Thick points for generalized Gaussian fields under different cut-offs

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- $H = H_0^1(D)$, $\Lambda := (-\Delta_{|D})^{-1} \rightsquigarrow$ massless Gaussian Free Field on $D \subseteq \mathbb{R}^2$.

The white noise representation and Green's functions

White noise representation Let W be a standard complex white noise. Then formally

$$X(x) = \int_{\mathbb{R}^d} \mathrm{e}^{-i\pi(x,\xi)_{\mathbb{R}^d}} \sqrt{\widehat{\Lambda}(\xi)} W(\mathrm{d}\xi).$$

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- $K_0(\|x-y\|)$: Green's function for $(m\mathbb{I}-\Delta)^{\frac{d}{2}}$ on $\mathbb{R}^d$ (Matérn kernel);
- $G_D(x,y)$: Green's function for $-\Delta$ on D with Dirichlet boundary conditions.

The white noise/integral cut-offs

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$$X_\epsilon(x) := \int_{B(0,\epsilon^{-1})} \mathrm{e}^{-i\pi(x,\xi)_{\mathbb{R}^d}} \sqrt{\widehat{\Lambda}(\xi)}\, W(\mathrm{d}\xi).$$

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Allez, Kahane, Rhodes, Vargas...

Sphere averaging

It is possible to make sense of

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$\rightsquigarrow$ the sphere average does not have long-range correlations.

Hausdorff dimension of the thick points

The set of *a*-thick points is

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- *C-Hazra ('14+) : conditions (A)-(D).*

Upper bound

Theorem (C. - Hazra)

If $(X_\epsilon(x))_{\epsilon \geq 0, x \in D}$, *$d \geq 2$, is a centered Gaussian process satisfying*

(A) *For all* $R > 0$ *and for all* $x, y \in D$ *and* $\epsilon, \eta \geq 0$

$$\mathrm{E}\left[(X_\epsilon(x) - X_\eta(y))^2\right] \leq \frac{\|x - y\| + |\eta - \epsilon|}{\eta \wedge \epsilon},$$

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- *for* $a\leq\sqrt{2d}$*,* $\dim_H(T(a,\, D))\leq d-\frac{a^2}{2}$*,*

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- *for* $a \leq \sqrt{2d}$, $\dim_H(T(a,\, D)) \leq d - \frac{a^2}{2}$,
- *for* $a > \sqrt{2d}$ *we have* $T(a,\, D)$ *is empty.*

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Theorem (C. - Hazra)

Let $(X_n(x))_{n\in\mathbb{N},\, x\in[0,1]^d}$ *be a centered Gaussian process with covariance kernel* $\Lambda_n(x, y)$ *which satisfies:*

(C) *for* $x \neq y$, $\Lambda_n(x, y) \leq -\frac{1}{2}\log\|x - y\| + H(x, y)$ *where* $\sup_{x\neq y\in[0,1]^d} H(x, y) < C < \infty$,

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Let $0 < a \leq \sqrt{2d}$ *and consider*

$$T(a) = \left\{x \in [0,1]^d : \lim_{n\to\infty}\frac{X_n}{n} = a\right\}.$$

Then we have $\dim_H(T(a)) \geq d - \frac{a^2}{2}$ *almost surely.*

Sketch of proof

Kahane's strategy: Peyrière's or rooted measures.

- Construct the measures $\exp\left(aX_n(x) - \frac{a^2}{2}\Lambda_n(x,\, x)\right) \mathrm{d}x$.

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- Conclude by finite energy methods.

Towards comparison

Theorem (C. - Hazra)

Let X_ϵ and Y_ϵ be two cut-off families for the same GGF. Call $Z_\epsilon(x) := X_\epsilon(x) - Y_\epsilon(x)$. Suppose there exist constants C, C' such that

i. $\mathrm{E}\left[Z_\epsilon(x)^2\right] \leq C$,

ii. $\mathrm{E}\left[(Z_\epsilon(x) - Z_\epsilon(y))^2\right] \leq C' \frac{\|x-y\|}{\epsilon}$.

Then the thick points of X_ϵ and Y_ϵ have the same Hausdorff dimension almost surely.

Final remarks...

- Almost sure equivalence of cut-offs is not obvious f. eg. Liouville Quantum Gravity measure (Kahane, Rhodes-Vargas in law, Duplantier-Sheffield a. s.).

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... and questions.

- 2-d case: conformal invariance for all (reasonable) cut-offs?
- Higher-dimensional case: describe cut-offs for GGFs f. eg. Bilaplacian in $d = 4$.

Thank you for your attention!