The Metric Coalescent

joint with David Aldous

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Two Related Processes

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- Compulsive Gambler
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- Metric Coalescent
  - Measure-valued Markov process,
  - Defined for any metric space (S, d).

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- n agents; each in some state $X_i(t)$ for each time $t \geq 0$.
- Each pair of agents (i, j) meets at the times of a Poisson process of rate ν_{ij}.
- At meeting times t between pairs of agents (i, j), the states transition

$$(X_i(t-), X_j(t-)) \mapsto (X_i(t), X_j(t))$$

according to some deterministic or random rule.

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- The iPod Model, a variant of the Voter Model (L. '13).

Compulsive Gambler Process

Simple FMIE process with agents' state space $\mathbb{R}_{\geq 0}$, interpreted as money. When agents i and j meet they play a fair, winner take all game, i.e. the transition function is

$$(a, b) \mapsto \begin{cases} (a+b, 0) & \text{with prob. } \frac{a}{a+b} \\ (0, a+b) & \text{with prob. } \frac{b}{a+b} \end{cases}$$

In the finite agent setting, we assume the total initial (and thus for all $t \geq 0$) wealth is normalized

$$\sum_{i \in \text{Agents}} X_i(0) = 1.$$

Importantly this allows us to view the state of the process as a probability measure on the set of agents.

Compulsive Gambler Process

The CG first studied in the setting of d-regular graphs and Galton-Watson trees (Aldous, Salez, L. '14 [ALS14]). Results on the proportion of agents still "solvent" at a time $t > 0$, in particular $t = \infty$.

The rest of today's talk will focus on a very particular variant of the CG, one with dependent rates ν_{ij}.

Extending the CG Process

We can reformulate the CG as a measure-valued Markov process in terms of:

- A metric space (S, d),
- A function $\phi(x)\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, called the **rate** function (**think** of as $\phi(= \frac{1}{x})$.

We write:

- $P(S)$ for the space of Borel probability measures on S,
- $P_{\text{fs}}(S) \subset P(S)$ for the subspace of finitely supported measures.

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- The atoms $s_i, 1 \leq i \leq \#\mu$ of μ are identified as the agents,
- The masses $\mu(s_i)$ as their respective current wealth,
- The meeting rates between agents (i.e. atoms) i and j is given by $\phi(x)$ and the metric as

$$\nu_{ij} = \phi(d(s_i, s_j))$$

A Visualization

A simulation of the Metric Coalescent process on $S = [0,1]^2$ with the Euclidean metric, started from finitely supported approximations of the uniform measure:

- Link

Developed by Weijian Han.

Our Result

Goal: Make sense of the MC process for a more general class of measures.

We make the following assumptions on (S, d) and $\phi(x)$:

- (S, d) is locally compact and separable,
- $\lim_{x \downarrow 0} \phi(x) = \infty$.

Main Theorem

Main Theorem [Lan14]

There exists a unique, cadlag, Feller continuous $P(S)$-valued Markov process

$$\mu_t, t \geq 0$$

defined from any initial measure $\mu_0 \in P(S)$ s.t. if μ_0 is compactly supported:

- (**Coming Down from Infinity**) $\mu_t \in P_{\text{fs}}(S)$ for all $t > 0$, almost surely;
- (**Consistency**) For each $t_0 > 0$, the process

$$\mu_t, t \geq t_0$$

is distributed as the MC started at μ_{t_0}.

Proof Idea: Naive Approach

The "naive" proof idea for a generic $\mu \in P(S)$ is to approximate μ with a sequence of finitely supported measures $\mu^i \in P_{\mathrm{fs}}(S)$. Then for $t \geq 0$ define (the random measure) μ_t as the weak limit

$$\mu_t = \lim_i \mu^i_t.$$

Feller continuity in the Main Theorem retroactively shows that this sequence of random measures does converge, however – even ignoring the coupling issues here – this approach isn't so fruitful.

Some progress is made in [Lan14] following this idea using moment methods.

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- de Finetti (i.e. Kingman's Paintbox) arguments show that the partitions give rise to a $P(S)$-valued process $\mu_t, t \geq 0$.
- Analysis begins with the fact that for this new process and any $f : S \to \mathbb{R}$ the evaluation process

$$\int_S f \, d\mu_t, t \geq 0$$

is a martingale.

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- **Time Reversal:** A classical result on Kingman's Coalescent is its duality under a time reversal to a conditioned Yule process. Viewing the MC as a "geometrization" of KC, can something similar be said?

References

Thanks for listening!

For further information on the Compulsive Gambler and Metric Coalescent as well as a complete reference list:

[ALS14] D. Aldous, D. Lanoue, and J. Salez, *The Compulsive Gambler Process*, ArXiv e-prints (2014).

[Lan14] D. Lanoue, *The Metric Coalescent*, ArXiv e-prints (2014).