Observables for $O(n)$ model on the honeycomb lattice. New game.

Alexander Glazman

University of Geneva; PDMI RAS, St. Petersburg

June 26, 2014

joint work with Dmitry Chelkak and Stanislav Smirnov

Outline

- Introduction (known results, motivation, definition of the model)
- O(n) model on a rhombic tiling, critical weights
- Local deformations as the way to get new observables
- Convergence result for the Ising model

Observables as a tool

Observable is a function defined on a lattice. It contains information about particular statistical mechanical model and satisfies some local relations. If you have enough relations and can determine the observable on the boundary then you might hope to obtain convergence results.

2001, Conformal invariance of percolation on the triangular lattice (Smirnov)

2006, Conformal invariance of the Ising model on the square lattice (Smirnov)

2011, Universality of the 2D Ising model (Chelkak and Smirnov)

2011-..., other results in the Ising model (Hongler, Izyurov, Kytola)

2011, Connectivity constant of the honeycomb lattice (Duminil-Copin and Smirnov)

The main tool in all these papers is a specific observable — fermionic observable, spinor, parafermionic observable.

In the case of the Ising model, the limit of the fermionic observable was identified.

Observables as a tool

Observable is a function defined on a lattice. It contains information about particular statistical mechanical model and satisfies some local relations. If you have enough relations and can determine the observable on the boundary then you might hope to obtain convergence results.

2001, Conformal invariance of percolation on the triangular lattice (Smirnov)

2006, Conformal invariance of the Ising model on the square lattice (Smirnov)

2011, Universality of the 2D Ising model (Chelkak and Smirnov)

2011-..., other results in the Ising model (Hongler, Izyurov, Kytola)

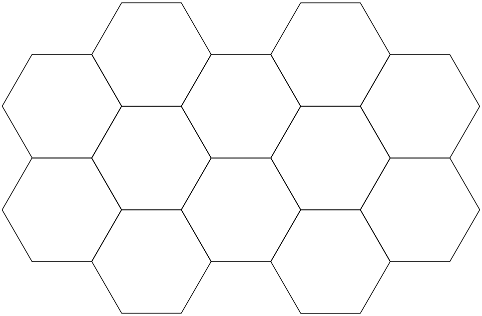
2011, Connectivity constant of the honeycomb lattice (Duminil-Copin and Smirnov)

The main tool in all these papers is a specific observable — fermionic observable, spinor, parafermionic observable.

In the case of the Ising model, the limit of the fermionic observable was identified.

(Loop representation of) $O(n)$ model on the honeycomb lattice

Honeycomb lattice, finite part.

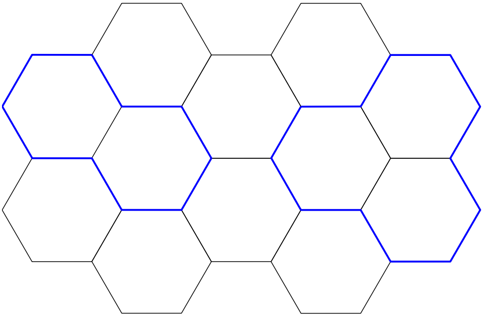


The configuration is a subgraph of this lattice where each vertex has degree 0 or 2. It can be divided into loops. The weight is calculated as follows:

$$\omega(conf) = x^{\#\text{edges}} \cdot n^{\#\text{loops}}$$

(Loop representation of) $O(n)$ model on the honeycomb lattice

Honeycomb lattice, finite part.

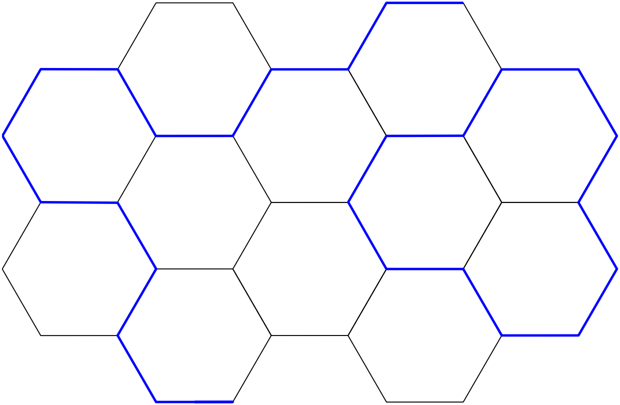


The configuration is a subgraph of this lattice where each vertex has degree 0 or 2. It can be divided into loops. The weight is calculated as follows:

$$\omega(conf) = x^{\#\text{edges}} \cdot n^{\#\text{loops}}$$

$O(n)$ model, boundary conditions

We can allow walks also. We just pick a pair of points on the boundary and say that they have degree 1 in our configuration.

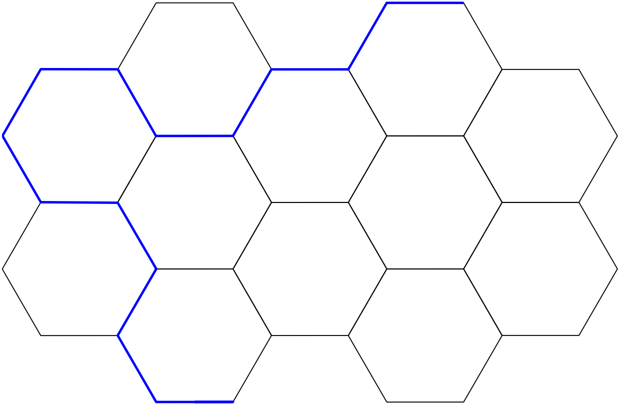


The weight is calculated in the same way:

$$\omega(conf) = x^{\#\text{edges}} \cdot n^{\#\text{loops}}$$

$n = 0$, self-avoiding walk

We pick boundary conditions with two vertices of degree 1 on the boundary and obtain the self-avoiding walk.

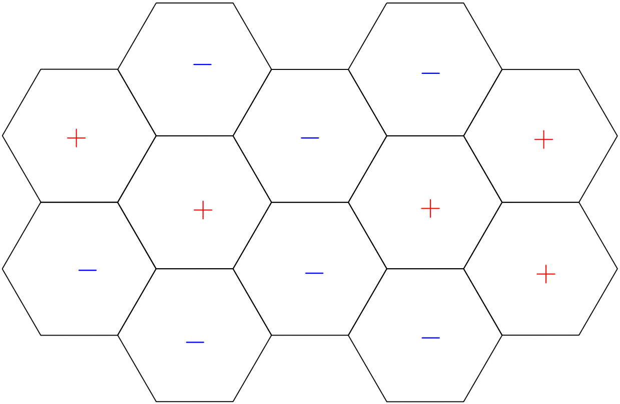


The weight of the configuration:

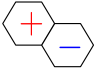
$$\omega(conf) = x^{length}$$

$n = 1$, Ising model

In this case, we do not count loops — Ising model. The configuration can be understood as walls between different adjacent spins.

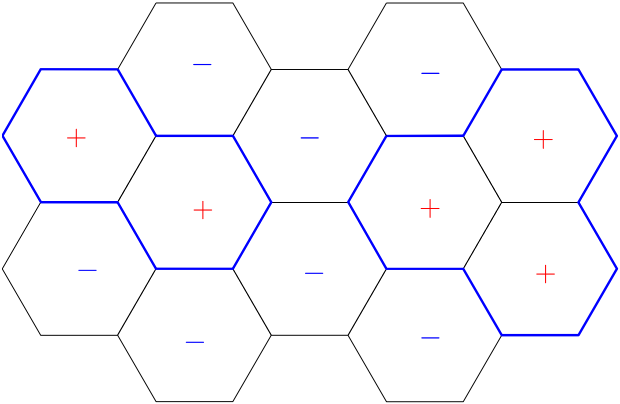


$$\omega(conf) = x^{\#}$$

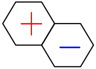


$n = 1$, Ising model

In this case, we do not count loops — Ising model. The configuration can be understood as walls between different adjacent spins.

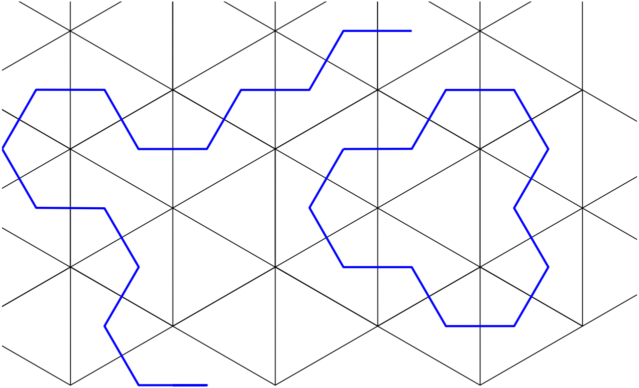


$$\omega(conf) = x^{\#}$$



$O(n)$ model on the dual lattice

Triangular lattice



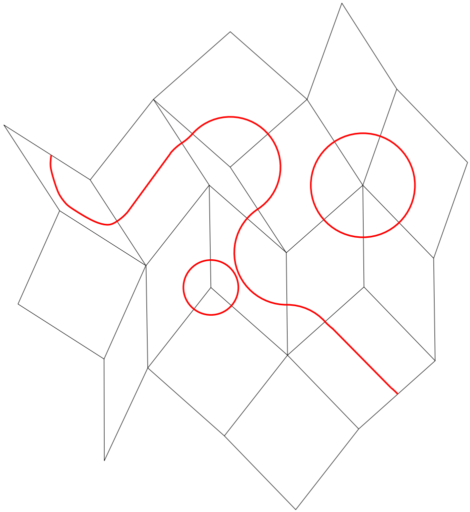
The weight of the configuration:

$$\omega(conf) = x^{length} \cdot n^{loops}$$

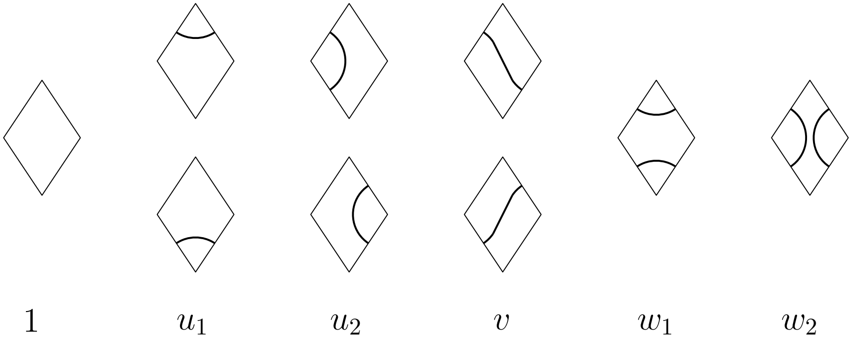
$O(n)$ model on a rhombic tiling

We consider the $O(n)$ model on any rhombic tiling of some domain.
The weight of the configuration:

$$\omega(conf) = \prod_{r\,-\,\text{rhombus}} \omega(r) \cdot n^{loops}$$

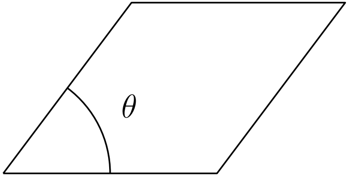


Weight of a rhombus



Critical weights, historical remark

We will consider a specific family of weights parametrized by angle θ of the rhombus. They were discovered as the weights, for which parafermionic observable satisfies half of Cauchy-Riemann equations. These weights satisfy also the Yang-Baxter equation.



First integrable weights were discovered by Nienhuis in 1990 as the solutions to Yang-Baxter equation.

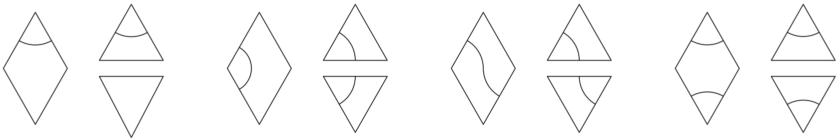
In 2009, Cardy and Ikhlef discovered the same weights as the weights for which the parafermionic observable satisfies some particular equation.

The case $\theta = \frac{\pi}{3}$

For $\theta \in (\frac{\pi}{3}, \frac{2\pi}{3})$, the weights are positive.
For $\theta = \frac{\pi}{3}$ the weights can be factorized, i.e.

$$u_2 = v = w_1 = (x_c)^2, \quad u_1 = x_c = \frac{1}{\sqrt{2+\sqrt{2-n}}} \quad \text{and} \quad w_2 = 0.$$

This is the critical $O(n)$ model on the honeycomb lattice.



$u_1 = x_c = \frac{1}{\sqrt{2+\sqrt{2-n}}}$ $\quad u_2 = x_c^2$ $\quad v = x_c^2$ $\quad w_1 = x_c^2$

Remarks about non-flat case

We can add equilateral triangles. The weight of a triangle is either 1 if it is empty or x_c if it contains an arc of the configuration.

This allows us to consider an infinitesimal local deformation of a triangular lattice. Then we take the derivative of the partition function and get an observable satisfying some local relations.

The important case for us — inserting conical singularities. This means that we allow the sum of the angles around some vertices of the lattice to be equal to $2\pi \pm \varepsilon$.
The infinitesimal deformation of this sort can be interpreted as a dicrete analogue of the stree-energy tensor in CFT.

Remarks about non-flat case

We can add equilateral triangles. The weight of a triangle is either 1 if it is empty or x_c if it contains an arc of the configuration.

This allows us to consider an infinitesimal local deformation of a triangular lattice. Then we take the derivative of the partition function and get an observable satisfying some local relations.

The important case for us — inserting conical singularities. This means that we allow the sum of the angles around some vertices of the lattice to be equal to $2\pi \pm \varepsilon$.
The infinitesimal deformation of this sort can be interpreted as a dicrete analogue of the stree-energy tensor in CFT.

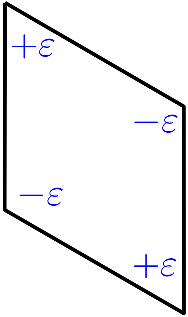
Remarks about non-flat case

We can add equilateral triangles. The weight of a triangle is either 1 if it is empty or x_c if it contains an arc of the configuration.

This allows us to consider an infinitesimal local deformation of a triangular lattice. Then we take the derivative of the partition function and get an observable satisfying some local relations.

The important case for us — inserting conical singularities. This means that we allow the sum of the angles around some vertices of the lattice to be equal to $2\pi \pm \varepsilon$.
The infinitesimal deformation of this sort can be interpreted as a dicrete analogue of the stree-energy tensor in CFT.

4 conical singularities



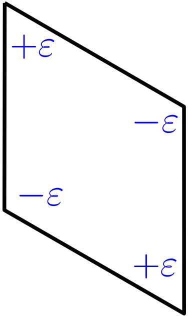
After such a deformation, we obtain another rhombus — with another integrable weights.

Consider the derivative of the partition function. It will be function of a pair of adjacent triangles, i. e. function on edges of the dual hexagonal lattice.

Hence, each configuration contributes:

$\omega(conf) \cdot \{$the logarithmic derivative of its weight in this rhombus at $\frac{\pi}{3}\}$

4 conical singularities



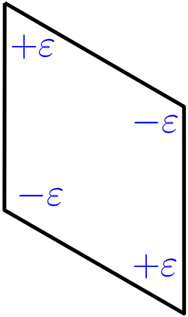
After such a deformation, we obtain another rhombus — with another integrable weights.

Consider the derivative of the partition function. It will be function of a pair of adjacent triangles, i. e. function on edges of the dual hexagonal lattice.

Hence, each configuration contributes:

$\omega(conf)\cdot\{$the logarithmic derivative of its weight in this rhombus at $\frac{\pi}{3}\}$

4 conical singularities



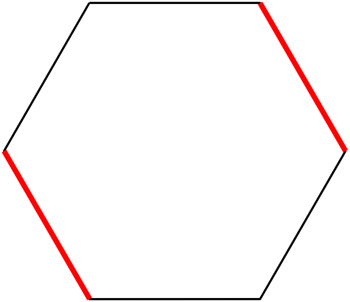
After such a deformation, we obtain another rhombus — with another integrable weights.

Consider the derivative of the partition function. It will be function of a pair of adjacent triangles, i. e. function on edges of the dual hexagonal lattice.

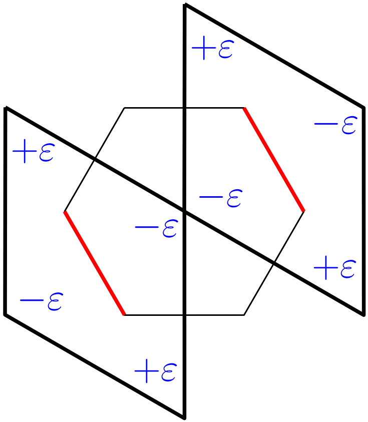
Hence, each configuration contributes:

$\omega(\mathit{conf})\cdot\{$the logarithmic derivative of its weight in this rhombus at $\frac{\pi}{3}\}$

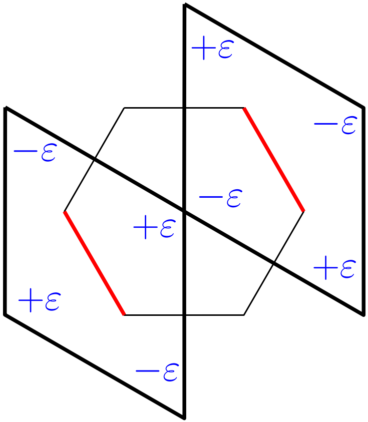
Local relations



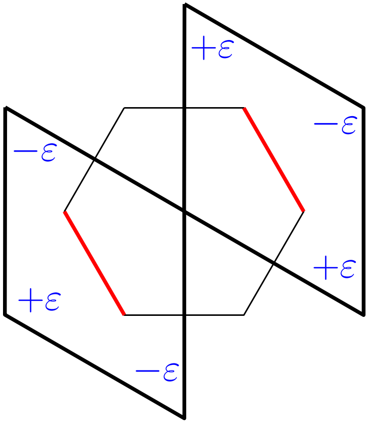
Local relations



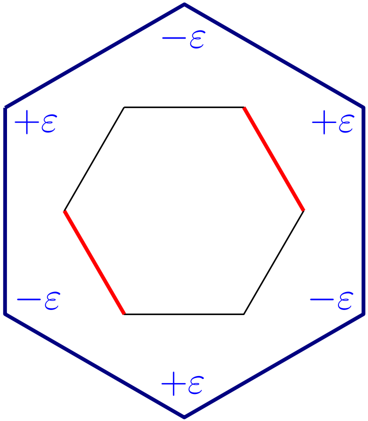
Local relations



Local relations

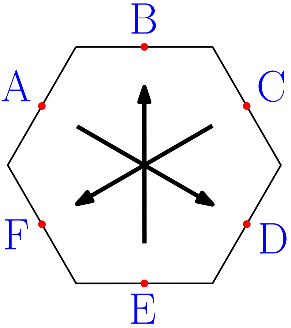


Local relations



Local relations

$$G(D) - G(A) = G(B) - G(E) = G(F) - G(C)$$



Theorem

Let Ω be a bounded planar simply connected domain and $\varphi : \Omega \to \mathbb{H}$ be a conformal map. Then for Dobrushin boundary conditions

$$\lim_{\delta\to 0} G(z)\delta^{-2} = \mathrm{Re}(const \cdot S\varphi(z) + const \cdot \left(\frac{\varphi'(z)}{\varphi(z)}\right)^2),$$

where $S\varphi(z) = \frac{\varphi'''(z)}{\varphi'(z)} - \frac{3}{2}\left(\frac{\varphi''(z)}{\varphi'(z)}\right)^2$ — the Schwarzian derivative of φ.

Thank you for your attention!