RANDOM WALK IN DYNAMIC RANDOM ENVIRONMENT

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§ BACKGROUND

Random walk in random environment is a topic of major interest in mathematics, physics, chemistry and biology. Over the years, both static and dynamic random environments have been investigated.

Most results require fast mixing properties of the random environment.

For dynamic random environments a typical assumption is that correlations decay rapidly in time and uniformly in the initial configuration.

§ LITERATURE

Three classes of dynamic random environments have been considered so far:

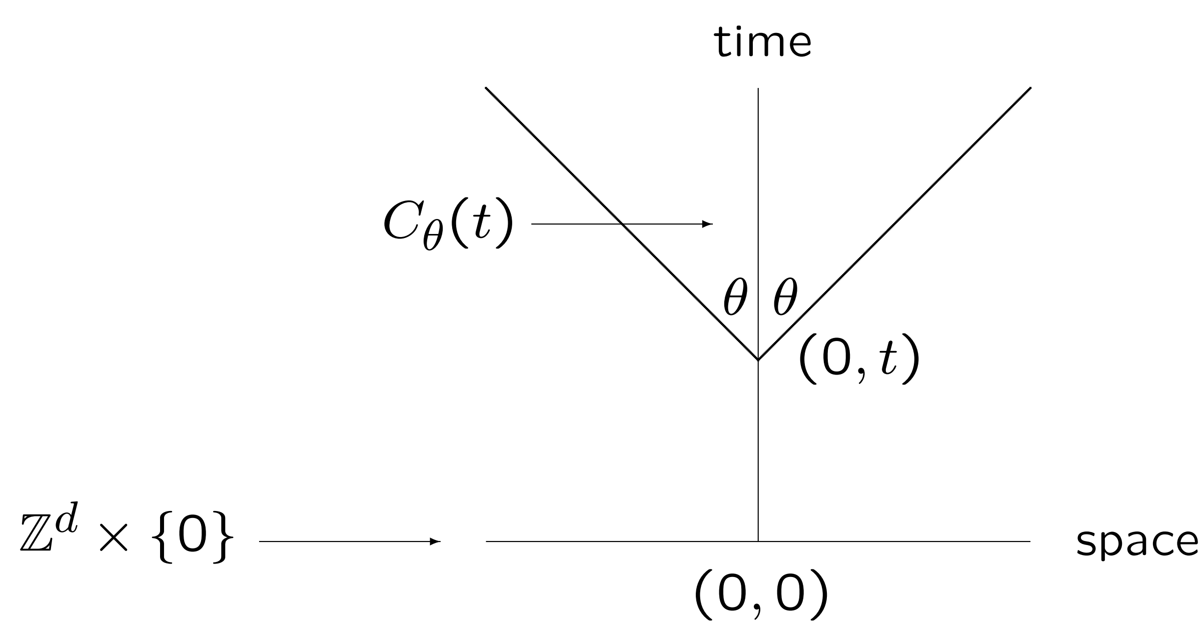
1. Independent in time: globally updated at each unit of time.
2. Independent in space: locally updated according to independent single-site Markov chains.
3. Dependent in space and time.

The homepage of Firas Rassoul-Agha contains an updated list of papers on random walk in static and dynamic random environment.

GENERAL THEOREM

The random walk satisfies the SLLN when the law P of the dynamic random environment is cone mixing, i.e.,

$$\lim_{t\to\infty} \sup_{\substack{A\in\mathcal{F}_{\mathbb{Z}^d\times\{0\}} \\ B\in\mathcal{F}_{C_\theta(t)}}} \Big| P(B|A) - P(B) \Big| = 0 \qquad \forall\, \theta \in (0, \tfrac{1}{2}\pi).$$



IDEA BEHIND GENERAL THEOREM

- Pick T large, and let π_T be a piece of path of length T whose probability is > 0 uniformly in the dynamic random environment.

- With probability 1, the random walk eventually performs π_T and afterwards stays confined in a cone with a large enough angle. When doing so, it experiences an approximate regeneration time, i.e., it enters into "fresh territory" up to an error that tends to zero as $T \to \infty$ by the cone mixing property.

- The frequency f_T at which the approximate regeneration times occur is > 0. Hence the displacement of the random walk at time t is the sum of t/f_T almost i.i.d. increments.

§ MODEL IN THIS TALK

Let $\{N(x)\colon x \in \mathbb{Z}\}$ be i.i.d. Poisson with mean $\rho \in (0, \infty)$. At time $n = 0$, for each $x \in \mathbb{Z}$ place $N(x)$ environment particles at site x. Subsequently, let these particles evolve independently as simple random walks on $\mathbb{Z}$.

Let $\mathcal{T}$ be the set of space-time points covered by the trajectories of the environment particles. The law of $\mathcal{T}$ is denoted by P^ρ. Note that $\mathcal{T}$ has slow mixing properties, e.g.

$$\mathrm{Cov}_\rho\Big(1_{(0,0)\in\mathcal{T}}, 1_{(0,n)\in\mathcal{T}}\Big) \sim \frac{c(\rho)}{\sqrt{n}}, \qquad n \to \infty,$$

where Cov_ρ denotes covariance with respect to P^ρ.

Given $\mathcal{T}$, let $X=(X_n)_{n\in\mathbb{N}_0}$ be the random walk on $\mathbb{Z}$ starting at the origin with transition probabilities

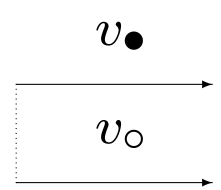
$$P^{\mathcal{T}}(X_{n+1}=x+1 \mid X_n=x)=\begin{cases} p_\circ, & \text{if } (x,n)\notin\mathcal{T},\\ p_\bullet, & \text{if } (x,n)\in\mathcal{T},\end{cases}$$

where $p_\circ, p_\bullet \in [0,1]$ are parameters and $P^{\mathcal{T}}$ stands for the law of X conditional on $\mathcal{T}$, called the quenched law. The annealed law is given by

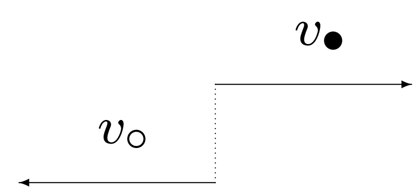
$$\mathbb{P}^\rho(\cdot)=\int P^{\mathcal{T}}(\cdot)\,P^\rho(d\mathcal{T}).$$

DEFINITION

Put $v_\circ = 2p_\circ - 1$ *and* $v_\bullet = 2p_\bullet - 1$. *The model is said to be non-nestling when* $v_\circ v_\bullet > 0$. *Otherwise it is said to be nestling.*



non-nestling
speeds



nestling
speeds

REMARK

By reflection symmetry, when $v_\bullet \neq 0$ we may assume without loss of generality that $v_\bullet > 0$.

§ MAIN THEOREMS

THEOREM 1

Let $v_\bullet > 0$ *and* $v_\circ \neq -1$. *Then there exists a* $\rho_\star \in [0,\infty)$ *such that for all* $\rho \in (\rho_\star,\infty)$ *there exist* $v = v(v_\circ, v_\bullet, \rho) \in [v_\circ \wedge v_\bullet, v_\circ \vee v_\bullet]$ *and* $\sigma = \sigma(v_\circ, v_\bullet, \rho) \in (0,\infty)$ *such that:*

(a) (SLLN) $\mathbb{P}^\rho$*-a.s.,*

$$\lim_{n\to\infty} n^{-1} X_n = v.$$

(b) (FCLT) In law under $\mathbb{P}^\rho$*,*

$$\left(\frac{X_{\lfloor nt \rfloor} - \lfloor nt \rfloor v}{\sqrt{n}\,\sigma}\right)_{t\geq 0} \overset{n\to\infty}{\longrightarrow} (B_t)_{t\geq 0}.$$

(c) (LDbound) There exists a $\gamma > 1$ *such that for all* $\varepsilon > 0$ *there exists a* $c = c(v_\circ, v_\bullet, \rho, \varepsilon) \in (0,\infty)$ *such that*

$$\mathbb{P}^\rho\Big(\exists\, n \geq m\colon\ |X_n - nv| > \varepsilon n\Big) \leq c^{-1} e^{-c \log^\gamma m} \qquad \forall\, m \in \mathbb{N}.$$

(d) In the non-nestling case $\rho_\star = 0$.

(e) Both v *and* σ *are continuous functions of* $v_\circ$, $v_\bullet$, ρ.

REMARK

Note that in the nestling case Theorem 1 requires that ρ is large enough.

HEURISTICS BEHIND THEOREM 1

- If $v_\bullet > 0$ and $v_\circ \neq -1$, then for ρ large enough the random walk stays to the right of a space-time line that moves at a strictly positive speed.

- Since the environment particles move diffusively, the random walk outruns the environment particles and experiences an approximate regeneration time at a strictly positive frequency.

- Control on the time lapses between the successive approximate regeneration times allows us to control the fluctuations of the random walk and to derive SLLN, FCLT, LDBound.

Theorem 1 is a consequence of Theorems 2–3 below. The following definition is central to our analysis.

DEFINITION

For fixed $v_\circ, v_\bullet, \rho$ *and given* $v_\star$*, we say that the* $v_\star$*-ballisticity condition holds when there exist* $c = c(v_\circ, v_\bullet, v_\star, \rho) \in (0, \infty)$ *and* $\gamma = \gamma(v_\circ, v_\bullet, v_\star, \rho) > 1$ *such that*

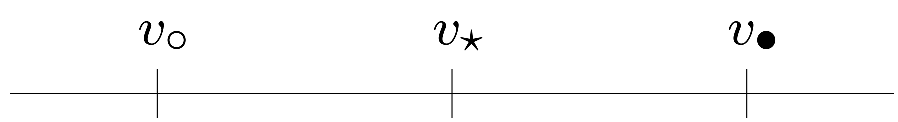
$$(\star) \qquad \mathbb{P}^\rho\Big(\exists\, n \in \mathbb{N}\colon\ X_n < nv_\star - L\Big) \le c^{-1} e^{-c \log^\gamma L} \qquad \forall\, L \in \mathbb{N}.$$

REMARK

Condition $(\star)$ is reminiscent of ballisticity conditions in the literature on random walk in static random environment, such as the (T')-condition of Sznitman.

THEOREM 2

If $v_\circ < v_\bullet$, then for all $v_\star \in [v_\circ, v_\bullet)$ there exist $\rho_\star = \rho_\star(v_\circ, v_\bullet, v_\star) \in (0, \infty)$ and $c = c(v_\circ, v_\bullet, v_\star) \in (0, \infty)$ such that $(\star)$ holds with $\gamma = \frac{3}{2}$ for all $\rho \in (\rho_\star, \infty)$.



THEOREM 3

Let $v_\circ, v_\bullet \neq -1$ and $\rho \in (0, \infty)$. Assume that $(\star)$ holds for some $v_\star \in (0, 1]$. Then the conclusions of Theorem 1 hold and $v \geq v_\star$.

REMARKS

- When $v_\circ \wedge v_\bullet > 0$, which corresponds to the non-nestling case, $(\star)$ holds for all $\rho \in (0,\infty)$ and $v_\star \in (0, v_\circ \wedge v_\bullet)$ by comparison with a homogeneous random walk with drift $v_\circ \wedge v_\bullet$.

- In the non-nestling case the bound in $(\star)$ can be taken exponentially small in L.

§ TECHNIQUES BEHIND THE PROOFS

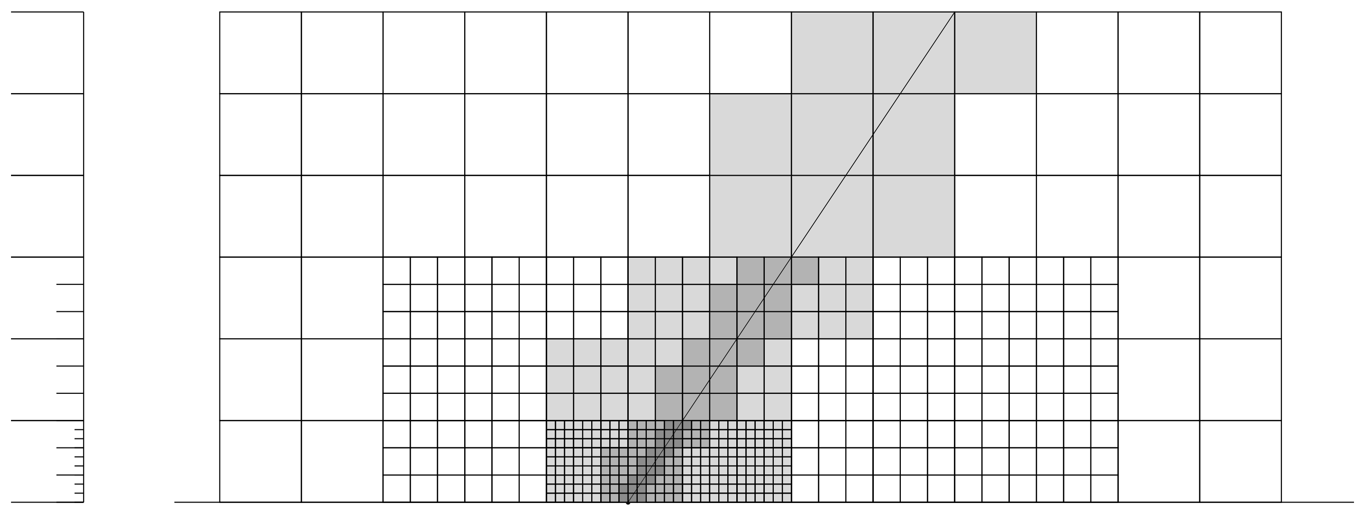
(I) The proof of Theorem 2 relies on a multiscale renormalisation analysis.

The key idea is that for large ρ the random walk spends most of its time on occupied sites and therefore moves at a speed close to $v_\bullet$.

(II) The proof of Theorem 3 relies on a construction of approximate regeneration times for the random walk trajectory.

The key idea is that $(\star)$ causes the random walk to outrun the environment particles and enter into “fresh territory” at random times.

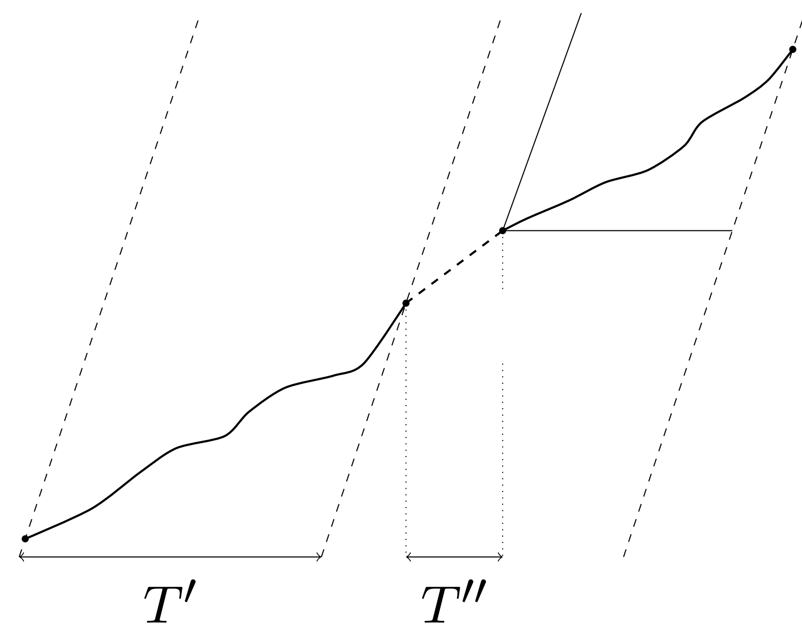
(I) Illustration of multi-scale renormalisation analysis:



Boxes of various sizes intersect a space-time line of constant speed $v_\star$. To prove $(\star)$, it is necessary to show that boxes above the line are unlikely to be hit by the trajectory of the random walk.

(II) Illustration of approximate regeneration times:

$T' = T^{\epsilon}$, $T'' = \delta \log T$ with $T \gg 1$, $0 < \delta, \epsilon \ll 1$:



Cones have angle $\frac{1}{2}v_\star$ with $v_\star$ the speed in the ballisticity condition in $(\star)$. Drawn is the trajectory of the random walk with space vertical and time horizontal.

§ CHALLENGES

- Prove the main theorem for all $\rho \in (0, \infty)$ in the nestling case.

- Extend the main theorem to models where the environment particles interact with each other.

The multi-scale renormalisation analysis is robust enough to imply that the ballisticity condition in $(\star)$ holds as long as the dynamic random environment has a very mild mixing property.

The approximate regeneration times are more delicate and heavily rely on model-specific features.