Normal varieties and singularities in algebraic geometry

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1 Introduction

In this report we shall investigate singularities in algebraic geometry, and study certain classes of varieties that do not have "too many" singularities. The most widely studied class will be that of normal varieties, but we shall also consider regular varieties, factorial varieties, and varieties that are regular in codimension one. The report is intended to be readable to a beginning PhD student. We assume that the reader has some familiarity with classical algebraic geometry, commutative algebra, and category theory. On the other hand, no expertise in any of these subjects is required, and advanced notions will be introduced as they are needed.

As far as possible, this report aims to be self-contained. Of course, in a subject as deep as algebraic geometry, it is hardly possible to prove everything from scratch. A list of the theorems from commutative algebra which we shall draw upon is given in §2 Preliminaries. Besides this, we shall sometimes need to rely on results from other areas of mathematics, especially field theory.

Advice to the reader

Since this report was written with a PhD student in mind, I have left a handful of assertions as exercises. However, all the theorems leading up to the main result, that normal varieties are regular in codimension one, are proven in full. Moreover, in some respects, working through the proofs themselves is an exercise. Although I have aimed to be less terse than the books from which I learnt this material, there are still many more details omitted than in a typical undergraduate course, say. Unfortunately, there seems to be no way around this: writing out every detail is both tedious and tends to obscure the underlying ideas that motivate the proofs. If the reader becomes truly stuck while reading this report, they may to wish to refer to [CL21] for commutative algebra: of all the books I have consulted, it contains the most detailed proofs, as well as including material on a number of topics that are not covered in the standard references, such as linear algebra over a commutative ring.

In the interests of transparency, I will record here the sources from which I learnt this material. After having taken courses in abstract and linear algebra from Bath, I learned the rudiments of category theory from the first chapter of [Alu16] in the summer after my second year at University. I was first exposed to algebraic geometry in my third year in a course on algebraic curves. After taking this course, I knew that algebraic geometry was what I wanted to do my final year report on. In preparation for this, I learned commutative algebra over the summer of my third year from a number of sources, mainly from [AM69], but also from [Alu16], [Eis95], and [CL21]. Finally, I learned more advanced algebraic geometry from [Sha13] and [Mil24], before eventually studying the theory of schemes from [Liu06], [GW20], and [Vak24]. Those familiar with these books will recognise the influence that they have had on the presentation here, and the only thing I can take full responsibility for are any errors that may remain.

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2 Preliminaries

We begin by reviewing some foundational concepts that shall be needed in our study of varieties. This will also give us a place to explain the notation and language used throughout the paper. The reader may prefer to skip immediately to §3 Motivation, referring to the following passages only as they are needed.

Conventions

The natural numbers $\mathbb{N}$ are taken to contain 0. We use $\subseteq$ to mean subset and $\subset$ to mean proper subset. If x is an element of a set A, we sometimes say that A *contains* x. On the other hand, if $A \subseteq B$, we say that B *includes* A. Throughout, we fix an algebraically closed field K.

Commutative algebra

We assume that the reader is familiar with the fundamental concepts in commutative algebra, say to the extent covered in the first three chapters of [AM69], along with Noetherian rings and modules. Throughout, we use "ring" as a shorthand for "commutative ring" (with identity). Maps of rings (i.e. ring homomorphisms) are required to preserve the multiplicative identity 1. If A and B are rings, B is a subset of A, and the inclusion $\iota : B \to A$ is a ring map, we say that B is a *subring* of A.

If k is a ring, then by a *k-algebra*, we mean a ring A equipped with a ring homomorphism $\rho : k \to A$. The map ρ induces a scalar multiplication $k \times A \to A$, $(\lambda, a) \mapsto \lambda a := \rho(\lambda)a$ which endows A with the structure of a k-module. A *sub-k-algebra* (or simply *subalgebra*) of A is a subring which is stable under scalar multiplication.

One notable example of algebras comes from localization. If A is a ring and $S \subseteq A$ is multiplicatively closed, then $S^{-1}A$ is an A-algebra under the universal map $\varphi : A \to S^{-1}A$. Note that φ is injective if and only if every element of S is regular (i.e. not a zero divisor). In such a situation, we sometimes suppress the map φ and in this vein regard A as a subring of $S^{-1}A$. In particular, we can embed an integral domain A into its field of fractions $\mathrm{Frac}(A)$.

Recall that a *local ring* is a ring with exactly one maximal ideal. If A is a local ring with maximal ideal $\mathfrak{m} \subset A$, then the *residue field of* A is defined as the quotient $k = A/\mathfrak{m}$. We often say "let $(A, \mathfrak{m})$ be a local ring" as a shorthand for "let A be a local ring with maximal ideal $\mathfrak{m} \subset A$". Similarly, we say "let $(A, \mathfrak{m}, k)$ be a local ring" to mean "let A be a local ring with maximal ideal $\mathfrak{m} \subset A$, and residue field k". If $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ are local rings, then a *local ring homomorphism* $A \to B$ is a ring homomorphism $\phi : A \to B$ such that $\phi(\mathfrak{m}) \subseteq \mathfrak{n}$.

We assume that the reader is familiar with the following elementary properties of local rings:

- A ring A is local if and only if the non-units of A form a maximal ideal.

- If A is a ring and $I \subset A$ is a proper ideal such that every element of $A \setminus I$ is a unit, then (A, I) is a local ring.
- If $(A, \mathfrak{m})$ is a local ring, then $A^\times = 1 + \mathfrak{m}$, where $A^\times$ is the group of units of A.

There are a limited number of results from commutative algebra which we shall use over and over, sometimes without comment:

(1) Every proper ideal of a ring is included in a maximal ideal. In particular, every nonzero ring has a maximal ideal (Theorem 1.3, [AM69]).

(2) Every nonzero ring has a minimal prime ideal. More generally, if $\mathfrak{a} \subseteq \mathfrak{b}$ are ideals in a ring A, and the ideal $\mathfrak{b}$ is prime, then the collection of prime ideals $\mathfrak{p}$ such that $\mathfrak{a} \subseteq \mathfrak{p} \subseteq \mathfrak{b}$ has a minimal element with respect to inclusion (Lemma 2.2.13, [CL21]).

(3) The nilradical of a ring (i.e. the radical of the zero ideal) is the intersection of its prime ideals (Proposition 1.8, [AM69]).

(4) If $I \subseteq A$ is an ideal, then there is an inclusion-preserving correspondence between ideals of A including I, and ideals of A/I.

(5) If $S \subseteq A$ is multiplicatively closed, then there is an inclusion-preserving correspondence between prime ideals of A which are disjoint from S, and prime ideals of $S^{-1}A$ (Proposition 2.27, [CL21]).

(6) If M is a finitely generated A-module, and $\varphi : M \to M$ is an endomorphism, then there is a monic polynomial $p \in A[t]$ such that $p(\varphi)$ is the zero endomorphism of M (Cayley–Hamilton, Theorem 4.3, [Eis95]).

(7) Suppose M is a finitely generated A-module, N is a submodule of M, and J is the Jacobson radical of A (i.e. the intersection of the maximal ideals of A). Then $M = JM + N$ implies $M = N$ (Nakayama's Lemma, Corollary 4.8, [Eis95]).[1]

(8) If A is a Noetherian ring, then so too is the polynomial ring $A[t_1, \ldots, t_n]$ (Hilbert's Basis Theorem, Theorem 1.2, [Eis95]).

(9) If M is a finitely generated module over a Noetherian ring, then M is Noetherian (Corollary III.6.8, [Alu16]).

(10) Suppose A is a nonzero finitely generated algebra over a field k. Then there exists a $d \geq 0$ and a finite injective k-algebra map $k[t_1, \ldots, t_d] \to A$ (Noether Normalization, Proposition 2.1.9, [Liu06]). [We shall discuss the definition of a finite ring homomorphism in §5 Commutative Algebra.]

(11) If A is finitely generated algebra over a field k, then it is Jacobson: that is, every prime ideal of A is the intersection of the maximal ideals including it (Theorem 1.7,

[1] In a number of sources, the name "Nakayama's Lemma" is reserved for the special case of this theorem where $N = 0$. However, the general case follows from this special case: consider the quotient module M/N.

[GW20]).

(12) If k is an algebraically closed field, then every maximal ideal of $k[z_1, \dots, z_n]$ is of the form $\langle z_1 - a_1, \dots, z_n - a_n \rangle$ for some $a_i \in k$ (Weak Nullstellensatz, Corollary 2.1.13, [Liu06]).

A fair number of these results are difficult, in the sense that a student having learned the relevant definitions could not reasonably be expected to supply the proofs themselves. However, in contrast to the theorems from commutative algebra which we shall encounter later, the proofs are reasonably self-contained: there are not swathes of preliminary lemmas that are needed in order to establish these theorems.

Classical algebraic geometry

We assume that the reader has some familiarity with the theory of affine algebraic sets over an algebraically closed field. Everything that we shall need is contained in the first two chapters of [Ful08], though some knowledge of singularities, the tangent space, and projective geometry also helps to motivate the material covered here. A student having taken the algebraic geometry course at Bath should have the requisite knowledge.

If $n \in \mathbb{N}$ and $S \subseteq K[z_1, \dots, z_n]$, then $Z(S)$ denotes the *vanishing locus* of S, i.e. the set of points $p \in K^n$ such that $f(p) = 0$ for all $p \in S$. (In the situation where S is a finite set $\{f_1, \dots, f_n\}$, we usually write $Z(f_1, \dots, f_n)$ in place of $Z(\{f_1, \dots, f_n\})$.) If $X \subseteq K^n$, we say that X is an *affine algebraic set* (or simply an *algebraic set*) if $X = Z(S)$ for some $S \subseteq K[z_1, \dots, z_n]$. If $X \subseteq K^n$, then $I(X)$ denotes the ideal in $K[z_1, \dots, z_n]$ consisting of those polynomials which vanish on X. By *affine n-space* $\mathbb{A}^n(K)$, we mean the point set K^n equipped with the *Zariski topology* – this is the topology whose closed sets are the algebraic sets $X \subseteq K^n$. The assignments $Z(-)$ and $I(-)$ are related by the *Nullstellensatz*, whose various forms are assumed to be known to the reader. If $X \subseteq \mathbb{A}^n(K)$ is an algebraic set, then $K[X]$ denotes the *coordinate ring* of X, i.e. the quotient of $K[z_1, \dots, z_n]$ by $I(X)$. Elements of $K[X]$ define polynomial functions $X \to K$.

Topology

If X is a topological space, then by an *open cover* of X, we mean a family $\{U_i\}_i$ of open sets in X such that $\bigcup_i U_i = X$. If E is a subset of X, then E inherits the subspace topology from X, and so we can also speak of an open cover of E. By a *neighbourhood* of a point $x \in X$, we mean a subset of X which includes an open set containing x. Thus, for us, a neighbourhood of x is not necessarily open.

We now recall a few topological notions that play a prominent role in algebraic geometry. A topological space X is *irreducible* if X cannot be expressed as a finite union of proper closed subsets. (In particular, the empty space is not irreducible, since it equals the empty union.) The following reformulation of the definition is often useful: X is irreducible if and only if X is nonempty and every nonempty open set $U \subseteq X$ is dense in X. If $X \subseteq \mathbb{A}^n(K)$

is an algebraic set, then X inherits the subspace topology from $\mathbb{A}^n(K)$, and we say that X is irreducible if it is irreducible as a topological space.

The following properties of irreducibility can be verified from the definition.[2]

(1) An irreducible space is connected.

(2) Every nonempty open subset of an irreducible space is irreducible.

(3) A nonempty space X is irreducible if and only if every pair of nonempty open subsets of X intersect.

(4) A subset $Y \subseteq X$ of a space X is irreducible if and only if the closure of Y in X is irreducible.

An irreducible subset of a space X which is maximal with respect to inclusion is known as an *irreducible component* of X. By (4), an irreducible component is necessarily closed in X.

Proposition 2.1. *Every irreducible subset of a topological space X is included in an irreducible component of X.*

Proof. This follows from Zorn's Lemma.[3] Suppose $E \subseteq X$ is irreducible. Let Σ be the collection of irreducible subsets of X including E. We order Σ by inclusion. Suppose $\mathcal{C}$ is a totally ordered subset of Σ. If $\mathcal{C}$ is empty, then E is an upper bound of $\mathcal{C}$. If $\mathcal{C}$ is nonempty, we claim that $F = \bigcup_{G \in \mathcal{C}} G$ is irreducible, and hence an upper bound of $\mathcal{C}$.

Indeed, suppose U and V are nonempty and open relative to F. Then, $U \cap G' \neq \varnothing$ and $V \cap G'' \neq \varnothing$ for two sets $G', G'' \in \mathcal{C}$. Since $\mathcal{C}$ is totally ordered, either $G' \subseteq G''$ or $G'' \subseteq G'$; without loss of generality, we may assume the former holds. Then, $U \cap G''$ and $V \cap G''$ are both nonempty and open relative to G'', hence they intersect. Thus, U and V intersect. We conclude that F is irreducible, and so Zorn produces a maximal element of Σ, as desired. □

It follows from Proposition 2.1 that if $x \in X$, then x is contained in an irreducible component of X. We thus derive the following corollary.

Corollary 2.2. *A topological space is the union of its irreducible components.*

An irreducible algebraic set is also known as an *affine variety*. An algebraic set $X \subseteq \mathbb{A}^n(K)$ is an affine variety if and only if $K[X]$ is an integral domain (which in turn holds if and only if $I(X) \subseteq K[z_1, \ldots, z_n]$ is a prime ideal). If $X \subseteq \mathbb{A}^n(K)$ is an affine variety, we can

[2] For further properties of irreducibility and complete proofs, the reader may wish to refer to Section 1.5 of [GW20].

[3] The reader unfamiliar with Zorn's Lemma may wish to refer to [Gow08]. For a proof of Zorn's Lemma by transfinite recursion, see for instance Proposition 1.5.4 of [MO18]. Any proof of Zorn's Lemma must make inescapable use of the axiom of choice, since they are known to be equivalent over ZF set theory. By a theorem of Wilfrid Hodges, the axiom of choice is also equivalent to the assertion that every nonzero ring has a maximal ideal.

form the field of fractions $K(X)$ of the coordinate ring $K[X]$. Elements of $K(X)$ are known as *rational functions* on X, though it is not in general possible to interpret the elements of $K(X)$ as being bona fide functions $X \to K$. We shall return to this in §4 Abstract varieties.

Suppose X is a set, and $Z_0, Z_1, \dots, Z_n$ is a nonempty finite sequence of subsets of X. We say that the sequence $\{Z_i\}_i$ is an *ascending chain of subsets* of X if $Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_n$. The *length* of such a chain is n. Thus, the term "length" refers to the number of inclusions, rather than the number of sets that appear in the sequence $\{Z_i\}_i$. The chain is *strictly ascending* if all of the inclusions $Z_i \subseteq Z_{i+1}$ are proper. We can similarly define descending and strictly descending chains of sets. If X is a topological space, then its *(Krull) dimension* is the supremum of lengths of strictly descending chains of irreducible closed subsets of X; here, the supremum is taken in the ordered set $\mathbb{R} \cup \{-\infty, +\infty\}$ of extended real numbers. In other words, the dimension $\dim(X)$ is defined by

$$\dim(X) = \sup \left\{ n \in \mathbb{N} \;\middle|\; \begin{array}{c} \text{there exists a strictly descending chain} \\ Z_0 \supset Z_1 \supset \cdots \supset Z_n \text{ of subsets of } X \text{ such that} \\ \text{each } Z_i \text{ is an irreducible closed subset of } X \end{array} \right\}.$$

Thus, the dimension of X is $-\infty$ (if and only if $X = \varnothing$), a natural number, or $+\infty$. Note that the dimension of Z equals the supremum of the dimensions of its irreducible components. Although dimension as defined here is a purely topological notion, it is not of much interest in general topology, and we shall only use it in the situation where X is an algebro-geometric object such as a variety or scheme.[4]

We can similarly define the Krull dimension of a ring A: it is the supremum of lengths of strictly ascending chains of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$. If $\mathfrak{p} \subset A$ is a prime ideal, we define its *height* $\mathrm{ht}(\mathfrak{p})$ to the supremum of lengths of strictly ascending chains of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ where $\mathfrak{p}_n \subseteq \mathfrak{p}$. Equivalently, $\mathrm{ht}(\mathfrak{p}) = \dim(A_{\mathfrak{p}})$.

If the above definition is to be taken seriously, it ought to be the case that the Krull dimension of $K[z_1, \dots, z_n]$ is n. Since

$$\langle 0 \rangle \subset \langle z_1 \rangle \subset \langle z_1, z_2 \rangle \subset \cdots \subset \langle z_1, \dots, z_n \rangle\,,$$

we have $\dim(K[z_1, \dots, z_n]) \geq n$. However, the opposite inequality is surprisingly difficult to prove. We shall discuss this further in Chapter 6.

If X is a topological space, and $Z \subseteq X$ is an irreducible closed subset, then we define the *codimension* $\mathrm{codim}(Z, X)$ of Z in X to be the supremum of lengths of strictly descending chains taking the form

$$Z_0 \supset Z_1 \supset \cdots \supset Z_n\,,$$

[4] Similarly, irreducibility is a somewhat odd condition from a purely topological point of view. For example, a Hausdorff space is irreducible if and only if it is a singleton. As a result, $\mathbb{R}^n$ with the usual topology has Krull dimension zero. This indicates that Krull dimension has little to do with the other notions of dimension in topology.

with each Z_i irreducible and closed in X, and $Z_n \supseteq Z$. We can extend this definition to arbitrary closed subsets Z of X by setting $\operatorname{codim}(Z,X) = \inf_W \operatorname{codim}(W,X)$, where W runs over the irreducible components of Z.

It follows from the definition of codimension that $\operatorname{codim}(Z,X)+\dim(Z) \leq \dim(X)$. Perhaps surprisingly, this inequality is strict in general. However, for the topological spaces we are interested in, such pathologies do not occur. Indeed, Theorem 6.8 tells us that if X is an affine variety, and $Y \subseteq X$ is a closed subvariety, then $\dim(Y) + \operatorname{codim}(Y,X) = \dim(X)$. Thus, the two plausible definitions of codimension are equivalent for our purposes.

Category theory

We assume that the reader is familiar with the definition of a category, along with the following notions: morphisms (monomorphisms, epimorphisms, isomorphisms, endomorphisms, and automorphisms), opposite categories, subcategories, initial and final objects, universal properties, products, coproducts, functors, natural transformations, equivalences of categories, projective limits, and inductive limits. All of the relevant definitions can be found in Appendix A of [GW20]. If the reader is not familiar with category theory, then a good place to start is the first chapter of [Alu16]. Other learning resources include [Ber15] and the appendix to [Eis95].

Recall that a *preordered set* is a set I equipped with a reflexive and transitive relation $\leq$. Any preordered set can be viewed as a category. More precisely, if I is a preordered set, then it gives rise to a category $\mathcal{C}$ defined as follows: the objects of $\mathcal{C}$ are elements of I, and given two elements $a, b \in I$, there is exactly one morphism from a to b if $a \leq b$, and no morphisms between a and b otherwise.[5] There is only one way to define the composition so as to give $\mathcal{C}$ the structure of a category. We shall generally use the same letter to denote a preordered set and the corresponding category.

[5]If $a \leq b$, then what the morphism from a to b *is*, set-theoretically, is unimportant, though for the sake of definiteness we may set $\operatorname{Hom}_{\mathcal{C}}(a,b) = \{(a,b)\}$.

3 Motivation

Suppose $X \subseteq \mathbb{A}^n(K)$ is an affine variety. Recall that if $p = (a_1, \dots, a_n) \in X$, then the *tangent space* T_pX is the affine subspace of K^n defined by[6]

$$T_pX = \bigcap_{f \in I(X)} Z\left(\left.\frac{\partial f}{\partial z_1}\right|_p (z_1 - a_1) + \cdots + \left.\frac{\partial f}{\partial z_n}\right|_p (z_n - a_n) \right).$$

We say that X is *regular* (or *non-singular*) at $p \in X$ if $\dim(X) = \dim_K(T_pX)$. Otherwise, we say that X is *singular* at p. The set of regular points of X is denoted by X_{reg}, and the set of singular points by X_{sing}. If X is regular at all of its points, then we say that X itself is regular. Otherwise, we say that X is singular.

The use of the word "singular" suggests that singular points are rare in some sense, and there is indeed a precise way of stating this.

Theorem 3.1. *If $X \subseteq \mathbb{A}^n(K)$ is an affine variety, then X_{reg} is open and dense (relative to X). Equivalently, X_{sing} is closed and has empty interior.*

Since X is assumed irreducible, proving this theorem amounts to showing that (i) X_{reg} is nonempty, and (ii) X_{sing} is closed. Neither of these assertions are easy to prove, at least with the definitions of "regular" and "dimension" that we have chosen. Later, we shall discover a characterisation of regularity (Theorem 7.12) that makes Theorem 3.1 much easier to prove.

In spite of Theorem 3.1, there is another sense in which the singular locus of an affine variety can be disturbingly large. The classic example is given by the nodal cubic curve.

Example 3.2. Consider the affine plane $\mathbb{A}^2(K)$ with coordinates x, y. If $f \in K[x, y]$ is an irreducible polynomial, then its vanishing locus $X = Z(f)$ is a curve in the plane. Since $I(X) = \langle f \rangle$, the tangent space of X can be described very simply: indeed, if $p = (a, b) \in \mathbb{A}^2(K)$, then

$$T_pX = Z\left(\left.\frac{\partial f}{\partial x}\right|_p (x - a) + \left.\frac{\partial f}{\partial y}\right|_p (y - b) \right).$$

If the partial derivatives of f do not both vanish at p, then T_pX is a line passing through the point (a, b). However, in the degenerate case where both partials vanish, $T_pX = \mathbb{A}^2(K)$. Let us examine the case where $f = y^2 - x^2(x + 1)$. At the point $p = (0, 0)$, the variety X is singular, whereas it is regular at all other points. It follows that $\operatorname{codim}(X_{\text{sing}}, X) = 1$.[7]

[6]The terms "affine" and "affine subspace" unfortunately have multiple inequivalent definitions in mathematics. Here, when we say T_pX is an affine subspace, we simply mean that T_pX is a translation of a linear subspace of K^n, i.e. $T_pX = p + V$ for some linear subspace $V \subseteq K^n$. It thus makes sense to speak of the dimension $\dim_K(T_pX)$: it is the vector space dimension of V.

[7]We are tacitly using the fact that $\mathbb{A}^2(K)$ has dimension 2, X has dimension one, and $\operatorname{codim}(X_{\text{sing}}, X) = \dim(X) - \dim(X_{\text{sing}})$. Although these assertions are geometrically reasonable, we shall have to wait until Chapter 6 to prove them rigorously.

The upshot of this example is that although the singular locus is always small in the topological sense, its dimension can be almost as large as the variety itself. Affine varieties X which do not this kind of pathology have a special name: we say that X is *regular in codimension one* if $\operatorname{codim}(X_{\text{sing}}, X) \geq 2$. The aim of this report is to study certain classes of varieties that are regular in codimension one, and examine their properties.

Perhaps the most important class of affine varieties which are regular in codimension one are normal varieties. The definition of a normal variety makes it far from obvious that they have this property, however. In fact, on the surface, it is not clear what the geometric content of the definition of a normal variety is at all. We give the definition here, but its significance will only gradually become apparent.

Definition 3.3. To say that an affine variety $X \subseteq \mathbb{A}^n(K)$ is *normal* means: for every rational function $f \in K(X)$ and $n \in \mathbb{N}$, if there exist polynomial functions $g_0, g_1, \ldots, g_{n-1} \in K[X]$ such that $g_0 + g_1 f + \cdots + g_{n-1} f^{n-1} + f^n = 0$, then $f \in K[X]$.

The definition of normality can be recast in the language of commutative algebra. To do this, we introduce the following notions. Suppose $\rho : A \to B$ is a ring homomorphism giving B the structure of an A-algebra. (Very often we shall be interested in the special case where A is a subring of B, and ρ is the inclusion.) If $f \in A[t]$ and $b \in B$, then we define $f(b)$ in the natural way: writing $f = a_0 + a_1 t + \cdots + a_n t^n$, we set $f(b) = \rho(a_0) + \rho(a_1) b + \cdots + \rho(a_n) b^n \in B$. We say that $x \in B$ is *integral* over A if x is a root of a monic polynomial $f \in A[t]$. The set of elements of B which are integral over A is known as the *integral closure* of A in B.

The notions of integral elements and integral closure may be viewed as the ring-theoretic counterparts of algebraic elements and algebraic closure. Indeed, given a field extension $L \to L'$, an element $\alpha \in L'$ is algebraic over L if and only if α is integral over L. The set of elements in L' which are algebraic over L is sometimes known as the *algebraic closure* of L in L'; this, however, should not be confused with the "absolute" notion of algebraic closure, i.e. an algebraic extension which is algebraically closed.

If B is an A-algebra with structural map $\rho : A \to B$, then $\rho(a)$ is clearly integral over A for all $a \in A$. If every element of B which is integral over A has this form, we say that A is *integrally closed* in B; that is, A is integrally closed in B if its integral closure equals $\rho(A)$. Given an integral domain A, the field of fractions $\operatorname{Frac}(A)$ carries an A-algebra structure via the map $a \mapsto a/1$, and we say that A is a *normal domain* if A is integrally closed in $\operatorname{Frac}(A)$. We can finally restate the definition of a normal variety as follows: an affine variety $X \subseteq \mathbb{A}^n(K)$ is a normal variety if and only if its coordinate ring $K[X]$ is a normal domain.

The notions of normality and integral closure make an unexpected appearance in school algebra, as can be seen from the following proposition.

Proposition 3.4 (Rational Root Test)**.** *Suppose $f = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$ is a nonconstant polynomial of degree n with integer coefficients a_i. If $r = p/q$ is a rational root of f, where p and q are coprime integers, then p divides a_0 and q divides a_n.*

Thus, if a rational number r is a root of a monic polynomial with integer coefficients, then r is in fact an integer. That is, $\mathbb{Z}$ is a normal domain.

Proof. By substituting $t = p/q$ into f and clearing denominators, it follows that

$$a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0\,;$$

upon subtracting $a_0 q^n$ from both sides, we see that p divides $a_0 q^n$. But p is coprime to q and therefore to q^n. Hence, p divides a_0. A similar argument shows that q divides a_n. □

Let us examine this theorem and proof a little more carefully. If we try to replace $\mathbb{Z}$ with an arbitrary integral domain A, what might go wrong? The only nontrivial fact used about $\mathbb{Z}$ in the proof is Euclid's lemma: if a divides bc and a is coprime to b, then a divides c. Euclid's lemma does not hold for arbitrary integral domains A (where $a, b \in A$ are said to be *coprime* if 1 is a greatest common divisor of a and b). One counterexample comes from algebraic number theory: if $A = \mathbb{Z}[\sqrt{-5}]$ and $a = 2$, $b = 1 + \sqrt{-5}$, $c = 1 - \sqrt{-5}$, then a divides bc and a is coprime to b, but a does not divide c.

On the other hand, Euclid's lemma *does* hold for factorial domains (also known as unique factorisation domains, or UFDs).[8] Thus, by imitating the proof of the rational root test, we arrive at the following proposition.

Proposition 3.5. *Suppose $f = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$ is a nonconstant polynomial of degree n with coefficients a_i in a factorial domain A. If $r = p/q$ is a root of f, where p and q are coprime elements of A, then p divides a_0 and q divides a_n.*

If A is factorial, then any fraction $r = p/q \in \mathrm{Frac}(A)$ can be written in lowest terms, simply by cancelling by a greatest common divisor of p and q. By combining this fact with Proposition 3.5, we derive the following corollary.

Corollary 3.6. *Every factorial domain is a normal domain.*

An affine variety whose coordinate ring is factorial is sometimes called a *factorial* variety. The geometric content of Corollary 3.6 is that factorial varieties are normal. We shall not study factorial varieties in much detail, but they do form an important class of examples of normal varieties. For instance, without having developed any theory whatsoever, we already know that $\mathbb{A}^n(K)$ is a normal variety (since its coordinate ring $K[z_1, \ldots, z_n]$ is factorial).

[8] The proof is an induction on the number of irreducible factors of c which the reader may wish to try for themselves; otherwise, see Proposition 2.5.17 of [CL21].

4 Abstract varieties

In this chapter we begin to develop the machinery that will be used to discuss varieties, singularities, and dimension theory. Most of the technical tools introduced here are borrowed from Grothendieck's theory of schemes, but we shall not discuss schemes in any detail. By remaining in the familiar context of varieties, it is hoped that the reader can more readily develop an intuition for the geometric ideas at play.

In classical algebraic geometry, the principal objects of study are affine and projective varieties. At the most elementary level, an affine variety X is defined as an irreducible Zariski-closed subset of $\mathbb{A}^n(K)$. This definition is in many ways too inflexible, and we shall spend a little time developing the formalism needed to give an "abstract" definition of an affine variety. The most immediate problem with the elementary definition is that it requires X to literally be an irreducible closed subset of $\mathbb{A}^n(K)$, rather than merely being "isomorphic to" such a subset. Affine varieties arising in nature are often not explicitly embedded in $\mathbb{A}^n(K)$. For example, if $X \subseteq \mathbb{A}^n(K)$ is an affine variety in the elementary sense, and $f \in K[z_1, \dots, z_n]$ is a polynomial which does not vanish on the whole of X, then it is both natural and conventional to "view" the open subset $U_f = \{p \in X \mid f(p) \neq 0\} \subseteq X$ as an affine variety. But without an abstract definition of an affine variety, it is difficult to put this "viewing" on rigorous footing.

Example 4.1. Suppose X and f are as above. Write X as the zero locus of $\{f_\alpha\}_\alpha$. Let $\Sigma \subseteq \mathbb{A}^{n+1}(K)$ be the algebraic set cut out by the f_α together with $g = z_{n+1}f - 1$ (where f and the f_α are now regarded as polynomials in $K[z_1, \dots, z_n, z_{n+1}]$). The map $\varphi : (p_1, \dots, p_n) \mapsto (p_1, \dots, p_n, 1/f(p_1, \dots, p_n))$ is a homeomorphism of U_f onto Σ, with inverse $\psi : (q_1, \dots, q_n, q_{n+1}) \mapsto (q_1, \dots, q_n)$. Since U_f is irreducible (being a nonempty open subset of an irreducible space), so too is Σ. Later, we shall define what it means for a

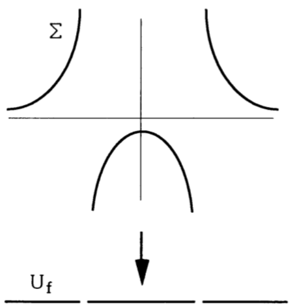


Figure 1: The case where $X = \mathbb{A}^1(K)$ and $f = x^2 - 1$.

rational function $f \in K(X)$ to be regular on an open subset $U \subseteq X$. Not only are Σ and U_f homeomorphic, they have isomorphic rings of regular functions. It thus seems entirely reasonable to say that U_f "is" an affine variety. However, in general, U_f is not closed in $\mathbb{A}^n(K)$, and hence it does not satisfy the requirements of the elementary definition.

In light of this example, our provisional definition of an abstract affine variety is that it is a topological space, together with a collection of functions defined on X, which is isomorphic to an elementary affine variety. To make this precise, we need to axiomatise the notion of a space with functions. It is for precisely this purpose that the concept of a ringed space was introduced.

Ringed and locally ringed spaces

In the following section, we assume that the reader is familiar with the definition and basic properties of presheaves and sheaves (of rings). If this is not the case, then a self-contained introduction is given in the appendix.

A *ringed space* is a pair consisting of a topological space Y and a sheaf of rings on Y. If X is a ringed space, then its underlying topological space is sometimes written $|X|$, and its sheaf of rings on $|X|$ is denoted $\mathcal{O}_X$, so that $X = (|X|, \mathcal{O}_X)$. In practice, however, one often uses the same letter to denote both the ringed space and the underlying space.

Suppose $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are ringed spaces. A *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ consisting of a continuous map $f : X \to Y$ and a morphism of sheaves $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$. Sometimes the pair $(f, f^\sharp)$ is denoted simply f, with the morphism $f^\sharp$ left implicit (but fixed).

Example 4.2. Suppose X and Y are topological spaces, and $\mathcal{O}_X$ and $\mathcal{O}_Y$ are the sheaves of continuous functions on X and Y respectively. Given a continuous map $f : X \to Y$, define $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ by $f^\sharp_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U)),\ g \mapsto g \circ f$.[9] The morphism of sheaves $f^\sharp$ "pulls back" functions defined on Y to functions defined on X. The pair $(f, f^\sharp)$ is the prototypical example of a morphism of ringed spaces.

A ringed space $(X, \mathcal{O}_X)$ is said to be *locally ringed* if for all $x \in X$, the stalk $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ at x is a local ring. If $(X, \mathcal{O}_X)$ is a locally ringed space, and $x \in X$, then the maximal ideal of $\mathcal{O}_{X,x}$ is denoted by $\mathfrak{m}_{X,x}$ or simply $\mathfrak{m}_x$. The residue field of $\mathcal{O}_{X,x}$ is denoted $\kappa_X(x)$ or simply $\kappa(x)$.

Suppose $\mathcal{O}$ is a sheaf on X, $f : X \to Y$ is a continuous map, and $x \in X$. There is a canonical map $(f_*\mathcal{O})_{f(x)} \to \mathcal{O}_x$ which can be described as follows. Given an element $\xi \in (f_*\mathcal{O})_{f(x)}$, let (s, U) be a representative of ξ, where $U \subseteq Y$ is an open neighbourhood of $f(x)$ and $s \in (f_*\mathcal{O})(U)$. Then, by definition $s \in \mathcal{O}(f^{-1}(U))$, so $(s, f^{-1}(U))$ represents a germ of $\mathcal{O}$ at x. We send ξ to this germ.

Exercise 4.3. Verify that the above definition is independent of the choice of representative of ξ, so that we have indeed defined a map $(f_*\mathcal{O})_{f(x)} \to \mathcal{O}_x$.

Suppose $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed spaces, and $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces. Then, $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves on Y, so for all $y \in Y$, there is an induced map on stalks $f^\sharp_y : \mathcal{O}_{Y,y} \to (f_*\mathcal{O}_X)_y$. For all $x \in X$, there is

[9] By $g \circ f$ we really mean $g \circ h$, where h is the restriction of f to a map $f^{-1}(U) \to U$.

a canonical map $(f_*\mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$, as described in the previous paragraph. We say that $(f, f^\sharp)$ is a *morphism of locally ringed spaces* if the composite

$$\mathcal{O}_{Y,f(x)} \xrightarrow{f^\sharp_{f(x)}} (f_*\mathcal{O}_X)_{f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is a local ring homomorphism. The map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is denoted by $f^\sharp_x$.

Example 4.4. Let X be a topological space and $\mathcal{O}_X$ the sheaf of continuous functions on X. The ringed space $(X, \mathcal{O}_X)$ is locally ringed. Indeed, fix $p \in X$ and consider the ring $A = \mathcal{O}_{X,p}$ of germs of continuous functions at p. Let $\mathfrak{m} = \{\xi \in \mathcal{O}_{X,p} \mid \xi(p) = 0\}$. We claim that $\mathfrak{m}$ is the unique maximal ideal of A. Evidently, $\mathfrak{m}$ is a proper ideal of A, so it suffices to show that every element of $A \setminus \mathfrak{m}$ is a unit. Suppose $\xi \in A \setminus \mathfrak{m}$ is represented by (f, U), where U is an open neighbourhood of p and $f : U \to \mathbb{R}$ is a continuous function. Since $f(p) \neq 0$ and f is continuous, there is an open neighbourhood $V \subseteq U$ of p on which f does not vanish. The germ represented by the function $1/f : V \to \mathbb{R}$ is an inverse to ξ.

Exercise 4.5. Let us keep the notation of the previous example. Show that the map $\mathcal{O}_{X,p} \to \mathbb{R},\ \xi \mapsto \xi(p)$ induces an isomorphism of $\kappa(p)$ onto $\mathbb{R}$.

Exercise 4.6. Show that the morphism of ringed spaces given in Example 4.2 is a morphism of locally ringed spaces.

Remark 4.7. A morphism of ringed spaces between two locally ringed spaces need not be a morphism of locally ringed spaces. The simplest example is given by taking X to be a point. Then defining a sheaf on X amounts to specifying what the ring $\mathcal{F}(X)$ is, and defining a locally ringed space structure on X amounts to specifying a local ring. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X, and $(X, \mathcal{F})$ and $(X, \mathcal{G})$ are locally ringed spaces, then a morphism of ringed spaces between them "is" a ring homomorphism, whereas a morphism of locally ringed spaces "is" a local ring homomorphism.

Suppose $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(g, g^\sharp) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ are morphisms of ringed spaces. The composition $(g, g^\sharp) \circ (f, f^\sharp)$ is defined as the morphism $(h, h^\sharp) : (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$ given by $h = g \circ f$ and $h^\sharp = g_* f^\sharp \circ g^\sharp$. If $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, and $(Z, \mathcal{O}_Z)$ are locally ringed, and $(f, f^\sharp)$ and $(g, g^\sharp)$ morphisms of locally ringed spaces, then their composition is defined in the same way.[10]

We have now defined the objects, morphisms, and composition in both the category of ringed spaces, and the category of locally ringed spaces. (Verifying that the category of ringed spaces is indeed a category is routine, but rather tedious.) The category of locally ringed spaces is a subcategory of the category of ringed spaces, but not a full subcategory (Remark 4.7). Isomorphisms of [locally] ringed spaces are defined as they are defined in any category: they are the invertible morphisms. If $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, then it is an isomorphism if and only if f is a homeomorphism and $f^\sharp$ is an

[10] If this definition of composition feels unmotivated, then the reader would be well advised to return to the example of locally ringed spaces of continuous functions.

isomorphism of sheaves (another tedious verification). If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed spaces, then a morphism of locally ringed spaces is an isomorphism if and only if it is an isomorphism when viewed as a morphism of ringed spaces. This follows from the fact that an isomorphism of rings between two local rings is an isomorphism of local rings.

Remark 4.8. Learning sheaf theory has a high "start-up" cost. There are many abstract notions that one has to get a handle on, and many of the proofs involve long and tedious verifications that do not provide much insight. However, once the language *has* been established, it becomes very useful in discussing different geometric objects. For example, a *smooth manifold of dimension n* can be defined as a locally ringed space $(M, \mathcal{O}_M)$ such that M is Hausdorff and second countable, and for every point $p \in M$ there is an open neighbourhood U of p such that $(U, \mathcal{O}_M|_U)$ is isomorphic to the locally ringed space of smooth functions on $\mathbb{R}^n$. Moreover, a smooth map between smooth manifolds is nothing more than a morphism of locally ringed spaces.

Classical varieties

Despite introducing all of the formalism above, we have not as yet explained how to view an affine variety as being a locally ringed space. Let us correct this immediately.

If $X \subseteq \mathbb{A}^n(K)$ is an irreducible Zariski-closed set, and $p \in X$, we let $m_p \subset K[X]$ be the maximal ideal consisting of those $f \in K[X]$ such that $f(p) = 0$. (This notation should not be confused with $\mathfrak{m}_p$, as defined above.) Since $K[X]$ is an integral domain, the localization $K[X]_{m_p}$ may be realised as a subring of $K(X)$, namely as the collection of those $f \in K(X)$ for which there exists $g, h \in K[X]$ such that $f = g/h$ and $h(p) \neq 0$. If a rational function $f \in K(X)$ belongs to $K[X]_{m_p}$, we say that f is *regular at p*. If $f \in K(X)$ is regular at every point of a nonempty open set $U \subseteq X$, we say that X is *regular on U*. We define a sheaf of rings on X by letting $\mathcal{O}_X(U)$ be the collection of rational functions which are regular on U, so that

$$\mathcal{O}_X(U) = \bigcap_{p \in U} k[X]_{m_p} \subseteq K(X)\,,$$

for every nonempty open set $U \subseteq X$, and $\mathcal{O}_X(\varnothing) = 0$. Note that if $V \subseteq U$ are nonempty open sets, then $\mathcal{O}_X(U) \subseteq \mathcal{O}_X(V)$; the restriction maps $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ are defined as the inclusions. The ringed space $(X, \mathcal{O}_X)$ is locally ringed, but we defer the proof until the end of the section (Lemma 4.11).

There are a few points to be made about these definitions. First, if $f \in K(X)$ is regular at $p \in X$, then it makes sense to speak of the *value* of f at p: let g and h be any two elements of $K[X]$ such that $f = g/h$ and $h(p) \neq 0$; we set $f(p) = g(p)/h(p)$; this definition is independent of the choice of g and h. Second, if $f \in K(X)$ is regular on an open set $U \subseteq X$, then it is not necessarily the case that there exist $g, h \in k[X]$ such that $f = g/h$ and h does not vanish on U. A counter-example is given in [Mum99]: let x, y, z, w be the coordinates on $\mathbb{A}^4(K)$, let $X = Z(xw - yz)$, and let $U \subseteq X$ be the set of points $p = (a, b, c, d) \in X$ such that $c = 0$ or $d = 0$. Then, the function $x/y \in k(X)$ is regular on

U, but does not admit a representation f/g, with $f, g \in k[X]$ and g nowhere vanishing on U. We omit the proof. Third, if $f \in K(X)$ is regular on the whole of X, then $f \in K[X]$. We shall prove this shortly (Lemma 4.10).

Suppose X is an irreducible algebraic set, and $U \subseteq X$ is an open subset. We have seen in the previous paragraph that a regular function $f \in K(X)$ on U defines a function $U \to K$, so that we have a map of rings $\varphi : \mathcal{O}_X(U) \to K^U$.

Lemma 4.9. *Let us keep the notation above. The map φ is injective.*

Proof. This is intuitively obvious but a little slippery to prove rigorously. We may assume that U is nonempty. Suppose $f \in \mathcal{O}_X(U)$ and $\varphi(f) = 0$, i.e. f defines the zero function. Let $p \in U$. Since f is regular at p, there exist $g, h \in K[X]$ such that $f = g/h$ and $h(p) \neq 0$. Since f vanishes on the open set $U_h = \{p \in X \mid h(p) \neq 0\}$, it must be the case that g vanishes on U_h. Hence the closed set $\{p \in X \mid g(p) = 0\}$ includes U_h. But since X is irreducible, U_h is dense, hence g is identically zero on X. □

By virtue of this lemma, there is no real harm in identifying an $f \in \mathcal{O}_X(U)$ with the function $U \to K$ that it defines. The restriction maps $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ then correspond to literal restriction of functions.

Suppose $\mathfrak{p} \subset K[z_1, \ldots, z_n]$ is a prime ideal, and $X = Z(\mathfrak{p})$ is the corresponding irreducible algebraic set. Let $\pi : K[z_1, \ldots, z_n] \to K[z_1, \ldots, z_n]/\mathfrak{p}$ be the quotient map. We define a map Z_X by

$$Z_X(J) := \{p \in X \mid f(p) = 0 \text{ for all } f \in J\} = Z(\pi^{-1}(J)) \cap X$$

for every ideal $J \subseteq K[X]$. In the other direction, we define I_X by

$$I_X(Y) := \{f \in K[X] \mid f(p) = 0 \text{ for all } p \in Y\} = \pi(I(Y))$$

for every closed subset $Y \subseteq X$. The maps Z_X and I_X establish a correspondence

$$\{\text{ideals of } K[X]\} \longleftrightarrow \{\text{closed subsets of } X\}$$

This correspondence generalises that between the ideals of $K[z_1, \ldots, z_n]$ and algebraic sets in $\mathbb{A}^n(K)$. It is not hard to see that an analogue of the *Nullstellensatz* still holds: $I_X(Z_X(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ for every ideal $\mathfrak{a} \subseteq K[X]$.

For the next lemma, we recall that if A is a ring and $f \in A$, then A_f denotes the localization of A at the multiplicative set $S = \{1, f, f^2, \ldots\}$.

Lemma 4.10. *Suppose $X \subseteq \mathbb{A}^n(K)$ is an irreducible algebraic set, and $f \in K[X]$. Then,*

$$\mathcal{O}_X(U_f) = K[X]_f$$

as subsets of $K(X)$. In particular, by taking $f = 1$, we see that $\mathcal{O}_X(X) = K[X]$.

Proof. If $p \in U_f$ then the multiplicative set $\{1, f, f^2, \dots\}$ is a subset of $K[X] \setminus m_p$, so that $K[X]_f \subseteq K[X]_{m_p}$. Hence $K[X]_f \subseteq \mathcal{O}_X(U_f)$. Conversely, suppose $g \in \mathcal{O}_X(U_f)$. Set $\mathfrak{a} = \{h \in K[X] \mid hg \in K[X]\}$. We must show that $f \in \sqrt{\mathfrak{a}}$. By the *Nullstellensatz* (see above), $\sqrt{\mathfrak{a}} = I_X(Z_X(\mathfrak{a}))$. It thus suffices to show that f vanishes on the set

$$Z_X(\mathfrak{a}) = \{p \in X \mid h(p) = 0 \text{ for all } h \in \mathfrak{a}\}.$$

Suppose $p \in X$ and $f(p) \neq 0$. Then, g is regular at p, so that we can write $g = h/k$ with $h, k \in K[X]$ and $k(p) \neq 0$. Then $k \in \mathfrak{a}$. But $k(p) \neq 0$, so $p \notin Z_X(\mathfrak{a})$. Thus $p \in Z_X(\mathfrak{a}) \Rightarrow f(p) = 0$ for all $p \in X$, completing the proof. □

Lemma 4.11. *Suppose $X \subseteq \mathbb{A}^n(K)$ is an irreducible algebraic set, and $p \in X$. There is a canonical isomorphism $K[X]_{m_p} \cong \mathcal{O}_{X,p}$.*

Proof. If $f \in K[X]_{m_p}$, then $f = g/h$ for some $g, h \in K[X]$ with $h(p) \neq 0$. There is an open neighbourhood U of p on which h does not vanish. Send f to the germ of (f, U) at p. It is easy to see that this does not depend on the choice of U. In the other direction, if $\xi \in \mathcal{O}_{X,p}$ is a germ represented by (f, U), send ξ to the function f. These maps are mutually inverse. □

It follows that if $X \subseteq \mathbb{A}^n(K)$ is an irreducible algebraic set, then X, together with the sheaf of rings $\mathcal{O}_X$ described above, is a locally ringed space. It thus seems reasonable to define a classical affine variety to be a locally ringed space which is isomorphic to a locally ringed space of this form. This definition is morally correct but suffers from a technical deficiency which we shall address in the next section.

Scheme-theoretic varieties

If A is a K-algebra, then following [Eis95], we say that A is an *affine ring* if A is finitely generated; an affine ring is said to be an *affine domain* if it is an integral domain.[11]

In the previous section, we started with an irreducible algebraic set $X \subseteq \mathbb{A}^n(K)$, and constructed a sheaf of rings on X. However, we could just as well have started with an affine domain A, and constructed a sheaf of rings on the maximal spectrum $X = \operatorname{mSpec} A$ of A, i.e. the collection of maximal ideals of A. The set X has a canonical topological structure given by letting $Z_X(\mathfrak{a}) := \{\mathfrak{m} \in \operatorname{mSpec}(A) \mid \mathfrak{m} \supseteq \mathfrak{a}\}$ be the closed sets of X, where $\mathfrak{a}$ runs over the ideals of A. We define a sheaf on X by setting

$$\mathcal{O}_X(U) = \bigcap_{\mathfrak{m} \in U} A_{\mathfrak{m}} \subseteq \operatorname{Frac}(A)$$

[11] We are assuming that the reader is familiar with the definition of a finitely generated K-algebra. If this is not the case, the reader should consult §5 Commutative algebra, Finite versus finite type algebras.

for nonempty open sets $U \subseteq X$, and $\mathcal{O}_X(\varnothing) = 0$. By a similar argument to that of Lemma 4.11, there is a canonical isomorphism $\mathcal{O}_{X,\mathfrak{p}} \cong A_{\mathfrak{p}}$ for every point $\mathfrak{p} \in X$. Thus $(X, \mathcal{O}_X)$ is a locally ringed space.

To see the connection between this and the previous construction, note that if A is the quotient of $K[z_1, \dots, z_n]$ by a prime ideal $I \subset K[z_1, \dots, z_n]$, then the maximal ideals of A are in correspondence with the maximal ideals of $K[z_1, \dots, z_n]$ including I, and by the *Nullstellensatz*, these are in correspondence with the points in $Z(I) \subseteq \mathbb{A}^n(K)$. Hence one can show that the locally ringed space $(X, \mathcal{O}_X)$ is isomorphic to $Z(I)$ (where $Z(I)$ is equipped with the usual sheaf of rings). Every affine domain can be realised as a quotient of $K[z_1, \dots, z_n]$ for some $n \geq 0$, but the above construction is more "intrinsic" in that it does not require us to choose a set of generators of A.

However, one of the greatest insights of scheme theory is that often, the geometric space that we wish to associate to an affine domain A is not its maximal spectrum, but rather its prime spectrum $\operatorname{Spec} A$, i.e. the collection of prime ideals of A. The set $X = \operatorname{Spec} A$ has a canonical topological structure given by letting $V(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supseteq \mathfrak{a}\}$ be the closed sets, where $\mathfrak{a}$ runs over the ideals of A. We can define a sheaf of rings on X by setting

$$\mathcal{O}_X(U) = \bigcap_{\mathfrak{p} \in A} A_{\mathfrak{p}}$$

for nonempty open sets $U \subseteq X$, and $\mathcal{O}_X(\varnothing) = 0$. Again, there is a canonical isomorphism $\mathcal{O}_{X,\mathfrak{p}} \cong A_{\mathfrak{p}}$.

There are several reasons to prefer working with the prime spectrum rather than the maximal spectrum. The first is that if we wish to do algebraic geometry over a more general ring A (as opposed to an affine domain A), then this is the only viable approach – roughly speaking, a general ring A does not have enough maximal ideals for it to be sensible to work with $\operatorname{mSpec} A$. However, the reason that is most relevant to us is that $\operatorname{Spec} A$ has a nicer topological structure than $\operatorname{mSpec} A$, and this is true even in the case where A is an affine domain.

In the context of scheme theory, we define *affine n-space* to be the locally ringed space $\mathbb{A}^n_K := \operatorname{Spec}(K[z_1, \dots, z_n])$. The similarity between the notations $\mathbb{A}^n(K)$ and $\mathbb{A}^n_K$ is no coincidence. By the *Nullstellensatz*, we can identify the points in $\mathbb{A}^n(K)$ with the maximal ideals of $K[z_1, \dots, z_n]$. Under this identification, $\mathbb{A}^n(K) \subseteq \mathbb{A}^n_K$. Moreover, the subspace topology on $\mathbb{A}^n(K)$ coincides with the Zariski topology (as defined in §2 Preliminaries, Classical algebraic geometry). More generally, if $I \subset K[z_1, \dots, z_n]$ is a prime ideal, then we have $Z(I) \subseteq V(I)$ by identifying the points in K^n in the vanishing locus of I with the maximal ideals of $K[z_1, \dots, z_n]$ including I.

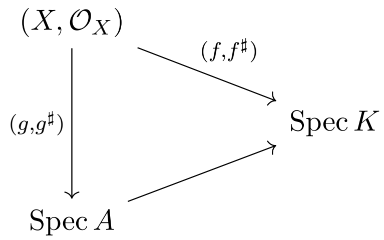
In summary, then, in scheme theory the geometric space which we associate to an affine domain A is "larger" than the space which we associate to A in classical algebraic geometry. In classical algebraic geometry, the geometric object associated to A is (or can be identified with) the maximal spectrum of A, whereas in scheme theory the geometric object associated to A is its prime spectrum. One potentially confusing aspect of this is that the elements of $X = \operatorname{Spec} A$ fulfil a dual role: they are both the prime ideals of A, and points in the topological space X. If we are thinking of the elements of $\operatorname{Spec} A$ as being prime ideals, we often denote them using upper case Roman letters like I or J; if we are thinking of them as being points in X, we denote them by lower case Roman letters like x or y, or Greek letters like ξ. Sometimes we shall think of the elements of $\operatorname{Spec} A$ as being *both* prime ideals of A, and points in X, in which case we shall denote them using Fraktur letters like $\mathfrak{p}$ or $\mathfrak{q}$.

Now that we have developed the formalism of locally ringed spaces, we can give precise definitions of both classical and scheme-theoretic affine varieties. Let us begin with scheme-theoretic affine varieties. Our provisional definition is that a scheme-theoretic affine variety is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to $\operatorname{Spec} A$ for some affine domain A (where $\operatorname{Spec} A$ has been equipped with the sheaf described above). There is one technical issue with this definition: an affine domain A is not just a ring, but a ring *equipped with* a homomorphism $K \to A$. We want the geometric object associated to A to "remember" the map $K \to A$. We thus make the following definition.

Definition 4.12. Suppose A is an affine domain with structural map $\rho : K \to A$. We define the *(scheme-theoretic) affine variety over K associated to A* as follows: it is the locally ringed space $\operatorname{Spec} A$ defined above, together with the morphism of locally ringed spaces $(f, f^\sharp) : \operatorname{Spec} A \to \operatorname{Spec} K = \{\mathrm{pt}\}$ such that $\rho = f^\sharp_{\{\mathrm{pt}\}} : K \to A$.

The motivation behind this definition is that since $\operatorname{Spec} K$ only has a single point $\{\mathrm{pt}\}$, there is only one continuous map $f : \operatorname{Spec} A \to \operatorname{Spec} K$, and the morphism of sheaves $f^\sharp : \mathcal{O}_{\operatorname{Spec} K} \to f_* \mathcal{O}_{\operatorname{Spec} A}$ is determined by its value at the open set $U = \{\mathrm{pt}\}$. The morphism $f^\sharp$ records what the structural map $\rho : K \to A$ is.

Definition 4.13. Suppose $(X, \mathcal{O}_X)$ is a locally ringed space equipped with a morphism $(f, f^\sharp) : (X, \mathcal{O}_X) \to \operatorname{Spec} K$. We say that X is a *(scheme-theoretic) affine variety (over K)* if there exists an affine domain A and an isomorphism $(g, g^\sharp)$ of locally ringed spaces $(X, \mathcal{O}_X) \to \operatorname{Spec} A$ such that the following diagram commutes:



where the morphism $\operatorname{Spec} A \to \operatorname{Spec} K$ is defined as above.

In practice, we often suppress the morphism $(f, f^\sharp) : (X, \mathcal{O}_X) \to \operatorname{Spec} K$, and use phrases such as "let X be an affine variety over K" or even "let X be an affine variety". This is similar to how when we say "let B be an A-algebra", the structural map $A \to B$ goes unmentioned.

The definition of a classical affine variety is similar: it is a locally ringed space $(X, \mathcal{O}_X)$ together with a morphism $(X, \mathcal{O}_X) \to \operatorname{Spec} K = \operatorname{mSpec} K$ such that there is an isomorphism $(X, \mathcal{O}_X) \to \operatorname{mSpec} A$ for some affine domain A that makes the obvious diagram commute. From this point onwards, when we say "affine variety" without any further qualification, we mean a scheme-theoretic affine variety. Most of the notions we introduce from this point onwards also have classical analogues, but we shall not always mention them.

There is still a lot to say about scheme-theoretic varieties. For example, if $X = \operatorname{Spec} A$ then the elements of A can be thought as functions on X, just as in the case of classical varieties. Unfortunately, for reasons of space, we are not able to elaborate on this, as it is mostly irrelevant to where we are headed. For more details on classical varieties and the corresponding scheme-theoretic notions, we refer to Chapter 2 of [GW20]. The proof that normal varieties are regular in codimension one makes use of generic points, whose definition and basic properties are discussed in that chapter.

A (scheme-theoretic) affine variety $(X, \mathcal{O}_X)$ is said to be *normal* if $\mathcal{O}_X(X)$ is a normal domain. To understand the motivation behind this definition, note that if $X = \operatorname{Spec} A$ for some affine domain A, then $\mathcal{O}_X(X) = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} A_{\mathfrak{p}} = A$ (see the proof of Lemma 5.3).

5 Commutative algebra

In this chapter we collect the results from commutative algebra that are needed to establish that normal varieties are regular in codimension one. Some of the results here are among the most technical in the report, and require a great detail of preliminary work to be established. To help the reader navigate this chapter, we have therefore adopted the following policy. If a result is used in subsequent chapters, it is given the title "Theorem" (otherwise it will be called a "Lemma", "Proposition", or "Corollary"). There are only three such theorems in this chapter, namely 5.3, 5.17, and 5.29. The first establishes that normality behaves well under localization; the second gives a characterisation of local principal ideal domains that proves to be essential in showing that normal varieties are regular in codimension one; and the third does most of the heavy lifting in dimension theory (Chapter 6).

Normality is a local condition

Suppose A is a ring, and $S \subseteq A$ is multiplicatively closed. If A is a subring of a ring B, then S is also multiplicatively closed as a subset of B, and we have a canonical injection $S^{-1}A \hookrightarrow S^{-1}B$. We may thus view $S^{-1}A$ as a subring of $S^{-1}B$. It is under this identification that the following lemma should be understood.

Lemma 5.1. *Suppose A is a subring of B. Let C be the integral closure of A in B. If $S \subseteq A$ is multiplicatively closed, then the integral closure of $S^{-1}A$ in $S^{-1}B$ is $S^{-1}C$.*

Proof. Note that every element of $S^{-1}C$ is integral over $S^{-1}A$. Conversely, suppose $x \in S^{-1}B$ is integral over $S^{-1}A$. Write $x = b/s$, where $b \in B$ and $s \in S$. By assumption,

$$\left(\frac{b}{s}\right)^n + \left(\frac{a_1}{s_1}\right)\left(\frac{b}{s}\right)^{n-1} + \cdots + \left(\frac{a_{n-1}}{s_{n-1}}\right)\left(\frac{b}{s}\right) + \frac{a_n}{s_n} = 0\,, \qquad (\star)$$

for some $a_i \in A$ and $s_i \in S$. Let $t = s_1 \cdots s_n$. By multiplying $(\star)$ by $s^n t^n$, we see that bt is integral over A. Hence, $bt \in C$ and $b/s = bt/st \in S^{-1}C$. □

Corollary 5.2. *Suppose A is an integral domain, and $S \subseteq A$ is a multiplicative set not containing 0, so that $S^{-1}A$ is an integral domain. If A is a normal domain, then so too is $S^{-1}A$.*

Proof. By Lemma 5.1, we see that $S^{-1}A$ is integrally closed in $S^{-1}\operatorname{Frac}(A)$. The map $S^{-1}A \to S^{-1}\operatorname{Frac}(A)$ extends uniquely to an isomorphism of $\operatorname{Frac}(S^{-1}A)$ onto $S^{-1}\operatorname{Frac}(A)$. Thus, $S^{-1}A$ is a normal domain. □

Theorem 5.3. *Normality is a local condition. That is, if A is an integral domain, the following are equivalent:*

(1) *A is a normal domain.*

(2) $A_{\mathfrak{p}}$ *is a normal domain for every prime ideal* $\mathfrak{p} \subset A$*.*

(3) $A_{\mathfrak{m}}$ *is a normal domain for every maximal ideal* $\mathfrak{m} \subset A$*.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Corollary 5.2, while the implication (2) $\Rightarrow$ (3) is obvious. It remains to be shown that (3) $\Rightarrow$ (1).

Fix an integral domain A. We claim that $\bigcap_{\mathfrak{m}} A_{\mathfrak{m}} = A$, where $\mathfrak{m}$ runs over the maximal ideals of A. Indeed, suppose $x \in A_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subset A$. Then, $I = \{a \in A \mid ax \in A\}$ is an ideal of A which is not included in any maximal ideal of A. Thus, I is the unit ideal of A. Hence $\bigcap_{\mathfrak{m}} A_{\mathfrak{m}} \subseteq A$, and the opposite inclusion is trivial.

Now suppose that (3) holds. If $y \in \operatorname{Frac}(A)$ is integral over A, then y is integral over $A_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subset A$, hence $y \in A_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subset A$. Hence $y \in A$, completing the proof. $\square$

This theorem is typical of those in commutative algebra in that its statement abstracts away its geometric content. The real significance of Theorem 5.3, at least to us, is that if X is a normal affine variety, then the local rings $\mathcal{O}_{X,x}$ are normal for all points $x \in X$. Moreover, to establish that X is normal, it suffices to check that $\mathcal{O}_{X,x}$ is normal for all *closed* points $x \in X$.

Finite versus finite type algebras

Suppose M is an A-module. Recall that if S is a subset of M, then the *submodule of* M *generated by* S, denoted by $\langle S \rangle$, is the smallest (with respect to inclusion) submodule of M which includes S. Similarly, if B is an A-algebra, and S is a subset of B, then the *subalgebra of* B *generated by* S, denoted by $\langle S \rangle$, is the smallest subalgebra of B which includes S. If S is a finite set $\{\alpha_1, \dots, \alpha_r\}$, we also write $A[\alpha_1, \dots, \alpha_r]$ for the subalgebra of B generated by S.

An A-module M is *finitely generated*, or of *finite type*, if there exists a finite set $S \subseteq M$ such that the submodule of M generated by S equals M. An A-algebra B with structural map $\rho : A \to B$ is said to be *finitely generated*, or of *finite type*, if there exists a finite set $S \subseteq B$ such that the subalgebra of B generated by S equals B. It is sometimes the map ρ which is said to be finitely generated or of finite type.

Recall that if B is an A-algebra with structural map $\rho : A \to B$, then B can also be regarded as an A-module with scalar multiplication $(a, b) \mapsto \rho(a)b$. We say that B is *finite* if B is finitely generated as an A-module. Again, it is sometimes the map ρ itself which is said to be finite.

Note that an A-algebra B is finite if and only B (when viewed as an A-module) is a quotient of the module $A^n = A \oplus \cdots \oplus A$ for some $n \in \mathbb{N}$. On the other hand, B is of finite type if and only if B is a quotient of the A-algebra $A[t_1, \dots, t_n]$ for some $n \in \mathbb{N}$. In categorical language,

A^n is the free object on n elements in the category of A-modules, whereas $A[t_1, \ldots, t_n]$ is the free object on n elements in the category of A-algebras.[12]

Recall from §3 Motivation that an element $x \in B$ is *integral over* A if x is a root of a monic polynomial with coefficients in A. We say that B is *integral over* A if every element of B is integral over A; that is, the integral closure of A in B equals B. We claim that a map of rings $A \to B$ is finite if and only if it is finitely generated and integral. The key step is the following lemma.

Lemma 5.4. *Let B be an A-algebra and let $b \in B$. The following are equivalent:*

(1) *b is integral over A.*

(2) *$A[b]$ is finite over A.*

(3) *There exists a faithful $A[b]$-module M which is finite over A. (Recall that an R-module M is faithful if the annihilator of M is trivial.)*

Proof. (1) $\Rightarrow$ (2) Let $p \in A[t]$ be a monic polynomial such that $p(b) = 0$. Let n be the degree of p. We claim that $1, b, \ldots, b^{n-1}$ generate $A[b]$ as an A-module. Indeed, suppose $y \in A[b]$. Then, $y = f(b)$ for some polynomial $f \in A[t]$. We can write $f = pq + r$ for some $q, r \in A[t]$ with $\deg(r) < n$. Then,

$$y = f(b) = p(b)q(b) + r(b) = r(b)\,,$$

so that y is an A-linear combination of the elements $1, b, \ldots, b^{n-1}$.

(2) $\Rightarrow$ (3) We can take $M = A[b]$.

(3) $\Rightarrow$ (1) Let $\varphi : M \to M$ be the map $x \mapsto bx$. By the Cayley–Hamilton Theorem (§2 Preliminaries, Commutative algebra), there is a monic polynomial $f \in A[t]$ such that $f(\varphi) = 0$. Since M is faithful, it follows that $f(b) = 0$. □

If $A \subseteq B \subseteq C$ are rings, C is finite over B, and B is finite over A, then it is easy to see that C is finite over A. Armed with this fact, we can now prove the characterisation of finite ring homomorphisms.

Corollary 5.5. *Let B be an A-algebra. Then, B is finite over A if and only if B is finitely generated and integral over A.*

Proof. By replacing A with its image in B, we may assume that A is a subring of B, and the structural map $A \to B$ is the inclusion. Suppose B is finite over A. Evidently, B is finitely generated over A. To show that B is integral over A, let $b \in B$. Consider B as an $A[b]$-module. Then, B is trivially faithful, hence $b \in B$ is integral over A by Lemma 5.4.

[12] See, for instance, Definition 7.8.3. of [Ber15] for the definition of a free object in a concrete category C.

Conversely, suppose B is finitely generated and integral over A. Suppose $b_1, \ldots, b_n$ generate B as an A-algebra, so that $B = A[b_1, \ldots, b_n]$. We have already seen that $A[b_1]$ is finite over A (Lemma 5.4), and similarly $A[b_1, b_2] = A[b_1][b_2]$ is finite over $A[b_1]$, hence it is finite over A. Inductively, it follows that B is finite over A. □

Discrete valuation rings

Among the normal domains, arguably the most important class are the discrete valuation rings. These rings make an appearance in number theory, complex analysis, and indeed algebraic geometry. In this section, we give a characterisation (Theorem 5.17) of discrete valuation rings that relates them to both principal ideal domains, and to one dimensional normal domains. In some sense, this characterisation is the algebraic analogue of the theorem that normal varieties are regular in codimension one, and accordingly it plays a central role in our treatment in Chapter 8.

Let k be a field. A surjective function $\nu : k \to \mathbb{Z} \cup \{\infty\}$ is said to be a *discrete valuation on* k if it satisfies the following axioms for all $x, y \in k$:

(1) $\nu(xy) = \nu(x) + \nu(y)$,

(2) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$,

(3) $\nu(x) = \infty \iff x = 0$.

The map ν restricts to a group homomorphism $k^\times \to \mathbb{Z}$, where $k^\times = k \setminus \{0\}$ is the group of units of k.

Given a discrete valuation $\nu : k \to \mathbb{Z} \cup \{\infty\}$ of a field k, the subring $A_\nu := \{x \in k \mid \nu(x) \geq 0\}$ of k is known as the *valuation ring of* ν. The ring A_ν is in fact a local integral domain, with maximal ideal $\mathfrak{m}_\nu := \{x \in k \mid \nu(x) > 0\}$ and groups of units $A_\nu^\times = \{x \in k \mid \nu(x) = 0\}$. An integral domain A is a *discrete valuation ring* if there is a discrete valuation ν on $\mathrm{Frac}(A)$ such that $A = A_\nu$.[13]

Example 5.6. If k is a field, then the ring $A = k[[t]]$ of formal power series in one variable t is a discrete valuation ring. The field of fractions of A is denoted $k((t))$. There is a valuation $\nu : k((t)) \to \mathbb{Z} \cup \{\infty\}$ which assigns to each nonzero series $f = a_0 + a_1 t + a_2 t^2 + \cdots$ the least index i such that $a_i \neq 0$. If f and g are series with $g \neq 0$, then $\nu(f/g) = \nu(f) - \nu(g)$. The maximal ideal $\mathfrak{m} \subset A$ is generated by t.

Remark 5.7. More generally, one can define a *valuation* on a field k as a function $\nu : k \to \Gamma \cup \{\infty\}$ with values in a totally ordered abelian group Γ satisfying the same axioms as above. The subring $\{x \in k \mid \nu(x) \geq 0\}$ is known as the *valuation ring of* ν. Some of the lemmas given below apply more generally to valuation rings. However, it can be shown (Chapter 8 of [Rei95]) that the discrete valuation rings are precisely the valuation rings which are

[13] By the equality $A = A_\nu$, we really mean that the image of A under the canonical map $A \to \mathrm{Frac}(A)$ coincides with A_ν, but we shall identify A with its image in $\mathrm{Frac}(A)$ for simplicity.

Noetherian. Since all of the rings which we shall encounter later will be Noetherian, there is no need for us to prove the results below in any more generality.

Lemma 5.8. *If A is a discrete valuation ring with field of fractions k, then for every nonzero element $x \in k$, either $x \in A$ or $x^{-1} \in A$.*

Proof. Let ν be a discrete valuation on k with valuation ring A. For every nonzero element $x \in k$,

$$\nu(1) = \nu(xx^{-1}) = \nu(x) + \nu(x^{-1}) = 0\,.$$

At least one of $\nu(x)$ and $\nu(x^{-1})$ is ≥ 0, from which the claim follows. □

Lemma 5.9. *Discrete valuation rings are normal: If A is a discrete valuation ring with field of fractions k, then A is a normal domain.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of A. Suppose $x \in k$ is integral over A, say

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_0 = 0\,.$$

If $x \notin A$, then $\nu(x) < 0$, hence $\nu(x^{-1}) > 0$, hence $x^{-1} \in \mathfrak{m}$. But then

$$1 + \left(a_1 x^{-1} + \cdots + a_{n-1}x^{-(n-1)} + a_n x^{-n}\right) = 0\,,$$

which is absurd. □

Lemma 5.10. *Suppose A is a discrete valuation ring with field of fractions k. Then, the collection of ideals of A is totally ordered by inclusion. Hence, every finitely generated ideal of A is principal.*

Proof. We begin by showing that the collection of *principal* ideals is totally ordered by inclusion. This amounts to showing that for all $a, b \in A$, either $a|b$ or $b|a$. But *this* follows from the fact that for all $x \in k^\times$, either $x \in A$ or $x^{-1} \in A$ (Lemma 5.8).

Now suppose I and J are arbitrary ideals in A. If $I \not\subseteq J$, there is an $x \in I$ such that $x \notin J$. We claim that $J \subseteq I$. Let $y \in J$. Since $\langle x\rangle \not\subseteq \langle y\rangle$, it must be the case that $\langle y\rangle \subseteq \langle x\rangle$. Hence $\langle y\rangle \subseteq \langle x\rangle \subseteq I$, so that $y \in I$.

Finally, if $x, y \in A$, then either $\langle x, y\rangle = \langle x\rangle$ or $\langle x, y\rangle = \langle y\rangle$, according as $\langle y\rangle \subseteq \langle x\rangle$ or $\langle x\rangle \subseteq \langle y\rangle$. It follows that every finitely generated ideal is principal. □

We are now in a position to prove that the discrete valuation rings are precisely the local principal ideal domains which are not fields (Lemma 5.14). We begin with three easy results.

Lemma 5.11. *Suppose A is a Noetherian integral domain, and $t \in A$ is a non-unit. Then, $\bigcap_{n=1}^{\infty}\langle t^n\rangle = 0$.*

Proof. We leave this as an exercise to the reader. (The reader not interested in doing exercises may wish to wait until we prove Krull's Intersection Theorem (Corollary 5.25), of which Lemma 5.11 is a special case.) □

Lemma 5.12. *The maximal ideal* $\mathfrak{m}$ *of a discrete valuation ring* A *is principal. Indeed, if* $\nu : k = \mathrm{Frac}(A) \to \mathbb{Z} \cup \{\infty\}$ *is a discrete valuation with valuation ring* A*, and* $t \in A$ *satisfies* $\nu(t) = 1$*, then* t *generates* $\mathfrak{m}$*. More precisely, if* x *is a nonzero element of* A*, and* $\nu(x) = n$*, then* $x = t^n u$ *for some unit* $u \in A$*.*

Proof. Since

$$\nu(x/t^n) = \nu(x) - n\nu(t) = 0\,,$$

it follows that x/t^n is a unit, whence the result. □

Corollary 5.13. *Suppose* $(A, \mathfrak{m})$ *is a local integral domain, and* ν *is a discrete valuation on the field of fractions of* A *with valuation ring* A*. For all nonzero* $x \in A$*, the number* $\nu(x)$ *may be characterised as the unique* $n \in \mathbb{N}$ *for which* $x \in \mathfrak{m}^n$ *and* $x \notin \mathfrak{m}^{n+1}$*.*

Proof. This follows from Lemma 5.12. □

Lemma 5.14. *Let* A *be a ring which is not a field. The following are equivalent:*

(1) *A is a discrete valuation ring.*

(2) *A is a local principal ideal domain.*

(3) *A is a Noetherian local domain whose maximal ideal is principal.*

Moreover, if these conditions hold, then writing $\mathfrak{m} \subset A$ *for the maximal ideal of* A*, every nonzero ideal of* A *is of the form* $\mathfrak{m}^n$ *for some* $n \geq 0$*.*

Proof. (1) ⇒ (2) Every discrete valuation ring is a local integral domain, so it suffices to show that every ideal in a discrete valuation ring $(A, \mathfrak{m})$ is principal. Let $k = \mathrm{Frac}(A)$, and suppose that A is the valuation ring of $\nu : k \to \mathbb{Z} \cup \{\infty\}$. Let $I \subseteq A$ be a nonzero proper ideal. Then, $\nu(I) \subseteq \mathbb{N} \cup \{\infty\}$ has a least element, say $n \in \mathbb{N}$. Let $t \in \mathfrak{m}$ be such that $\nu(t) = 1$. Since $I \subseteq \mathfrak{m}$, it follows from Lemma 5.12 that every nonzero element of I can be expressed as $t^m u$ for some unit $u \in A$ and $m \geq n$. Hence, $I = \langle t^n \rangle$. This also takes care of the final statement in the lemma.

(2) ⇒ (3) Clear.

(3) ⇒ (1) In light of Corollary 5.13, there is only one candidate for what the discrete valuation on $\mathrm{Frac}(A)$ could be. Let $\widetilde{\nu} : A \to \mathbb{Z} \cup \{\infty\}$ be the map defined by

$$\widetilde{\nu}(x) = \begin{cases} \infty & \text{if } x = 0\,, \\ \ell & \text{if } x \neq 0 \text{ and } x \in \mathfrak{m}^\ell \text{ and } x \notin \mathfrak{m}^{\ell+1}\,. \end{cases}$$

Since $\bigcap_{n\geq 1}\mathfrak{m}^n = 0$ by Lemma 5.11, the definition of $\widetilde{\nu}$ is coherent. The map $\widetilde{\nu} : A \to \mathbb{Z}\cup\{\infty\}$ extends to a discrete valuation $\nu : \mathrm{Frac}(A) \to \mathbb{Z}\cup\{\infty\}$ satisfying $\nu(x/y) = \widetilde{\nu}(x) - \widetilde{\nu}(y)$ for all $x, y \in A$ with $y \neq 0$. Since A is not a field and $\bigcap_{n\geq 1}\mathfrak{m}^n = 0$, it must be the case that $\mathfrak{m}^2$ is a proper subset of $\mathfrak{m}$. Hence, 1 is in the image of ν, so that ν is surjective. □

An integral domain A is said to be a *Dedekind domain* if it is normal, one-dimensional, and Noetherian.[14] The results we have established up to this point show that a discrete valuation ring is a local Dedekind domain. In fact, the converse holds. This is the deepest characterisation of discrete valuation rings, and it plays a pivotal role in the proof that normal varieties are regular in codimension one.

Lemma 5.15. *Suppose A is a Noetherian integral domain, and $I \subseteq A$ is a nonzero ideal. Then, $B = \{f \in \mathrm{Frac}(A) \mid fI \subseteq I\}$ is a subalgebra of* $\mathrm{Frac}(A)$ *which is finite over A.*

Proof. Evidently, B is a subalgebra of $\mathrm{Frac}(A)$. To establish that B is finite over A, we first show that $\mathrm{Hom}_{A\text{-}\mathsf{Mod}}(I, I)$ is a Noetherian A-module. Write $I = \langle a_1, \ldots, a_n\rangle$ for some $a_i \in A$, and let $\phi : A^n \to I$ be the A-module homomorphism sending e_i to a_i. Consider the pullback $\phi^* : \mathrm{Hom}_{A\text{-}\mathsf{Mod}}(I, I) \to \mathrm{Hom}_{A\text{-}\mathsf{Mod}}(A^n, I)$, $f \mapsto \phi^*(f) = f \circ \phi$. It is not hard to check that ϕ^* is injective. But $\mathrm{Hom}_{A\text{-}\mathsf{Mod}}(A^n, I) \cong I^n$ and I^n is a Noetherian module, so $\mathrm{Hom}_{A\text{-}\mathsf{Mod}}(I, I)$ must also be Noetherian.[15]

To prove that B is finite over A, it thus suffices to show that B embeds into $\mathrm{Hom}_{A\text{-}\mathsf{Mod}}(I, I)$. Let $\psi : B \to \mathrm{Hom}_{A\text{-}\mathsf{Mod}}(I, I)$ be the A-module homomorphism given by $\psi(f) = \alpha \mapsto f\alpha$. Since A is an integral domain and $I \neq 0$, it follows that ψ is injective, completing the proof. □

Lemma 5.16. *Suppose A is a local Dedekind domain, i.e. an integral domain which is local, normal, Noetherian, and of dimension one. Then, A is a discrete valuation ring.*

Proof. It suffices to show that the maximal ideal of A is principal. The result will then follow from the characterisation of discrete valuation rings given in Lemma 5.14.

From Corollary 5.13, we learn that if $(R, \mathfrak{m})$ is a local integral domain, and ν is a discrete valuation on $\mathrm{Frac}(R)$ with valuation ring R, then for all nonzero $x \in R$, the number $\nu(x)$ may be characterised as the unique $n \in \mathbb{N}$ for which $x \in \mathfrak{m}^n$ and $x \notin \mathfrak{m}^{n+1}$. Moreover, by Lemma 5.12, every element $t \in R$ satisfying $\nu(t) = 1$ is a generator of $\mathfrak{m}$.

Let $\mathfrak{m}$ be the maximal ideal of A. In light of the preceding paragraph, our task is to show that $\mathfrak{m} \neq \mathfrak{m}^2$ and that every element of $\mathfrak{m}\setminus\mathfrak{m}^2$ is a generator of $\mathfrak{m}$. First, $\mathfrak{m} \neq \mathfrak{m}^2$. Reason: if $\mathfrak{m}$ were equal to $\mathfrak{m}^2$, then $\mathfrak{m} = 0$ by Nakayama's Lemma; hence, A would be a field, contrary

[14] Some authors merely require Dedekind domains to have dimension at most one. This amounts to allowing fields to be Dedekind domains. However, for our purposes, it is convenient to exclude fields.

[15] By I^n, we mean the direct sum $I \oplus \cdots \oplus I$, not the product of ideals $I \cdots I$. The module I^n is Noetherian by virtue of the fact is a finitely generated module over a Noetherian ring (§2 Preliminaries, Commutative algebra).

to our hypothesis that $\dim(A) = 1$. Now let $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Since 0 and $\mathfrak{m}$ are the only prime ideals of A, and $x \neq 0$, it follows that $A/\langle x\rangle$ has a single prime ideal, namely the image of $\mathfrak{m}$. Hence $\mathfrak{m} = \sqrt{\langle x\rangle}$.

The subalgebra $B = \{f \in \mathrm{Frac}(A) \mid f\mathfrak{m} \subseteq \mathfrak{m}\}$ of $\mathrm{Frac}(A)$ is finite over A by Lemma 5.15. But then B is integral over A by Corollary 5.5, and the normality of A implies that $A = B$.

Since $\mathfrak{m}$ is finitely generated and $\mathfrak{m} = \sqrt{\langle x\rangle}$, there is an integer $r \geq 2$ such that $\mathfrak{m}^r \subseteq \langle x\rangle$. Let $y \in \mathfrak{m}^{r-1}$. Then, $\frac{y}{x}\mathfrak{m} \subseteq \frac{1}{x}\mathfrak{m}^r \subseteq A$, so that $\frac{y}{x}\mathfrak{m}$ is an ideal of A. The ideal $\frac{y}{x}\mathfrak{m}$ is proper, so that $\frac{y}{x}\mathfrak{m} \subseteq \mathfrak{m}$. Hence, $\frac{y}{x} \in B = A$. This implies that $y \in \langle x\rangle$. Consequently, $\mathfrak{m}^{r-1} \subseteq \langle x\rangle$. Since $\mathfrak{m}^r \subseteq \langle x\rangle \Rightarrow \mathfrak{m}^{r-1} \subseteq \langle x\rangle$ for all integers $r \geq 2$, we see that $\mathfrak{m} \subseteq \langle x\rangle$, and at last we are finished. □

Putting all of these results together, we obtain the following characterisation of discrete valuation rings.

Theorem 5.17. *Let A be a ring which is not a field. The following are equivalent:*

(1) *A is a discrete valuation ring.*

(2) *A is a local principal ideal domain.*

(3) *A is a Noetherian local domain whose maximal ideal is principal.*

(4) *A is a local Dedekind domain, i.e. a local integral domain which is normal, one-dimensional, and Noetherian.*

Moreover, if these conditions hold, then writing $\mathfrak{m}$ for the maximal ideal of A, every nonzero ideal of A is of the form $\mathfrak{m}^n$ for some $n \geq 0$. There is a unique discrete valuation ν on $\mathrm{Frac}(A)$ with valuation ring A, and for all nonzero $x \in A$, the number $\nu(x)$ may be characterised as the unique $n \in \mathbb{N}$ for which $x \in \mathfrak{m}^n$ and $x \notin \mathfrak{m}^{n+1}$.

Topologies, filtrations, and the Artin–Rees Lemma

In this section, we change course entirely. We introduce topological groups, rings, and modules, and prove a deep theorem, the Artin–Rees Lemma, which establishes that the subspace topology on a submodule interacts well with its algebraic structure. Why we are doing all this will probably be a little mysterious at first to the reader; suffice to say, these results are the first stepping stone in understanding the dimension theory of varieties.

By a *topological group*, we mean an abelian group G endowed with a topology for which the operations $G \times G \to G$, $(x, y) \mapsto x + y$ and $G \to G$, $x \mapsto -x$ are continuous.[16] A morphism of topological groups is defined as a continuous group homomorphism. If G and H are topological groups, then a group homomorphism $G \to H$ is continuous if and only if it is continuous at 0.

[16] By our convention, topological groups are abelian by definition.

It follows from the definition of a topological group that for any fixed $c \in G$, the map $G \to G$, $x \mapsto x + c$ is a homeomorphism. This means that the neighbourhoods of c are just the translates of the neighbourhoods of 0, so that the topology on G is determined by the neighbourhoods of 0.

If G is an abelian group, then a *filtration* of G is defined as an infinite descending chain of subgroups $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$ of G which begins with G. The importance of filtrations is given by the following proposition.

Proposition 5.18. *Suppose G is an abelian group, and $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$ is a filtration of G. Then, there is a unique topological group structure on G in which $\{G_n\}_{n \geq 0}$ is a neighbourhood base of* 0.

(Recall that if X is a topological space, and $x \in X$, then a *neighbourhood base* of x is a collection $\mathcal{C}$ of neighbourhoods of x such that, for every neighbourhood U of x, there exists $V \in \mathcal{C}$ such that $V \subseteq U$. From a neighbourhood base of x, we can recover the collection $\mathcal{N}$ of neighbourhoods of x: indeed, $\mathcal{N} = \{E \subseteq X \mid \text{there exists a } V \in \mathcal{C} \text{ such that } V \subseteq E\}$.)

Proof. The uniqueness assertion follows from the above discussion, so we just need to prove existence. Consider the collection $\mathcal{B}$ of cosets $x + G_n$ ($x \in G$, $n \in \mathbb{N}$). We claim that $\mathcal{B}$ is a base of open sets of a topology on G. To establish this, we need to check that the following conditions are satisfied:

(1) $\mathcal{B}$ covers G,

(2) For all $U, V \in \mathcal{B}$ and $x \in G$, if $x \in U \cap V$ then there is a $W \in \mathcal{B}$ such that $x \in W \subseteq U \cap V$.

Evidently, (1) holds. That (2) holds follows from the fact that if $x, y \in G$ and $n \in \mathbb{N}$, then the cosets $x + G_n$ and $y + G_n$ are either disjoint or equal. Hence, $\mathcal{B}$ is a base of a topology on G; in particular, $\{G_n\}_{n \geq 0}$ is a neighbourhood base of 0.

To check that the group operations are continuous, it suffices to show that the mapping $f : (x, y) \mapsto x - y$ is continuous. If $n \in \mathbb{N}$ and $a \in G$, then

$$f^{-1}(a + G_n) = \bigcup_{(x,y) \in G \times G} (a + x + G_n) \times (y + G_n),$$

so that the preimages of the basic open sets are open. This completes the proof. □

In summary, if G is an abelian group, then a filtration $\{G_n\}_{n \geq 0}$ of G induces a topology on G with the following property: $E \subseteq G$ is a neighbourhood of 0 if and only if $G_n \subseteq E$ for sufficiently large n.

We can similarly define topological rings and topological modules. A *topological ring* is a ring A endowed with a topology such that A is a topological group under its additive structure and $A \times A \to A$, $(x, y) \mapsto xy$ is continuous. If A is a topological ring, then a *topological*

A-module is an A-module M endowed with a topology such that M is a topological group under its additive structure, and $A \times M \to M$, $(a, m) \mapsto am$ is continuous. Filtrations of rings and modules are defined analogously to filtrations of abelian groups, with subgroups being replaced by ideals and submodules, respectively. Morphisms of topological rings and topological modules are defined in the obvious way.

Proposition 5.18 is true almost verbatim for topological rings. Suppose A is a ring, and $\langle 1 \rangle = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ is a filtration of A. Then, there is a unique topological ring structure on A in which the ideals I_n form a neighbourhood base of 0.[17]

Example 5.19. Consider the case where $A = \mathbb{Z}$. Given a prime number p, the topology defined by the filtration $\langle 1 \rangle \supseteq \langle p \rangle \supseteq \langle p^2 \rangle \supseteq \cdots$ is known as the *p-adic topology* on $\mathbb{Z}$. A subset $E \subseteq \mathbb{Z}$ is a neighbourhood of 0 if and only if there is a number $n \in \mathbb{N}$ such that all multiples of p^n belong to E.

More generally, if A is a ring and $I \subseteq A$ is an ideal, then the *I-adic filtration* of A is given by $\{I^j\}_{j \geq 0}$; the resulting topology is known as the *I-adic topology* on A. This topology plays an especially important role when A is a local ring, and I is its maximal ideal.

Suppose A is a ring, $I \subseteq A$ is an ideal, and M is an A-module. A filtration $\{M_n\}_{n \geq 0}$ of M is known as an *I-filtration* if $IM_n \subseteq M_{n+1}$ for all $n \in \mathbb{N}$. If we regard A as a topological ring with the I-adic topology, then M has a canonical topological A-module structure induced by an I-filtration $\{M_n\}_n$: again, we take $\{M_n\}_n$ to be a neighbourhood base of 0.

Two different filtrations may well define the same topology. For example, in the case of topological groups, we could take $G = \mathbb{Z}$, $G_n = 2^n\mathbb{Z}$, $G'_n = 4^n\mathbb{Z}$. It is possible to say precisely when this phenomenon occurs. Suppose $\{G_n\}_n$ and $\{G'_m\}_m$ are two filtrations of an abelian group G, with induced topologies τ and τ' respectively. Let $\mathcal{N}$ and $\mathcal{N}'$ be the respective collections of neighbourhoods of 0. Note that $\tau \subseteq \tau'$ if and only if $\mathcal{N} \subseteq \mathcal{N}'$, which in turn holds if and only if for every $n \in \mathbb{N}$, there is an $m \in \mathbb{N}$ such that $G_n \supseteq G'_m$. An analogous criterion applies to rings and modules.

Once again, suppose A is a ring, $I \subseteq A$ is an ideal, and M is an A-module. We say that an I-filtration $\{M_n\}_n$ is *stable* if $IM_n = M_{n+1}$ for sufficiently large n. The simplest example of a stable I-filtration is given by $M_n = I^nM$. As in the case of topological rings, this filtration is known as the *I-adic filtration*, and the resulting topology is known as the *I-adic topology*. Using the criterion described above, one can show that all stable I-filtrations induce the I-adic topology on M.

We are now in a position to state the most famous result about topological modules.

Lemma (Artin–Rees)**.** *Suppose M is a finitely generated module over a Noetherian ring*

[17] The proof follows the same lines as that of Proposition 5.18; we leave the details to the reader. To show that multiplication is continuous, it is perhaps easiest to use the following criterion for continuity: a function $f : X \to Y$ is continuous at x if for every neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subseteq V$.

A, and N is a submodule of M. Let I be an ideal of A. For every stable I-filtration $\{M_n\}_n$ on M, we have

$$I(M_n \cap N) = M_{n+1} \cap N$$

for sufficiently large n. Hence, $\{M_n \cap N\}_n$ is a stable I-filtration on N.

One can think of the Artin–Rees Lemma as follows. If A is a ring with the I-adic topology, and M is a module over A with the I-adic topology, then there are two natural topologies that one can put on a submodule $N \subseteq M$: the subspace topology, and the I-adic topology. The subspace topology on N is induced by the I-filtration $\{M_n \cap N\}_n$, and the I-adic topology by $\{I^n N\}_n$. The Artin–Rees Lemma tells us that, under some mild hypotheses, these two topologies coincide.

One cannot drop the assumption that A is Noetherian, nor the assumption that M is finitely generated, as the following examples show.

Example 5.20. In the special case where $M = A$, $N = \bigcap_{n\geq 0} I^n$, and $M_n = I^n$, the Artin–Rees Lemma asserts that

$$I \bigcap_{n\geq 0} I^n = \bigcap_{n\geq 0} I^n .$$

This equality does not hold in general for non-Noetherian rings A. On the other hand, since non-Noetherian rings are a little difficult to come by, any counter-example is fairly complicated. Perhaps the simplest is due to [Car11]: let

$$A = \frac{\mathbb{Q}[x, z, y_1, y_2, y_3, \dots]}{\langle x - zy_1, x - z^2 y_2, x - z^3 y_3, \dots \rangle} ,$$

and let $I = \langle z \rangle$. Then, $\bigcap_{n\geq 0} I^n = \langle x \rangle$, but $z\langle x \rangle \neq \langle x \rangle$.

Example 5.21. Suppose $A = \mathbb{Z}$ and $I = \langle 2 \rangle$. Then, the I-adic topology on $M = \mathbb{Q}$ is just the indiscrete topology, and the restriction of this topology to $N = \mathbb{Z}$ is also indiscrete. This does not coincide with the I-adic topology on $\mathbb{Z}$.

To prove the Artin–Rees Lemma, we introduce one more concept: gradations. If A is a ring, then a *grading* on A is a family $\{A_n\}_{n\geq 0}$ of subgroups of the additive group of A which satisfies the following two properties:

(1) As abelian groups, $A = \bigoplus_{n\geq 0} A_n$ (where $\oplus$ denotes the internal direct sum),

(2) $A_m A_n \subseteq A_{m+n}$ for all $m, n \in \mathbb{N}$.

These conditions imply that A_0 is a subring of A, and each A_n has an A_0-module structure induced by the ring multiplication of A.

A *graded ring* is a ring A endowed with a grading $\{A_n\}_{n\geq 0}$. The prototypical example is given by the polynomial ring $A = k[x_1, \dots, x_r]$ over a field k: we take A_n to be the subgroup of homogenous polynomials of degree n. Given a graded ring A, a *grading* on an A-module

M is a family $\{M_n\}_{n\geq 0}$ of subgroups of M such that $M = \bigoplus_{n\geq 0} M_n$ and $A_m M_n \subseteq M_{m+n}$ for all $m, n \in \mathbb{N}$. A *graded A-module* is an A-module endowed with a grading $\{M_n\}_{n\geq 0}$.

The importance of gradations to us is given by the following special case. Let A be a Noetherian ring and M be an A-module. Consider the ring $A[tI] := \sum_{n\geq 0} I^n t^n \subseteq A[t]$, graded by degree (i.e. $A[tI] = A \oplus tI \oplus t^2 I \oplus \cdots$). An I-filtration $\{M_n\}_n$ of M gives rise to a graded $A[tI]$-module $\widetilde{M} := \sum_{n\geq 0} M_n t^n = M_0 \oplus M_1 t + M_2 t^2 \oplus \cdots$.

Lemma 5.22. *Let us keep the notation above. The I-filtration $\{M_n\}_n$ is stable if and only if the module $\widetilde{M}$ is finitely generated.*

Proof. For each $n \in \mathbb{N}$, let Q_n be the subgroup $M_0 \oplus M_1 t \oplus M_2 t^2 \oplus \cdots \oplus M_n t^n \subseteq \widetilde{M}$. Note that the submodule $\langle Q_n \rangle$ of $\widetilde{M}$ generated by Q_n equals

$$M_0 \oplus M_1 t \oplus \cdots \oplus M_n t^n \oplus I M_n t^{n+1} \oplus I^2 M_n t^{n+2} \oplus \cdots .$$

($\Rightarrow$) If $\{M_n\}_n$ is stable, then there is an $n_0 \in \mathbb{N}$ such that $\widetilde{M} = \langle Q_{n_0} \rangle$. Choose a finite set of generators for each of $M_0, M_1 t \ldots, M_{n_0} t^{n_0}$. It is not hard to check that the union U of these sets generates $\widetilde{M}$. For example, $I M_{n_0} t^{n+1} \subseteq \langle U \rangle$, since if $a \in I$ and $m \in M_{n_0}$, then $amt^{n+1} = (at)(mt^n)$ and $mt^n \in \langle U \rangle$.

($\Leftarrow$) Conversely, suppose $\widetilde{M}$ is finitely generated, say by $x_1, \ldots, x_s$. Then, $x_1, \ldots, x_s \in Q_{n_0}$ for some $n_0 \in \mathbb{N}$. Hence,

$$\begin{aligned} \langle x_1, \ldots, x_s \rangle &= M_0 \oplus M_1 t \oplus \cdots \oplus M_{n_0} t^{n_0} \oplus I M_{n_0} t^{n_0+1} \oplus I^2 M_{n_0} t^{n_0+2} \oplus \cdots \\ &= M_0 \oplus M_1 t \oplus \cdots \oplus M_{n_0} t^{n_0} \oplus M_{n_0+1} t^{n_0+1} \oplus M_{n_0+2} t^{n_0+2} \oplus \cdots = \widetilde{M}\,, \end{aligned}$$

so that $I M_n = M_{n+1}$ for $n \geq n_0$. □

Lemma 5.23. *Let A be a Noetherian ring. Then, the ring $A[tI] = A \oplus tI \oplus t^2 I \oplus \cdots$ is also Noetherian.*

Proof. Since A is Noetherian, we can write $I = \langle a_1, \ldots, a_k \rangle$ for some $a_i \in A$. The elements $a_i t$ generate $A[tI]$ as an A-algebra. Hence, $A[tI]$ is a quotient of the polynomial ring $A[x_1, \ldots, x_k]$, so that it is Noetherian by Hilbert's Basis Theorem. □

Lemma 5.24 (Artin–Rees)**.** *Suppose M is a finitely generated module over a Noetherian ring A, and N is a submodule of M. Let I be an ideal of A. For every stable I-filtration $\{M_n\}_n$ on M, we have*

$$I(M_n \cap N) = M_{n+1} \cap N$$

for sufficiently large n. Hence, $\{M_n \cap N\}_n$ is a stable I-filtration on N.

Proof. Since $\{M_n\}_n$ is stable, the module $\widetilde{M}$ is finitely generated. By Lemma 5.23, the submodule $\widetilde{N} = \bigoplus_{i\geq 0}(M_i \cap N)t^i$ must also be finitely generated. Finally, the I-filtration $\{M_n \cap N\}$ is stable by Lemma 5.22. □

The following corollary of the Artin–Rees Lemma is frequently used in applications.

Corollary 5.25 (Krull's Intersection Theorem)**.** *Suppose M is a finitely generated module over a Noetherian ring A. Let I be an ideal of A. The intersection $\bigcap_{n\geq 0} I^n M$ consists precisely of those elements of M which are annihilated by an element of $1 + I$.*

Proof. Suppose $x \in M$ is annihilated by an element of $1 + I$. Then, $(1 - \alpha)x = 0$ for some $\alpha \in I$, hence $x = \alpha x = \alpha^2 x = \cdots$, hence $x \in \bigcap_{n\geq 0} I^n M$.

Conversely, suppose $x \in \bigcap_{n\geq 0} I^n M$ and $N = xA$. By applying the Artin–Rees Lemma with $M_n = I^n M$, we see that $I^n M \cap N \subseteq IN$ for sufficiently large n, so that $x \in IN = xI$. □

Exercise 5.26. Suppose $(A, \mathfrak{m})$ is a Noetherian local ring. By taking $M = A$ and $I = \mathfrak{m}$ in Krull's Intersection Theorem, we see that $\bigcap_{n\geq 0} \mathfrak{m}^n = 0$. This result is sometimes described as the algebraic analogue of the fact that a complex analytic function is determined by the coefficients of its Taylor expansion. Look up why.

The Hauptidealsatz

Our efforts to prove the Artin–Rees Lemma, and in particular Krull's Intersection Theorem, were not in vain. We shall use these results to derive perhaps the single most important theorem in commutative algebra, namely Krull's *Hauptidealsatz* ("Principal Ideal Theorem"). This theorem plays an indispensable role in the foundations of algebraic geometry, providing the formal justification for "obvious" assertions such as "$\mathbb{A}^n$ has dimension n" or "a variety cut out by a single nonconstant function has codimension one". We give a tour of some of these geometric implications in the next chapter.

A ring A is *Artinian* if it satisfies the descending chain condition on ideals, i.e. every descending chain $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ of ideals of A eventually stabilises.

Lemma 5.27. *Suppose $(A, \mathfrak{m}, k)$ is a Noetherian local ring. The following are equivalent:*

(1) $\mathfrak{m}$ *coincides with the nilradical of A.*

(2) *There exists $q \geq 1$ such that $\mathfrak{m}^q = 0$.*

(3) *A is Artinian.*

Proof. Let us begin by noting that for each $r \geq 0$, we may view the A-module $\mathfrak{m}^r/\mathfrak{m}^{r+1}$ as a vector space over k (since $\mathfrak{m}$ annihilates $\mathfrak{m}^r/\mathfrak{m}^{r+1}$). As A is Noetherian, the space $\mathfrak{m}^r/\mathfrak{m}^{r+1}$ is finite-dimensional.

(1) $\Rightarrow$ (2) Suppose $\mathfrak{m} = \sqrt{\langle 0 \rangle}$. Write $\mathfrak{m} = \langle a_1, \ldots, a_s \rangle$. There is a $t \geq 1$ such that $a_i^t = 0$ for each i. Hence, $\mathfrak{m}^{ts} = 0$.

(2) $\Rightarrow$ (3) This one is not easy. Suppose that $\mathfrak{m}^q = 0$ for some $q \geq 1$. Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ be a descending chain of ideals of A. Set $I_0 = A$. Fix a number $r \geq 0$ such that $r \leq q - 1$. For each $n \in \mathbb{N}$, let V_n be the k-vector space $(I_n \cap \mathfrak{m}^r)/(I_n \cap \mathfrak{m}^{r+1})$, and let $\varphi_n : V_{n+1} \to V_n$ be the linear map induced by the embedding of $I_{n+1} \cap \mathfrak{m}^r$ into $I_n \cap \mathfrak{m}^r$.

Each of the maps φ_n is injective, and so loosely speaking, we have a descending chain of subspaces of $\mathfrak{m}^r/\mathfrak{m}^{r+1}$ given by

$$\frac{\mathfrak{m}^r}{\mathfrak{m}^{r+1}} \supseteq \frac{I_1 \cap \mathfrak{m}^r}{I_1 \cap \mathfrak{m}^{r+1}} \supseteq \frac{I_2 \cap \mathfrak{m}^r}{I_2 \cap \mathfrak{m}^{r+1}} \supseteq \frac{I_3 \cap \mathfrak{m}^r}{I_3 \cap \mathfrak{m}^{r+1}} \supseteq \cdots$$

Since $\mathfrak{m}^r/\mathfrak{m}^{r+1}$ is finite-dimensional, this chain must eventually stabilise; that is, there is an $n_0 \in \mathbb{N}$ such that φ_n is an isomorphism for all $n \geq n_0$. Let $n \geq n_0$. It follows from the surjectivity of φ_n that

$$I_n \cap \mathfrak{m}^r = (I_{n+1} \cap \mathfrak{m}^r) + (I_n \cap \mathfrak{m}^{r+1}) .$$

Hence, given any $r \geq 0$ such that $r \leq q - 1$, we have $I_n \cap \mathfrak{m}^r \subseteq I_{n+1} + (I_n \cap \mathfrak{m}^{r+1})$ for sufficiently large n. By repeatedly using this relation, we see that for sufficiently large n,

$$I_n \subseteq I_{n+1} + (I_n \cap \mathfrak{m}) \subseteq I_{n+1} + (I_n \cap \mathfrak{m}^2) \subseteq \cdots \subseteq I_{n+1} + (I_n \cap \mathfrak{m}^q) = I_{n+1} ,$$

so that the chain $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ eventually stabilises.

(3) $\Rightarrow$ (1) If A is Artinian, then the chain $\mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \mathfrak{m}^3 \supseteq \cdots$ eventually stabilises, so that there is a $q \geq 1$ for which $\mathfrak{m}^q = \mathfrak{m}^{q+1}$. Hence, $\mathfrak{m}^q = 0$ by Nakayama's Lemma. It follows that $\mathfrak{m} = \sqrt{\langle 0 \rangle}$. □

Since the nilradical of a ring is the intersection of its prime ideals, the condition that $\mathfrak{m} = \sqrt{\langle 0 \rangle}$ is equivalent to A having a unique prime ideal, which in turn is equivalent to $\dim(A) = 0$ (since A is local). We thus derive the following corollary.

Corollary 5.28. *A Noetherian local ring of dimension zero is Artinian.*

More generally, it can be shown that a nonzero ring A is Artinian if and only if A is Noetherian and of dimension zero ([AM69], Theorem 8.5). This shows that the apparent symmetry between the ascending and descending chain conditions is in fact misleading.

Theorem 5.29 (Krull's *Hauptidealsatz*)**.** *Suppose A is a Noetherian ring, and $f \in A$. For every prime ideal $\mathfrak{p} \subset A$ that is minimal among those containing f, we have* $\mathrm{ht}(\mathfrak{p}) \leq 1$.

Proof. Note that the height of $\mathfrak{p}$ in A equals the height of the maximal ideal of $A_{\mathfrak{p}}$. It thus suffices to prove the theorem in the case where A is a local ring, and $\mathfrak{p} \subset A$ is its maximal ideal. Via a similar reduction, we may further assume that A is an integral domain (by passing to the quotient of A by a minimal prime ideal). We must show that if $\mathfrak{q}$ is a prime ideal which is properly included in $\mathfrak{p}$, then $\mathfrak{q} = 0$.

Since $\mathfrak{p}$ is both a maximal ideal, and minimal among those prime ideals which contain f, it follows that $\mathfrak{p}$ is the *only* prime ideal which contains f. Thus, $A/\langle f \rangle$ has a unique prime ideal, so that it is Artinian by Corollary 5.28. Consider the descending sequence of ideals $\mathfrak{q}_n = \mathfrak{q}^n A_f \cap A$ in A. The image of this sequence in $A/\langle f \rangle$ must eventually stabilise. Hence, $\mathfrak{q}_n + \langle f \rangle = \mathfrak{q}_{n+1} + \langle f \rangle$ for sufficiently large n. In particular, there is an $n_0 \in \mathbb{N}$ such that $\mathfrak{q}_n \subseteq \mathfrak{q}_{n+1} + \langle f \rangle$ for all $n \geq n_0$.

Let $n \geq n_0$. Let $x \in \mathfrak{q}_n$. Then, $x - fy \in \mathfrak{q}_{n+1}$ for some $y \in A$. Hence, $fy \in \mathfrak{q}_n$, so that $fy = (s_1 \cdots s_n)/f^t$ for some $s_i \in \mathfrak{q}$ and $t \geq 1$. Thus, $y \in \mathfrak{q}_n$. This shows that $\mathfrak{q}_n \subseteq \mathfrak{q}_{n+1} + f\mathfrak{q}_n \subseteq \mathfrak{q}_{n+1} + \mathfrak{p}\mathfrak{q}_n$, hence $\mathfrak{q}_n = \mathfrak{q}_{n+1} + \mathfrak{p}\mathfrak{q}_n$.

We are on the final stretch now. From the equality $\mathfrak{q}_n = \mathfrak{q}_{n+1} + \mathfrak{p}\mathfrak{q}_n$, it follows that $\mathfrak{q}_n = \mathfrak{q}_{n+1}$ by Nakayama's Lemma. Since $\mathfrak{q}_n A_f = \mathfrak{q}^n A_f$, we see that $\mathfrak{q}^{n_0} A_f = \mathfrak{q}^{n_0+1} A_f = \mathfrak{q}^{n_0+2} A_f = \cdots$. On the other hand,

$$\mathfrak{q}^{n_0} A_f = \bigcap_{n \geq 0} \mathfrak{q}^n A_f = 0\,,$$

by Krull's Intersection Theorem (Corollary 5.25). Hence, $\mathfrak{q} = 0$. □

Remark 5.30. If I is a proper principal ideal of a Noetherian ring A, then the *Hauptidealsatz* tells us that I has height at most one. This explains what the *Hauptidealsatz* has to do with *principal* ideals.

Exercise 5.31. Let R be a Noetherian integral domain. Prove, using the *Hauptidealsatz*, that R is factorial if and only if every prime ideal of height one is principal. [The forward implication is routine, and does not even require the Noetherian hypothesis. Conversely, suppose every prime ideal of height one is principal. Use Zorn's Lemma to show that for every nonzero element f of A which is not a unit, there is a prime ideal $\mathfrak{p} \subset R$ which is minimal among those containing f. Then apply the *Hauptidealsatz*.]

6 Dimension theory

Algebraic geometry is a subject which is well-known for having a number of geometrically "obvious" facts that are difficult to justify without using rather sophisticated algebraic tools. For example, it seems reasonable that if X is a variety cut out by r equations in $\mathbb{A}^n_K$, then X has dimension $n - r$. Unfortunately, this is not true in general, and to say exactly when it is true is a little delicate. However, the arcane results of the previous chapter now allow us to make serious progress on this issue. Our treatment of dimension theory mostly follows that of Chapter 2 in [Liu06], with a few more details given in the proofs.

Let us begin by extending the *Hauptidealsatz*. We need the following somewhat technical lemma.

Lemma 6.1. *Let A be a Noetherian ring, and $f \in A$. For every strictly ascending chain of prime ideals $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ in A (with $n \geq 1$), if $f \in \mathfrak{p}_n$ then there exists a strictly ascending chain of prime ideals $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n$ in A such that $\mathfrak{q}_n = \mathfrak{p}_n$ and $f \in \mathfrak{q}_1$.*

Proof. By induction on n. If $n = 1$ this is trivial. Suppose $n \geq 2$ and the assertion has been established for $1, \ldots, n-1$. Let $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_{n-1} \subset \mathfrak{p}_n$ be a strictly ascending chain of prime ideals in A with $f \in \mathfrak{p}_n$.

If $f \in \mathfrak{p}_{n-1}$, then by the inductive hypothesis, there is a strictly ascending chain $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_{n-1} = \mathfrak{p}_{n-1}$ with $f \in \mathfrak{q}_1$; by adjoining $\mathfrak{p}_n$ to the end of this chain, we obtain the desired chain.

If $f \notin \mathfrak{p}_{n-1}$, then consider the collection $\mathcal{C} = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p}_{n-2} + \langle f \rangle \subseteq \mathfrak{p} \subseteq \mathfrak{p}_n\}$ of prime ideals of A. The collection $\mathcal{C}$ has a minimal element with respect to inclusion.[18] Let $\mathfrak{q}_{n-1}$ be a minimal element of $\mathcal{C}$. The ideal $\mathfrak{p}_{n-2}$ is a proper subset of $\mathfrak{q}_{n-1}$. By applying the inductive hypothesis to the chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_{n-2} \subset \mathfrak{q}_{n-1}$, we obtain a strictly ascending chain $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_{n-1}$ with $f \in \mathfrak{q}_1$. To establish the lemma, it thus suffices to show that $\mathfrak{q}_{n-1}$ is a proper subset of $\mathfrak{p}_n$.

Consider the image $\overline{\mathfrak{q}_{n-1}}$ of $\mathfrak{q}_{n-1}$ in $A/\mathfrak{p}_{n-2}$. We claim that $\overline{\mathfrak{q}_{n-1}}$ is minimal among those prime ideals in $A/\mathfrak{p}_{n-2}$ containing $\bar{f}$. Indeed, if $\mathfrak{p} \subset A$ is a prime ideal and $\bar{f} \in \bar{\mathfrak{p}} \subseteq \overline{\mathfrak{q}_{n-1}}$, then $\mathfrak{p} + \mathfrak{p}_{n-2} \subseteq \mathfrak{q}_{n-1} + \mathfrak{p}_{n-2} = \mathfrak{q}_{n-1}$. Since $\mathfrak{p} + \mathfrak{p}_{n-2} \in \mathcal{C}$ and $\mathfrak{q}_{n-1}$ is minimal, it follows that $\mathfrak{p} + \mathfrak{p}_{n-2} = \mathfrak{q}_{n-1}$, hence $\bar{\mathfrak{p}} = \overline{\mathfrak{q}_{n-1}}$. Then $\operatorname{ht}(\overline{\mathfrak{q}_{n-1}}) = 1$ by the *Hauptidealsatz* (Theorem 5.29), whereas $\operatorname{ht}(\overline{\mathfrak{p}_n}) \geq 2$ by virtue of the strict inclusions $\mathfrak{p}_{n-2} \subset \mathfrak{p}_{n-1} \subset \mathfrak{p}_n$. Hence $\mathfrak{q}_{n-1} \neq \mathfrak{p}_n$, completing the proof. □

Theorem 6.2 (Generalised *Hauptidealsatz*)**.** *Suppose A be a Noetherian ring and $I \subset A$ is a proper ideal generated by r elements. Let $\mathfrak{p} \subset A$ be a prime ideal that is minimal among those including I. Then,* $\operatorname{ht}(\mathfrak{p}) \leq r$. *Hence* $\operatorname{ht}(I) \leq r$.

[18] This is a standard application of Zorn's Lemma. See §2 Preliminaries, Commutative algebra for a reference.

If $\mathfrak{p} \subset A$ is a prime ideal which is minimal among those including I, we sometimes say that $\mathfrak{p}$ is a *minimal prime ideal over* I.

Proof. By induction on r. If $r = 0$ then I is the zero ideal and a minimal prime ideal over I is just a minimal prime ideal. Evidently, a minimal prime ideal must have height zero. The case $r = 1$ is the *Hauptidealsatz* (Theorem 5.29).

Suppose $r \geq 2$ and the assertion has been established for $0, 1, \ldots, r-1$. Let $f_1, \ldots, f_r$ be a family of generators of I. The image of I in $A/\langle f_r \rangle$ is generated by the images of $f_1, \ldots, f_{r-1}$. Moreover, the image of $\mathfrak{p}$ in $A/\langle f_r \rangle$ is minimal over $\bar{I}$. Hence $\mathrm{ht}(\bar{\mathfrak{p}}) \leq r-1$ by the inductive hypothesis.

Let n be the height of $\mathfrak{p}$. If $n = 0$ then we are done, so assume that $n \geq 1$. Let $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$ be a strictly ascending chain of prime ideals of A. Since $f_r \in \mathfrak{p}$, by Lemma 6.1 there exists a strictly ascending chain $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n = \mathfrak{p}$ such that $f_r \in \mathfrak{q}_1$. The image of this chain in $A/\langle f \rangle$ is also strictly ascending, hence $n - 1 \leq \mathrm{ht}(\bar{\mathfrak{p}}) \leq r - 1$, so that $n \leq r$. □

Suppose $(A, \mathfrak{m}, k)$ is a local ring. The A-module $\mathfrak{m}/\mathfrak{m}^2$ is annihilated by $\mathfrak{m}$, and thus we may view $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over $k = A/\mathfrak{m}$.

Corollary 6.3. *Let us keep the notation above. If A is Noetherian, then both* $\dim(A)$ *and* $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$ *are finite and* $\dim(A) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2)$.

Proof. Write $\mathfrak{m} = \langle x_1, \ldots, x_r \rangle$. The images of the x_i in $\mathfrak{m}/\mathfrak{m}^2$ span $\mathfrak{m}/\mathfrak{m}^2$. Hence the dimension of $\mathfrak{m}/\mathfrak{m}^2$ is at most r; in particular, it is finite.

Let $y_1, \ldots, y_n \in \mathfrak{m}$ be such that the images of the y_i form a basis of $\mathfrak{m}/\mathfrak{m}^2$. By Nakayama's Lemma, $y_1, \ldots, y_n$ generate $\mathfrak{m}$. Hence $\dim(A) = \mathrm{ht}(\mathfrak{m}) \leq n = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ by the Generalised *Hauptidealsatz* (Theorem 6.2). □

Remark 6.4. It is possible for a general Noetherian ring to have infinite Krull dimension. Nagata famously gave an example in the appendix to his commutative algebra book ([Nag62]). His example is also discussed in Chapter 9 of the modern source [Rei95].

Theorem 6.5. *Suppose A is a Noetherian local ring, and $f \in A$ is a non-unit. Then,* $\dim(A/\langle f \rangle) \geq \dim(A) - 1$. *Moreover, equality holds if f is not contained in any minimal prime ideal of A.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of A, and $n = \dim A$. Let us first show that $\dim(A/\langle f \rangle) \geq n - 1$. By hypothesis, there exists a strictly ascending chain of prime ideals of A

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{m}\,.$$

Since $f \in \mathfrak{m}$, by Lemma 6.1 there exists a strictly ascending chain of prime ideals

$$\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n = \mathfrak{m}\,.$$

The image of this sequence in $A/\langle f\rangle$ remains strictly ascending, so $\dim(A/\langle f\rangle) \geq n-1$.

Now suppose further that f is not contained in a minimal prime ideal of A. By the *Hauptidealsatz* (Theorem 5.29), every prime ideal $\mathfrak{p} \subset A$ that is minimal among those containing f must satisfy $\operatorname{ht}(\mathfrak{p}) = 1$ (note that $\operatorname{ht}(\mathfrak{p}) \neq 0$ since otherwise $\mathfrak{p}$ would be a minimal prime ideal).

To show that $\dim(A/\langle f\rangle) \leq n-1$, it suffices to show that given any strictly ascending chain of prime ideals in $A/\langle f\rangle$, there is a strictly ascending chain in A with one more prime ideal. If $\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \cdots \subset \mathfrak{q}_r$ is a chain in $A/\langle f\rangle$, then it lifts back to a chain in A such that $\langle f\rangle \subseteq \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_r$. If $\mathfrak{p}_1$ is not minimal among those prime ideals containing f, then the chain can be extended by adjoining a prime ideal $\mathfrak{p}_0$ such that $\langle f\rangle \subseteq \mathfrak{p}_0 \subset \mathfrak{p}_1$; if $\mathfrak{p}_1$ *is* minimal among those prime ideals containing f, then it has height one, so again the chain can be extended. □

Lemma 6.6. *Suppose A is a Noetherian ring, $\mathfrak{m} \subset A$ is a maximal ideal, $\mathfrak{n} \subset A[t_1, \ldots, t_n]$ is a maximal ideal, and $n \cap A = \mathfrak{m}$. Then* $\operatorname{ht}(\mathfrak{n}) = \operatorname{ht}(\mathfrak{m}) + n$.

Proof. Lemma 5.16. of [Liu06]. □

Corollary 6.7. $K[z_1, \ldots, z_n]$ *has dimension n.*

Proof. We have already seen that $\dim(K[z_1, \ldots, z_n]) \geq n$ (§2 Preliminaries, Topology). On the other hand, by taking $A = K$ in the above lemma, we see that $\dim(K[z_1, \ldots, z_n]) \leq n$. □

Theorem 6.8. *Suppose A is a finitely generated algebra over a field k. If A is an integral domain, and $\mathfrak{p} \subset A$ is a prime ideal, then*

- $\operatorname{ht}(\mathfrak{p}) + \dim(A/\mathfrak{p}) = \dim(A)$,
- *If $\mathfrak{p}$ is maximal, then* $\dim(A_{\mathfrak{p}}) = \dim A$.

Proof. We omit the proof, since it relies upon a field-theoretic characterisation of dimension that we do not have the space to develop here. See Proposition 5.23. of [Liu06]. □

Corollary 6.9. *Suppose X is an affine variety, and $\mathfrak{p} \in X$ is a closed point. Then* $\dim(X) = \dim(\mathcal{O}_{X,\mathfrak{p}})$.

Proof. Write $X = \operatorname{Spec} A$ for some affine domain A. Then $\dim(X) = \dim(A)$, whereas $\dim(\mathcal{O}_{X,\mathfrak{p}}) = \dim(A_{\mathfrak{p}})$. The result is now immediate from the above theorem. □

Example 6.10. Suppose $f \in A := K[z_1, \dots, z_n]$ is an irreducible polynomial. Then $\langle f \rangle$ has height one, so $\dim(A/\langle f \rangle) = \dim(A) - 1$. More generally, if $\mathfrak{p} \subset A$ is a prime ideal generated by r elements, then $\text{ht}(\mathfrak{p}) \leq r$, so $n - r \leq \dim(A/\mathfrak{p})$. On the other hand, $\dim(A/\mathfrak{p}) \leq n$, so $n - r \leq \dim(A/\mathfrak{p}) \leq n$.

7 The tangent space

In §3 Motivation, we defined the tangent space of a classical affine variety $X \subseteq \mathbb{A}^n(K)$ at a point $p = (a_1, \ldots, a_n) \in X$ to be the affine subspace $T_pX \subseteq K^n$ given by

$$T_pX = \bigcap_{f \in I(X)} Z\left(\left.\frac{\partial f}{\partial z_1}\right|_p (z_1 - a_1) + \cdots + \left.\frac{\partial f}{\partial z_n}\right|_p (z_n - a_n) \right).$$

This definition of the tangent space has the advantage that it is concrete and easy to motivate: it is the linear subvariety of $\mathbb{A}^n(K)$ that best approximates X near p, as in advanced calculus. It has the disadvantage that it requires X to live inside $\mathbb{A}^n(K)$.

We remedy this issue by proposing a more intrinsic definition of the tangent space of a (scheme-theoretic) affine variety X that does not require X to be embedded in affine space. In the special case where $X \subseteq \mathbb{A}^n_K$ is a closed subvariety and $\mathfrak{p} \in X$ is a closed point, we shall establish that the classical and modern definitions of the tangent space are essentially equivalent.

If $(A, \mathfrak{m}, k)$ is a Noetherian local ring, then the *(Zariski) cotangent space of* A is the k-vector space $\mathfrak{m}/\mathfrak{m}^2$. The *(Zariski) tangent space of* A is defined as the dual space $(\mathfrak{m}/\mathfrak{m}^2)^\vee$ of $\mathfrak{m}/\mathfrak{m}^2$. If X is an affine variety, and $x \in X$, then the *(Zariski) cotangent space of* X *at* x is simply the cotangent space of the local ring $\mathcal{O}_{X,x}$, i.e. the quotient $\mathfrak{m}_x/\mathfrak{m}_x^2$. The *(Zariski) tangent space of* X *at* x is the tangent space of $\mathcal{O}_{X,x}$; we denote it by $T_{X,x}$. We recall from Corollary 6.3 that the tangent space of a Noetherian local ring is finite-dimensional, so that the tangent and cotangent spaces are (non-canonically) isomorphic.

Roughly speaking, the motivation behind these mysterious definitions is as follows. If we interpret A as a ring of functions of some sort, then the passage from $\mathfrak{m}$ to $\mathfrak{m}/\mathfrak{m}^2$ kills the non-linear terms of the functions. This is entirely analogous to how in the classical definition of T_pX given above, we neglect the higher order terms in the Taylor expansion of f. As in differential geometry, the tangent and cotangent spaces are dual to each other. However, in algebraic geometry, it is the cotangent space which is in some sense the "basic" one: the tangent space is defined as the dual of the cotangent space, not the other way round.

Differentials

Let $p \in K^n$. If $f \in K[z_1, \ldots, z_n]$, we define the *differential of* f *at* p as the linear form $D_pf \in (K^n)^\vee$ given by

$$\begin{aligned} D_pf : K^n &\to K \\ (v_1, \ldots, v_n) &\mapsto \sum_{i=1}^n v_i \left.\frac{\partial f}{\partial z_i}\right|_p . \end{aligned}$$

The map $D_p : K[z_1, \ldots, z_n] \to (K^n)^\vee,\ f \mapsto D_p f$ is known as the *differential at* p. It is easy to see that D_p is K-linear and satisfies the product rule

$$D_p(fg) = f(p)D_p(g) + g(p)D_p(f)\,.$$

Suppose $I \subseteq K[z_1, \ldots, z_n]$ is a prime ideal, and $Y = Z(I) \subseteq \mathbb{A}^n(K)$ is the corresponding classical affine variety. If $p = (a_1, \ldots, a_n) \in Y$, then $T_p Y$ is just the translation of the linear subspace

$$W = \{v \in K^n \mid (D_p f)(v) = 0 \text{ for all } f \in I\} \subseteq K^n$$

to the point $p \in K^n$. Let $\mathfrak{p} \in X := \operatorname{Spec}(K[z_1, \ldots, z_n]/I)$ be the closed point corresponding to p, i.e. $\mathfrak{p}$ is the image of the ideal $\langle z_1 - a_1, \ldots, z_n - a_n\rangle \subseteq K[z_1, \ldots, z_n]$ under the map $K[z_1, \ldots, z_n] \to K[z_1, \ldots, z_n]/I$. We claim that there is a canonical linear isomorphism $W \cong T_{\mathfrak{p}} X$. This is the precise sense in which the classical and modern definitions of the tangent space are "the same".

The difficulty in proving this claim is understanding the cotangent space $\mathfrak{m}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^2$ to X at $\mathfrak{p}$. Let $A = K[z_1, \ldots, z_n]/I$. We learned in §4 Abstract varieties that there is an identification $\mathcal{O}_{X,\mathfrak{p}} = A_{\mathfrak{p}}$. Under this identification, $\mathfrak{m}_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$, and $\mathfrak{m}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^2 = \mathfrak{p}A_{\mathfrak{p}}/\mathfrak{p}^2 A_{\mathfrak{p}}$. Now the following theorem from commutative algebra comes to our aid.

Suppose A is a nonzero ring, $\mathfrak{p} \subset A$ is a maximal ideal, and $k = A/\mathfrak{p}$. Consider the A-module homomorphism $\varphi : \mathfrak{p} \to \mathfrak{p}A_{\mathfrak{p}}/\mathfrak{p}^2 A_{\mathfrak{p}}$ sending $x \in \mathfrak{p}$ to the class of $x/1$. The kernel of φ includes $\mathfrak{p}^2$, and so φ factors as

$$\mathfrak{p} \longrightarrow \mathfrak{p}/\mathfrak{p}^2 \xrightarrow{\widetilde{\varphi}} \frac{\mathfrak{p}A_{\mathfrak{p}}}{\mathfrak{p}^2 A_{\mathfrak{p}}}\,.$$

Since $\mathfrak{p}$ annihilates $\mathfrak{p}A_{\mathfrak{p}}/\mathfrak{p}^2 A_{\mathfrak{p}}$, the A-module $\mathfrak{p}A_{\mathfrak{p}}/\mathfrak{p}^2 A_{\mathfrak{p}}$ is naturally a k-vector space.

Lemma 7.1. *Let us keep the notation above. The map* $\widetilde{\varphi} : \mathfrak{p}/\mathfrak{p}^2 \to \mathfrak{p}A_{\mathfrak{p}}/\mathfrak{p}^2 A_{\mathfrak{p}}$ *is a linear isomorphism.*

Proof. This is surprisingly difficult to prove. In what follows, we assume that the reader is familiar with tensor products of modules: some references are Chapter 2 of [AM69], Chapter 1 of [Liu06], and Chapter 2 of [Eis95].

We first show that the unique A-algebra map $\psi : k \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is an isomorphism. We shall then show that $\widetilde{\varphi}$ is just the tensor product of ψ and the identity map on $\mathfrak{p}$, establishing the claim.

As k is a field and $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \neq 0$, the map ψ is injective. For surjectivity, let $b \in A$ and $s \in A \setminus \mathfrak{p}$. We claim that (the class of) b/s is in the image of ψ. Since $\mathfrak{p} + \langle s\rangle = \langle 1\rangle$, there exists an $a \in A$ and $m \in \mathfrak{p}$ such that $1 = as + m$. Hence in $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ we have

$$\frac{b}{s} = \frac{b}{s} - \frac{bm}{s} = \frac{b - bm}{s} = \frac{bas}{s} = \frac{ab}{1}\,.$$

But clearly $ab/1$ is in the image of ψ, so ψ is surjective.

Let 1 be the identity map on $\mathfrak{p}$. Since $\psi : k \to A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ and $1 : \mathfrak{p} \to \mathfrak{p}$ are both A-module isomorphisms, so too is their tensor product

$$\psi \otimes_A 1 : k \otimes_A \mathfrak{p} \to \frac{A_\mathfrak{p}}{\mathfrak{p}A_\mathfrak{p}} \otimes_A \mathfrak{p}\,.$$

Recall that if M is an A-module, and $I \subseteq A$ is an ideal, then the A-module M/IM, together with the bilinear map $f : A/I \times M \to M/IM$, $(\bar{r}, m) \mapsto \overline{rm}$, satisfies the universal property of the tensor product of A/I and M. Hence the tensor product $k \otimes_A \mathfrak{p}$ may be realised as $\mathfrak{p}/\mathfrak{p}^2$, and $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_A \mathfrak{p}$ may be realised as $\mathfrak{p}A_\mathfrak{p}/\mathfrak{p}^2A_\mathfrak{p}$. It is then not hard to check that $\psi \otimes_A 1 = \widetilde{\varphi}$, and the result follows.[19] □

Corollary 7.2. *Suppose $I \subseteq K[z_1, \ldots, z_n]$ is a prime ideal, $A = K[z_1, \ldots, z_n]/I$, and $X = \operatorname{Spec}(A)$. If $\mathfrak{p} \in X$ is a closed point, i.e. a maximal ideal $\mathfrak{p} \subset K[z_1, \ldots, z_n]/I$, then there is a canonical isomorphism $\mathfrak{p}/\mathfrak{p}^2 \cong \mathfrak{p}A_\mathfrak{p}/\mathfrak{p}^2A_\mathfrak{p}$. Hence the cotangent space of X at $\mathfrak{p}$ may be identified with the K-vector space $\mathfrak{p}/\mathfrak{p}^2$.*

Let $p = (a_1, \ldots, a_n) \in K^n$ be the point corresponding to $\mathfrak{p}$. In light of Corollary 7.2, our task is to exhibit a linear isomorphism from $W = \{v \in K^n \mid (D_p f)(v) = 0 \text{ for all } f \in I\}$ to $(\mathfrak{p}/\mathfrak{p}^2)^\vee$.

Lemma 7.3. *Let us keep the notation above. There is a linear isomorphism*

$$\begin{aligned} \phi : W &\to (\mathfrak{p}/\mathfrak{p}^2)^\vee \\ v &\mapsto \sum_{i=1}^n v_i \left.\frac{\partial}{\partial z_i}\right|_p . \end{aligned}$$

Hence the (modern) tangent space of $X \cong V(I) \subseteq \mathbb{A}^n_K$ at $\mathfrak{p}$ may be identified with the (classical) tangent space of $Y = Z(I) \subseteq \mathbb{A}^n(K)$ at p.

Proof. Let us first make sense of the definition of ϕ. If $v \in W$ and $f, g \in K[z_1, \ldots, z_n]$ are congruent modulo I, then it is immediate from the definition of W that $D_p(f)(v) = D_p(g)(v)$. Hence, given any $v \in W$, there is a map

$$\begin{aligned} \phi_v : \mathfrak{p} &\to K \\ f &\mapsto \sum_{i=1}^n v_i \left.\frac{\partial f}{\partial z_i}\right|_p . \end{aligned}$$

[19] Implicit in this proof is the following fundamental fact: If M and N are A-modules, then strictly speaking, there are many different pairs $(P, f : M \times N \to P)$ which satisfy the universal property of the tensor product of M and N. However, by abstract nonsense, any two such pairs are uniquely isomorphic to each other (in the appropriate category). Because of this, any reasonable theorem about tensor products is true regardless of how $M \otimes_A N$ is defined at the set-theoretic level.

The ideal $\mathfrak{p}^2$ is generated by the (images of the) quadratics $f_{ij} = (z_i - a_i)(z_j - a_j)$, where $i, j \in \{1, \ldots, n\}$. A straightforward computation shows that $\phi_v(f_{ij}) = 0$ for all i, j. Hence the kernel of ϕ_v includes $\mathfrak{p}^2$, and so we may regard ϕ_v as a linear form $\mathfrak{p}/\mathfrak{p}^2 \to K$. Then, by definition $\phi = v \mapsto \phi_v$.

We claim that there is an inverse ψ to ϕ given by

$$\psi : (\mathfrak{p}/\mathfrak{p}^2)^\vee \to W$$
$$t \mapsto (t(z_1 - a_1), \ldots, t(z_n - a_n)) .$$

Again, the main difficulty is in seeing that ψ is well-defined. We must show that if $t : \mathfrak{p}/\mathfrak{p}^2 \to K$ is a linear form, then $(t(z_1 - a_1), \ldots, t(z_n - a_n)) \in W$. Let $f \in I$. Then, by expanding f about the point $p \in K^n$, we can write

$$f = \left(\sum_{i=1}^{n} \left.\frac{\partial f}{\partial z_i}\right|_p (z_i - a_i) \right) + q \qquad (\star)$$

for some "higher-order" term $q \in K[z_1, \ldots, z_n]$; that is, when q is regarded as a polynomial in the variables $z_i - a_i$, all of its terms have total degree ≥ 2.

Let π be the map $K[z_1, \ldots, z_n] \to K[z_1, \ldots, z_n]/I$. Both of the summands on the right hand side of $(\star)$ lie in $\pi^{-1}(\mathfrak{p})$, and so we can think of the equality $(\star)$ as taking place in $\mathfrak{p}$. Then, by applying t to both sides, we see that

$$0 = t(f) = \left(\sum_{i=1}^{n} \left.\frac{\partial f}{\partial z_i}\right|_p t(z_i - a_i) \right) + t(q) .$$

But $q \in \mathfrak{p}^2$, hence

$$\left(\sum_{i=1}^{n} \left.\frac{\partial f}{\partial z_i}\right|_p t(z_i - a_i) \right) = 0 ,$$

and so $(t(z_1 - a_1), \ldots, t(z_n - a_n)) \in W$. It is then routine to check that ϕ and ψ are mutually inverse linear maps, and we are finished. $\square$

Example 7.4. Suppose C is the affine variety in $\mathbb{A}^2_K$ cut out by the equation $x^2 = y^3$, i.e. $C = \operatorname{Spec}(K[x, y]/\langle x^2 - y^3 \rangle)$. By Example 6.10, C is a curve. Let $\mathfrak{p}_0 \in C$ be the point corresponding to the origin of K^2. If $\mathfrak{p} \in C$ be a closed point with coordinates $p = (a, b) \in K^2$, then

$$D_p(x^2 - y^3) = (v, w) \mapsto 2av - 3b^2 w .$$

Hence

$$T_{C,\mathfrak{p}} \cong \{(v, w) \in K^2 \mid (D_p f)(v, w) = 0 \text{ for all } f \in \langle x^3 - y^3 \rangle\}$$
$$= \{(v, w) \in K^2 \mid 2av - 3b^2 w = 0\} ,$$

so that C is singular at $\mathfrak{p}$ if and only if $\mathfrak{p} = \mathfrak{p}_0$. In the case where $\mathfrak{p} = \mathfrak{p}_0$, the isomorphism of Lemma 7.3 sends the standard basis e_1, e_2 of K^2 to the basis $\left.\frac{\partial}{\partial x}\right|_{(0,0)}, \left.\frac{\partial}{\partial y}\right|_{(0,0)}$ of $(\mathfrak{p}/\mathfrak{p}^2)^\vee$.

Regularity

Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring. We have already seen (Corollary 6.3) that $\dim A$ is finite and $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \geq \dim(A)$. We say that A is a *regular local ring* if equality holds. (Note that regular local rings are Noetherian, by definition.) If $(A, \mathfrak{m})$ is a Noetherian local ring, then by Nakayama's Lemma, a family $x_1, \ldots, x_n$ of elements of $\mathfrak{m}$ generates $\mathfrak{m}$ if and only if the images of the x_i span $\mathfrak{m}/\mathfrak{m}^2$. We have thus established the following lemma.

Lemma 7.5. *A Noetherian local ring* $(A, \mathfrak{m})$ *is a regular local ring if and only if* $\mathfrak{m}$ *has a generating set with* $\dim(A)$ *elements.*

The use of the word "regular" in the above definition is no coincidence. If X is an affine variety, we say that X is *regular* at $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is a regular local ring. That is, X is regular at x if $\dim_{\kappa(x)}(T_{X,x}) = \dim(\mathcal{O}_{X,x})$. Notice that if $X \subseteq \mathbb{A}^n$ is a closed subvariety and $\mathfrak{p} \in X$ is a closed point, then in view of Corollary 6.9, to say that X is regular at $\mathfrak{p}$ simply means that $\dim_K(T_{X,\mathfrak{p}}) = \dim(X)$, whereas to say that X is singular at $\mathfrak{p}$ means that $\dim_K(T_{X,\mathfrak{p}}) > \dim(X)$. Thus the definition of regularity given here is a natural generalisation of the one given in §3 Motivation.

Lemma 7.6. *If R is a regular local ring, then R is an integral domain.*

Proof. We omit the proof for reasons of space, but it is not especially difficult. See Corollary 10.14 of [Eis95]. □

In accordance with the definitions given in §3, we say that an affine variety X is *singular* at $x \in X$ if X is not regular at x. To say that X is *regular* means that X is regular at all closed points; otherwise, we say that X is *singular*. The sets of regular and singular points are denoted by X_{reg} and X_{sing}, respectively.

Example 7.7. Suppose C is a curve in affine space. If C is normal, then by Theorem 5.3, for all $\mathfrak{p} \in C$ the local ring $\mathcal{O}_{C,\mathfrak{p}}$ is normal, one-dimensional, and Noetherian. Hence, C is a regular curve by Theorem 5.17. This illustrates the main idea in the proof that normal varieties are regular in codimension one. Conversely, suppose C is a regular curve. Then, C has one-dimensional regular local rings. Using Lemma 7.5 and Theorem 5.17 again, we see that C is normal.

Remark 7.8. Given that our varieties have non-closed points, it might seem natural to instead define a variety to be regular if it is regular at *all* points, not just all closed points. In fact, the definitions are equivalent. This is not too difficult to prove, provided that the reader is willing to accept on faith the following assertion.

Theorem 7.9. *The localization of a regular local ring at a prime ideal is again a regular local ring.*

This is a highly nontrivial result: it was first established in the 1950s by Auslander, Buchsbaum, and Serre, in one of the first major applications of homological methods to commutative algebra ([Ál12]). A proof can be found in [Mat86], Theorem 19.3.

Exercise 7.10. Prove that if an affine variety is regular at all closed points, then it is regular at all points. [Use Theorem 7.9 and the fact that every local ring at a non-closed point is the localization of a local ring at a closed point.]

It can be verified from the definition that an affine variety is regular at its generic point. In particular, every affine variety has at least one regular point. What is much less obvious is that every affine variety has a regular *closed* point. We shall use as a black box in this report, since the proof involves rather more field theory than we have needed so far.

Lemma 7.11. *Every affine variety has a regular closed point.*

Proof. Lemma 4.2.21 of [Liu06]. □

In practice, it is difficult to verify whether an affine variety X is regular at a point if one simply uses the definition. The following criterion plays a critical role in determining regularity, and is useful in both practical calculations and the proving of theorems.

Theorem 7.12 (Jacobian criterion)**.** *Suppose $I \subset K[z_1, \ldots, z_n]$ is a prime ideal, and $A = K[z_1, \ldots, z_n]/I$. Let $f_1, \ldots, f_r$ be a set of generators of I. Then, $X = \mathrm{Spec}(A)$ is regular at a closed point $\mathfrak{p} \in X$ with coordinates $p = (a_1, \ldots, a_n) \in K^n$ if and only if the Jacobian matrix*

$$J_p = \left(\frac{\partial f_i}{\partial z_j}\bigg|_p \right)_{1 \le i \le r,\ 1 \le j \le n}$$

satisfies the identity $\mathrm{rank}(J_p) = n - \dim(X)$. Moreover, X is singular at $\mathfrak{p}$ if and only if $\mathrm{rank}(J_p) < n - \dim(X)$.

Proof. Let us first do some linear algebra revision. First, if V is a finite-dimensional k-vector space, then there is a canonical isomorphism $V \cong V^{\vee\vee}$ given by sending $v \in V$ to the "evaluation map" $\mathrm{eval}_v : V^\vee \to k,\ f \mapsto f(v)$. Second, if E is a subspace of V, then the *annihilator* E^0 of E in V is the subspace of $V^\vee$ defined by

$$E^0 = \{f \in V^\vee \mid f(v) = 0 \text{ for all } v \in E\}.$$

If $f : V \to k$ is a linear form belonging to E^0, then f factors as $V \to V/E \xrightarrow{\widetilde{f}} k$. This gives rise to an isomorphism $E^0 \to (V/E)^\vee,\ f \mapsto \widetilde{f}$. From this we derive the formula

$$\dim_k(E) + \dim_k(E^0) = \dim_k(V).$$

Now set $k = K$, $V = (K^n)^\vee$, and $E = D_p I = \{D_p f \mid f \in I\}$. By the equivalence of the classical and modern tangent spaces (Lemma 7.3) and the above, we have

$$\begin{aligned} T_{\mathfrak{p}}X &\cong \{v \in K^n | (D_p f)(v) = 0 \text{ for all } f \in I\} \\ &= \{v \in K^n | \, \mathrm{eval}_v(D_p f) = 0 \text{ for all } f \in I\} \\ &\cong \{\Phi \in (K^n)^{\vee\vee} \mid \Phi(D_p f) = 0 \text{ for all } f \in I\} \\ &= (D_p I)^0 \,. \end{aligned}$$

Hence $\dim_K(D_p I) + \dim_K(T_p X) = n$. A little more linear algebra shows that the rank of J_p coincides with the vector space dimension of $D_p I$, whence the result. □

8 Normal varieties are regular in codimension one

We have at last developed enough theory to prove that normal varieties are regular in codimension one. Although we shall state and prove these results in the language of schemes, all of these results carry over to the case of classical varieties.

Let us first recall the algebraic analogue of the assertion that normal varieties are regular in codimension one (Theorem 5.17).

Theorem. *Let A be a ring which is not a field. The following are equivalent:*

(1) *A is a discrete valuation ring.*

(2) *A is a local principal ideal domain.*

(3) *A is a Noetherian local domain whose maximal ideal is principal.*

(4) *A is a local Dedekind domain, i.e. a local integral domain which is normal, one-dimensional, and Noetherian.*

Now that we have discussed regularity, we can add a fifth entry to the above list: A is a regular local ring of dimension one. This follows from the fact every regular local ring is an integral domain (Lemma 7.6), and a Noetherian local ring of dimension one is regular if and only if its maximal ideal is principal (Lemma 7.5).

Theorem 8.1. *Let A be a ring which is not a field. The following are equivalent:*

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(3) *A is a Noetherian local domain whose maximal ideal is principal.*

(4) *A is a local Dedekind domain, i.e. a local integral domain which is normal, one-dimensional, and Noetherian.*

(5) *A is a regular local ring of dimension one.*

More specifically, it is the implication (4) $\Rightarrow$ (5) that plays a critical role in the below proof – the other implications are important only to the extent that they helped us establish (4) $\Rightarrow$ (5).

The other key ingredient in the proof is the assertion that the regular locus of an affine variety is open and dense. Notice that without this assertion, it is not even clear what it *means* for normal varieties to be regular in codimension one – we only defined $\operatorname{codim}(Z, X)$ in the situation where $Z \subseteq X$ is a *closed* subset (§2 Preliminaries, Topology).

Lemma 8.2. *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring, $\mathfrak{p} \subset R$ is a prime ideal, and both $R/\mathfrak{p}$ and $R_\mathfrak{p}$ are regular local rings. Let $e = \dim(R_\mathfrak{p}) = \operatorname{ht}(\mathfrak{p})$. Suppose further that $\mathfrak{p}$ has a generating set with e elements, say $f_1, \ldots, f_e$. Then R is a regular local ring.*

Proof. By repeatedly applying Theorem 6.5, we see that

$$\dim(R/\mathfrak{p}) = \dim(R/\langle f_1, \ldots, f_e\rangle) \geq \dim(R/\langle f_1, \ldots, f_{e-1}\rangle) - 1 \geq \cdots \geq \dim(R) - e\,.$$

On the other hand, $\mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) \leq \dim(R)$, so $\dim(R/\langle\mathfrak{p}\rangle) \leq \dim(R) - e$. Hence $\dim(R/\mathfrak{p}) = \dim(R) - e$. Thus the image of $\mathfrak{m}$ in $R/\langle\mathfrak{p}\rangle$ can be generated by $\dim(R) - e$ elements (Lemma 7.5). But then $\mathfrak{m}$ can be generated by $\dim(R)$ elements, whence the result. □

Theorem 8.3. *The regular locus of an affine variety is open and dense.*

Proof. Suppose $X = \mathrm{Spec}(A)$, where $A = K[z_1, \ldots, z_n]/I$ for some prime ideal $I \subset K[z_1, \ldots, z_n]$. Let $f_1, \ldots, f_r$ generate I. From Example 6.10, we know that $n - r \leq \dim(X) \leq n$.

Consider the matrix M with coefficients in $K[z_1, \ldots, z_n]$ given by

$$M = \left(\frac{\partial f_i}{\partial z_j}\right)_{1\leq i\leq r,\ 1\leq j\leq n}.$$

Let J be the ideal of A generated by the images of the minors of M of order $n - \dim(X)$. We shall show that $X_{\mathrm{sing}} = V(J)$.

By the Jacobian criterion (Lemma 7.12), for every closed point $\mathfrak{p} \in \mathrm{Spec}\,A$ with coordinates $p = (a_1, \ldots, a_n) \in K^n$, we have

$$\begin{aligned}
\mathfrak{p} \supseteq J &\iff \text{the minors of order } n - \dim(X) \text{ of } M \text{ vanish at } p \\
&\iff \text{all } (n - \dim X) \times (n - \dim X) \text{ submatrices of } J_p \text{ have determinant zero} \\
&\iff \mathrm{rank}(J_p) < n - \dim(X) \\
&\iff X \text{ is singular at } \mathfrak{p}.
\end{aligned}$$

Now suppose $\mathfrak{q} \in X$ is a singular point. We claim that if $\mathfrak{m} \in X$ is a closed point such that $\mathfrak{m} \supseteq \mathfrak{q}$, then $\mathfrak{m}$ is a singular point. Indeed, if $\mathfrak{m}$ were a regular point, then $A_\mathfrak{m}$ would be a regular local ring, hence $A_\mathfrak{q} \cong (A_\mathfrak{m})_{\mathfrak{q}A_\mathfrak{m}}$ would be a regular local ring (Theorem 7.9). Since every prime ideal of A is the intersection of the maximal ideals including it (§2 Preliminaries, Commutative algebra), it follows that $X_{\mathrm{sing}} \subseteq V(J)$.

It remains to be shown that $X_{\mathrm{reg}} \subseteq X \setminus V(J)$. If $\mathfrak{p} \in X_{\mathrm{reg}}$, then the closed subvariety $V(\mathfrak{p}) \subseteq X$ must contain a regular closed point $\mathfrak{m}$ (Theorem 7.11). We claim that $\mathfrak{m}$ is regular when viewed as a point of X. Since $V(\mathfrak{p}) \cong \mathrm{Spec}(A/\mathfrak{p})$, it follows that $A_\mathfrak{m}/\mathfrak{p}A_\mathfrak{m} \cong (A/\mathfrak{p})_{\overline{\mathfrak{m}}}$ is a regular local ring. Moreover, $(A_\mathfrak{m})_{\mathfrak{p}A_\mathfrak{m}} \cong A_\mathfrak{p}$ is a regular local ring. Hence by Lemma 8.2, $A_\mathfrak{m}$ is a regular local ring, so that $\mathfrak{m} \not\supseteq J$ and in particular $\mathfrak{p} \not\supseteq J$. □

Theorem 8.4. *Normal varieties are regular in codimension one: if X is a normal affine variety, then* $\mathrm{codim}(X_{\mathrm{sing}}, X) \geq 2$.

Proof. Our hard won results from commutative algebra show that X is regular at every point of codimension one, i.e. if $x \in X$ is such that $\operatorname{codim}(\overline{\{x\}}, X) = 1$, then $x \in X_{\mathrm{reg}}$. Indeed, from the equality $\operatorname{codim}(\overline{\{x\}}, X) = \dim(\mathcal{O}_{X,x})$, it follows that $\dim(\mathcal{O}_{X,x}) = 1$. Since X is normal, so too are its local rings (Theorem 5.3). Hence $\mathcal{O}_{X,x}$ is a local Dedekind domain, so that it is a regular local ring by Theorem 8.1.

Since the singular locus is a proper closed subset of X (Theorem 8.3) and X is irreducible, we can rule out the possibility that $\operatorname{codim}(X_{\mathrm{sing}}, X) = 0$. Suppose for contradiction that $\operatorname{codim}(X_{\mathrm{sing}}, X) = 1$. Then, X_{sing} has an irreducible component Z such that $\operatorname{codim}(Z, X) = 1$. Let ξ be the generic point of Z. Then,

$$\operatorname{codim}(\overline{\{\xi\}}, X) = \operatorname{codim}(Z, X) = 1\,,$$

so that $\xi \in X_{\mathrm{reg}}$ by the preceding paragraph. This contradiction establishes the theorem. □

9 Appendix

Sheaves and presheaves

Suppose X is a topological space with topology τ. Note that τ is preordered under inclusion, and thus we may view τ as a category. By a *presheaf (of rings)* on X, we mean a contravariant functor $\mathcal{F}$ from τ to the category CRing of rings. Unwinding definitions, this means that

- To every open set U in X, the functor $\mathcal{F}$ assigns a ring $\mathcal{F}(U)$.
- To every pair $V \subseteq U$ of open sets in X, the functor $\mathcal{F}$ assigns a ring homomorphism $\mathcal{F}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$. It is conventional to denote $\mathcal{F}_{U,V}$ by some other symbol, say ρ_{UV}.

Moreover, $\mathcal{F}$ is required to satisfy the following axioms:

- For every open set U, the homomorphism $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map.
- If $W \subseteq V \subseteq U$ are open sets, then $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$.

(In a similar manner, one can define presheaves of sets, groups, etc. However we shall never need these in this report.)

Although, at the formal level, a presheaf is simply a certain kind of functor, the prototypical examples of presheaves come from geometry, and it is hardly sensible to ignore them.

Example 9.1. If X is an arbitrary topological space, then we can define a presheaf on X by letting $\mathcal{F}(U) = C(U, \mathbb{R})$, the ring of continuous functions $U \to \mathbb{R}$. If $V \subseteq U$ are open sets, then $\rho_{UV} : C(U, \mathbb{R}) \to C(V, \mathbb{R})$ is defined by $f \mapsto f|_V$.

Example 9.2. We can modify Example 9.1 by instead setting $\mathcal{F}(U) = C_b(U, \mathbb{R})$, the ring of continuous bounded functions $U \to \mathbb{R}$.

Example 9.3. If $X = \mathbb{R}$, we can let $\mathcal{F}(U)$ be the ring of differentiable/smooth/analytic functions $U \to \mathbb{R}$. If $X = \mathbb{C}$, we can let $\mathcal{F}(U)$ be the ring of holomorphic functions $U \to \mathbb{C}$. Again, the map ρ_{UV} is defined by $f \mapsto f|_V$.

These examples have a common thread: in every case, $\mathcal{F}(U)$ is a certain class of functions on U, and the map ρ_{UV} restricts the functions on U to functions on V. It should therefore come as no surprise to the reader that the homomorphism ρ_{UV} is suggestively called the *restriction map* from U to V. Moreover, if $f \in \mathcal{F}(U)$, we often write $f|_V$ for the image of ρ_{UV} under f.[20] On the other hand, there is no requirement in the definition of a presheaf that $\mathcal{F}(U)$ is a ring of functions, or that ρ_{UV} "restricts" the elements of $\mathcal{F}(U)$ in any literal sense. The next example illustrates this nicely.

[20] Note that in the notation $f|_V$, the dependence on U has been suppressed; if there is any chance of confusion, we shall thus revert to using $\rho_{UV}(f)$.

Example 9.4. Fix a ring A and a topological space X. We can define a presheaf on X by letting $\mathcal{F}(U) = A$ for every nonempty open set $U \subseteq X$; if U is empty, we let $\mathcal{F}(U)$ be the zero ring. Given a pair $V \subseteq U$ of open sets, let ρ_{UV} be the identity mapping on A if V is nonempty; if V is empty, let ρ_{UV} be the zero map.

In spite of this example, it is still useful to think of the elements of $\mathcal{F}(U)$ as being "functions" in some sense. From this point of view, the axioms of a presheaf assert that the restriction operator $\rho_{UV} = f \mapsto f|_V$ behaves formally like the ordinary restriction operator on functions.

A *sheaf (of rings)* on X is a presheaf $\mathcal{F}$ on X such that for every open set $U \subseteq X$ and open cover $\{U_i\}_i$ of U, the following conditions are satisfied:

(1) If $f \in \mathcal{F}(U)$ is such that $f|_{U_i} = 0$ for every index i, then $f = 0$.

(2) If $f_i \in \mathcal{F}(U_i)$ for each index i, and $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every pair of indices i, j, then there exists an $f \in \mathcal{F}(U)$ such that the restriction of f to U_i equals f_i for each i.

Roughly speaking, these properties mandate that the functions are determined by local data. The first says that if f vanishes locally on an open set, then it vanishes everywhere. The second says that functions can be glued along open sets.

Example 9.5. The concepts of continuity, differentiability, and smoothness are local, and correspondingly the presheaves of continuous/differentiable/smooth functions are in fact sheaves. On the other hand, the presheaf of continuous bounded functions is not a sheaf in general, since the gluing axiom (2) fails. For example, if $X = \mathbb{R}$, then the inclusion mappings $f_n : (-n, n) \to \mathbb{R}$, for $n \in \mathbb{N}$, cannot be glued together, since the identity mapping on $\mathbb{R}$ is unbounded. We leave it to the reader to decide for themselves whether the remaining examples of presheaves are sheaves.

Remark 9.6. Some authors explicitly state as an axiom that a sheaf $\mathcal{F}$ must send the empty set to the zero ring. A careful reading of the above definition shows that this axiom is in fact redundant. Indeed, consider the empty cover $\{U_i\}_{i \in \varnothing}$ of the empty set. Given an $f \in \mathcal{F}(\varnothing)$, it is vacuously true that $f|_{U_i} = 0$ for every index $i \in \varnothing$. Hence $f = 0$ by the first axiom of a sheaf. (Other authors require that a *presheaf* must send the empty set to the zero ring. However this only has the effect of making the category of presheaves on a space less well-behaved.)

Suppose $\mathcal{F}$ is a presheaf on X, and $x \in X$. Consider the collection $\mathcal{C}_x$ of pairs (f, U), where U is an open neighbourhood of x, and $f \in \mathcal{F}(U)$. We say that two pairs (f, U) and (g, V) in $\mathcal{C}_x$ are *equivalent* if there is an open neighbourhood W of x such that $W \subseteq U \cap V$ and $f|_W = g|_W$. The equivalence class of (f, U) is known as the *germ of* (f, U) *at* x. Provided that the open set U is clear from context, we often write f_x for the germ of (f, U) at x.

Example 9.7. The concept of germs is very natural in the setting of differential calculus. Suppose $\mathcal{F}$ is the sheaf of differentiable functions on $\mathbb{R}^n$, say, and U and V are open neighbourhoods of a point $p \in \mathbb{R}^n$. Then, two pairs $(f : U \to \mathbb{R}, U)$ and $(g : V \to \mathbb{R}, V)$ are equivalent if and only if there is an open neighbourhood $W \subseteq \mathbb{R}^n$ of p such that $f(x) = g(x)$ for all $x \in W$. It thus makes sense to speak of the derivative of a germ at p, since any two representatives of the germ will agree on some open neighbourhood of p, and hence their derivatives will coincide. In spite of this, it only makes sense to speak of the *value* of the germ at the point $p \in \mathbb{R}^n$.

The *stalk of $\mathcal{F}$ at x* is defined as the collection of germs at x; that is, it is the collection $\mathcal{C}_x$ modulo the equivalence relation defined above. We write $\mathcal{F}_x$ for the stalk at x. The stalk $\mathcal{F}_x$ inherits a ring structure from the presheaf $\mathcal{F}$: if ξ and η are germs at x, then let (f, U) and (g, V) be representatives of ξ and η respectively, and let $W \subseteq U \cap V$ be an open set on which f and g coincide; then, $\xi + \eta$ is defined as the germ of $(f + g, W)$. It is routine to check that this is independent of the choice of representatives. Multiplication of germs is defined in an analogous way.[21]

Suppose $\mathcal{F}$ and $\mathcal{G}$ are presheaves on X. Then, a *morphism of presheaves (on X)* from $\mathcal{F}$ to $\mathcal{G}$ is a natural transformation α from $\mathcal{F}$ to $\mathcal{G}$. Unwinding definitions again, this means that α is a family $\{\alpha_U\}_U$ of ring homomorphisms $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that the following diagram commutes for every pair of open sets $V \subseteq U$:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\rho_{UV}} & \mathcal{F}(V) \\ {\scriptstyle \alpha_U}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_V} \\ \mathcal{G}(U) & \xrightarrow[\widetilde{\rho}_{UV}]{} & \mathcal{G}(V) \end{array}$$

where $\widetilde{\rho}_{UV}$ denotes the restriction map of $\mathcal{G}$. If $\mathcal{F}$ and $\mathcal{G}$ both happen to be sheaves, we also say that α is a morphism of sheaves.

If $\alpha : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then there is a ring homomorphism $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ sending the germ of (f, U) at x to the germ of $(\alpha_U(f), U)$ at x; the commutativity of the above diagram ensures that α_x is well-defined.

If $\alpha : \mathcal{F} \to \mathcal{G}$ and $\beta : \mathcal{G} \to \mathcal{H}$ are morphisms of presheaves on a space X, then they can be composed in an obvious way, and the same is true of sheaves. We obtain the categories of sheaves and presheaves on a fixed space X, respectively. One potentially confusing feature of these categories is that their *objects* are defined as contravariant *functors* from the category of open sets on X to the category of rings.

[21] The stalk $\mathcal{F}_x$ also admits a categorical interpretation, as a certain inductive limit (§2 Preliminaries, Category theory). Let $\mathcal{C}$ be the collection of open neighbourhoods of x. Order $\mathcal{C}$ by reverse inclusion, so that $U \leq V \Leftrightarrow U \supseteq V$ for all $U, V \in \mathcal{C}$. Then

$$(\mathcal{F}(U)_{U \in \mathcal{C}}, (\rho_{UV})_{U \leq V})$$

is an inductive system in CRing. The stalk $\mathcal{F}_x$ (together with the maps $\mathcal{F}(U) \to \mathcal{F}_x, f \mapsto f_x$) is the inductive limit of this system.

Thus far, we have only considered sheaves and presheaves on a fixed topological space X. However, one powerful aspect of the theory is that presheaf structures can be transferred along continuous maps. Suppose $f : X \to Y$ is continuous, and $\mathcal{F}$ is a presheaf on X. The *pushforward*, or *direct image* of $\mathcal{F}$ under f is the presheaf $f_*\mathcal{F}$ on Y defined by $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ for every open set $U \subseteq X$, with restriction maps $\widetilde{\rho}_{UV} = \rho_{f^{-1}(U)f^{-1}(V)}$.[22]

Example 9.8. Suppose X and Y are topological spaces, and $f : X \to Y$ is a continuous map. Let $\mathcal{F}$ be the sheaf of continuous functions on X. Then, $f_*\mathcal{F}$ assigns to each open set $U \subseteq Y$ the ring $C(f^{-1}(U), \mathbb{R})$, and to each pair of open sets $V \subseteq U$, the restriction map (in the conventional sense) $C(f^{-1}(U), \mathbb{R}) \to C(f^{-1}(V), \mathbb{R})$.

Another simpler way to obtain presheaves on a different space is to restrict them. If X is a topological space, $U \subseteq X$ is an open subset, and $\mathcal{F}$ is a presheaf on X, then the *restriction of $\mathcal{F}$ to U* is the presheaf $\mathcal{F}|_U$ on U such that $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for every open subset $V \subseteq U$, and restriction maps $\widetilde{\rho}_{VW} = \rho_{VW}$ for every pair of open subsets $V \subseteq W$ in U. If $\mathcal{F}$ is a sheaf, then so too is $\mathcal{F}|_U$.

Exercise 9.9. Explain what might go wrong if we attempted to define the restriction of a presheaf to a non-open subset.

[22] Given a presheaf $\mathcal{G}$ on Y, we can also define the pullback (or inverse image) $f^*\mathcal{G}$. However, the construction is more involved, and we never need it in this report.

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