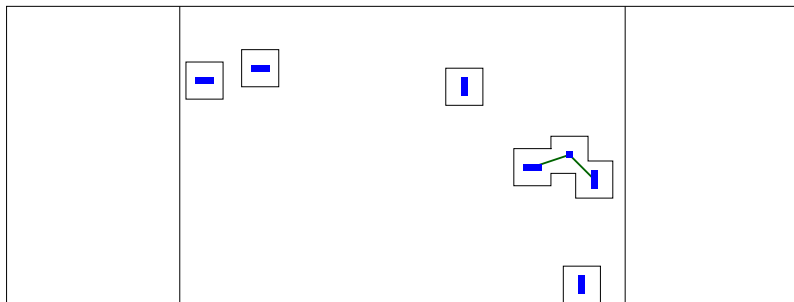
Annulus surrounding passage points but not containing passage points:  
6 arms or 4 half-plane arms in  $\omega$  (possibly with one defect).

$$\mathbb{P}_{p_c} \left( \begin{array}{c} \text{Diagram 1: A square with side length } R \text{ and } r. \text{ A solid red path winds through the square, crossing the boundary multiple times.} \\ \text{Diagram 2: A square with side length } R \text{ and } r. \text{ A solid red path winds through the square, crossing the boundary multiple times.} \end{array} \right) \leq \left( \frac{r}{R} \right)^{2+\lambda}$$

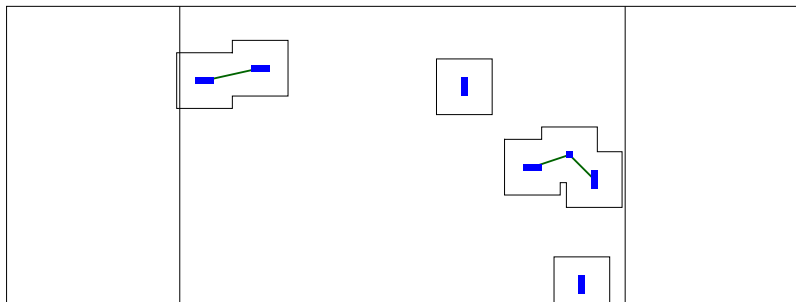


For a set  $X$ , what is  $\mathbb{P}_{\rho_c, \delta}(\mathcal{X} = X)$ ?

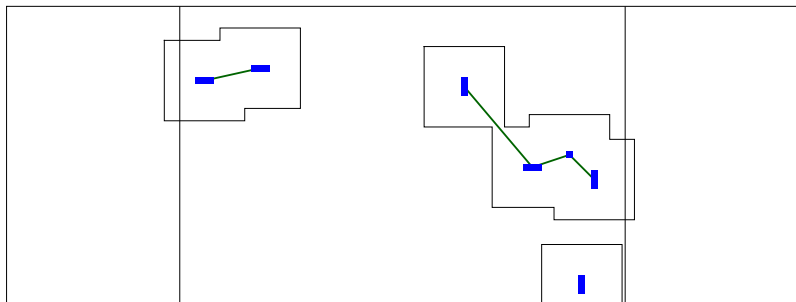




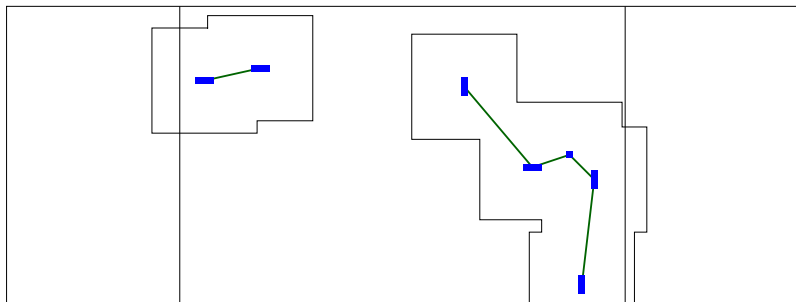
For a set  $X$ , what is  $\mathbb{P}_{\rho_c, \delta}(\mathcal{X} = X)$ ?



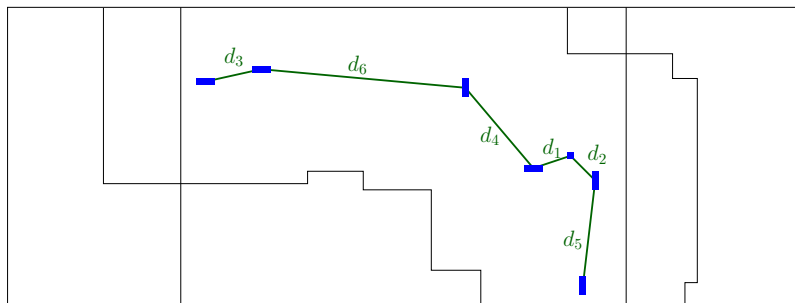
For a set  $X$ , what is  $\mathbb{P}_{\rho_c, \delta}(\mathcal{X} = X)$ ?



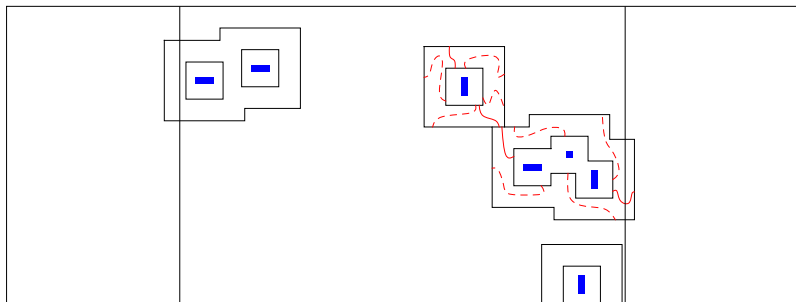
For a set  $X$ , what is  $\mathbb{P}_{\rho_c, \delta}(\mathcal{X} = X)$ ?



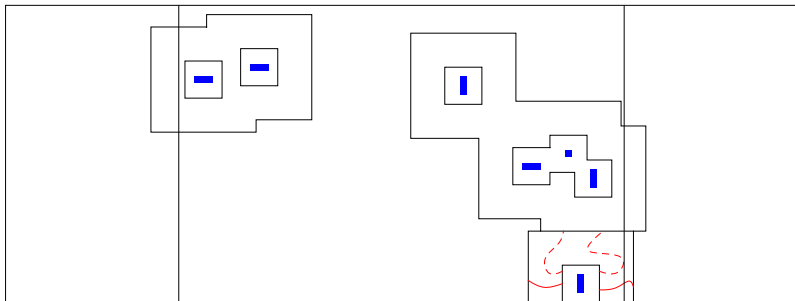
For a set  $X$ , what is  $\mathbb{P}_{\rho_c, \delta}(\mathcal{X} = X)$ ?



For a set  $X$ , what is  $\mathbb{P}_{\rho_c, \delta}(\mathcal{X} = X)$ ?



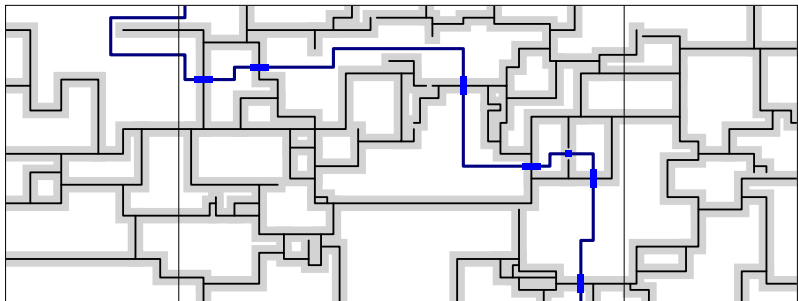
For a set  $X$ , what is  $\mathbb{P}_{\rho_c, \delta}(\mathcal{X} = X)$ ?



$$\mathbb{P}_{p,\delta}(\mathcal{X} = X) \leq c^k n^{-2-\lambda} \prod_j d_j^{-2-\lambda} \times \delta^k,$$

where  $d_1, \dots, d_k$  are the merger times of  $X$ .

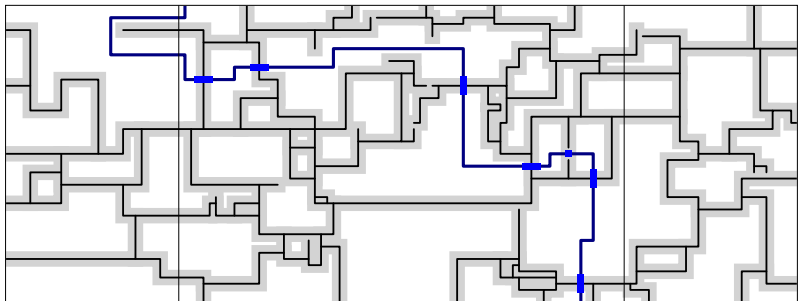
$$\#\{X \text{ with merger times } d_1, \dots, d_k\} \leq C^k n^2 \prod_j d_j.$$



$$\begin{aligned}
 \mathbb{P}(\text{vertical crossing in } \tilde{\omega}^\delta) &\leq n^{-\lambda} \sum_{\substack{k \geq 1 \\ d_1, \dots, d_k}} \left( \delta^k c^k \prod_k d_k^{-1-\lambda} \right) \\
 &= n^{-\lambda} \sum_{k \geq 1} \left( \delta c \sum_{d \geq 1} d^{-1-\lambda} \right)^k \rightarrow 0,
 \end{aligned}$$

for  $\delta > 0$  small.





$$\begin{aligned}
 \mathbb{P}(\text{vertical crossing in } \tilde{\omega}^\delta) &\leq n^{-\lambda} \sum_{\substack{k \geq 1 \\ d_1, \dots, d_k}} \left( \delta^k c^k \prod_k d_k^{-1-\lambda} \right) \\
 &= n^{-\lambda} \sum_{k \geq 1} \left( \delta c \sum_{d \geq 1} d^{-1-\lambda} \right)^k \rightarrow 0,
 \end{aligned}$$

for  $\delta > 0$  small.

Thank you!