Reconstructing certain quiver flag varieties from a tilting bundle

submitted by

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for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

November 2018

Declaration of any previous Submission of the Work

The material presented here for examination for the award of a higher degree by research has not been incorporated into a submission for another degree.

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James Green

Declaration of Authorship

I am the author of this thesis, and the work described herein was carried out by myself personally, with the exception of Chapter 3 which contains a research article that originated from collaboration with my supervisor, Alastair Craw.

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James Green

This declaration concerns the article entitled:										
Reconstructing toric quiver flag varieties from a tilting bundle										
Publication status (tick one)										
draft manuscript		**Submitted**		**In review**		**Accepted**		**Published**	✓	
Publication details (reference)	*European Journal of Mathematics* **4** (2018), no.1, 185-199									
Candidate's contribution to the paper (detailed, and also given as a percentage).	The candidate contributed to/ considerably contributed to/predominantly executed the… Formulation of ideas: 50% Design of methodology: 50% Experimental work: N/A Presentation of data in journal format: N/A									
Statement from Candidate	This paper reports on original research I conducted during the period of my Higher Degree by Research candidature.									
Signed						**Date**				

Abstract

Given a quiver flag variety Y equipped with a tilting bundle E, a construction of Craw, Ito and Karmazyn [CIK18] produces a closed immersion $f_E \colon Y \to \mathcal{M}(E)$, where $\mathcal{M}(E)$ is the fine moduli space of cyclic modules over the algebra $\mathrm{End}(E)$. In this thesis we present two classes of examples where f_E is an isomorphism. Firstly, when Y is toric and E is the tilting bundle from [Cra11]; this generalises the well-known fact that $\mathbb{P}^n$ can be recovered from the endomorphism algebra of $\bigoplus_{0 \leq i \leq n} \mathcal{O}_{\mathbb{P}^n}(i)$. Secondly, when $Y = \mathrm{Gr}(n, 2)$, the Grassmannian of 2-dimensional quotients of $\Bbbk^n$ and E is the tilting bundle from [Kap84]. In each case, we give a presentation of the tilting algebra $A = \mathrm{End}(E)$ by constructing a quiver Q' such that there is a surjective $\Bbbk$-algebra homomorphism $\Phi \colon \Bbbk Q' \to A$, and then give an explicit description of the kernel.

Acknowledgements

First and foremost, I would like to thank my supervisor, Alastair Craw. His enthusiasm for mathematics has been a constant source of inspiration that has ensured my motivation and enjoyment of the project from start to finish. This has stayed true despite extended periods of illness; may his current good health continue for a long time to come. I also owe a huge thanks to David Calderbank, who generously stepped in when Ali was unable to be there. I could not have hoped to work with a better pair of supervisors.

Many thanks to Ed Segal and Gregory Sankaran for their helpful comments, to Will Donovan for a useful conversation that brought an example of Buchweitz, Leuschke and Van den Bergh to our attention, and to Jerzy Weyman and Victor Reiner for some useful emails. I would also like to thank the EPSRC for funding this project.

Thanks to the memory of my grandfather's cousin Professor James Ron Leigh, who shared many fun and inspirational emails with me about mathematics. I will forever treasure our one meeting together.

I am fortunate to have been surrounded by so many wonderful people during the last four years in the Mathematics Department at Bath. Thanks to my office mates old and new, particularly Nathan Prabhu-Naik, James Roberts and Matthew Pressland during my first year and Leonard Hardiman more recently for many helpful conversations. Also, many thanks to the famous Wednesday group: Matt Thomas, Kieran Jarrett, Alice Davis, Bec Green, Leonard Hardiman, Dan Green, Joel Cawte and Jack Betteridge, for keeping me sane (i.e. not doing maths or playing bridge at least one night a week). We have shared a lot of great memories together.

Playing bridge has been a big part of my life over the last few years. Thanks to Alex Lawton for putting up with me in the early days, and to Ambrose Holmes-Mackie for answering my limitless supply of questions. The lovely people at Bath Bridge Club have been incredibly accommodating; many thanks to them, especially Paul Cooper and Pat Davies, for helping me create a bridge club at the university.

A special thanks to my good friends over the years: Richard Haslam, Richard Nash, Luke Deacon-Kent, Shaun Drury, Lauren Hucker, and Dorian Lidell.

Finally, my deepest thanks to my family for their endless love and support. To my parents Sal and John, sister Bec, and grandparents Pat, Jackie and Rowland: I couldn't have done this without you. Love you always.

For Mum, Dad and Bec

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Chapter 1

Introduction

We begin by briefly outlining some background material before discussing the main results. Throughout, let $\Bbbk$ be an algebraically closed field of characteristic zero.

Let Q be a finite acyclic quiver with a unique source, vertex set $Q_0 = \{0, \ldots, \ell\}$ and arrow set Q_1. Let $\underline{r} := (1, r_1, \ldots, r_\ell) \in \mathbb{N}^{\ell+1}$, and denote by $\mathrm{Rep}(Q, \underline{r})$ the space of representations of Q with dimension vector $\underline{r}$, of which the isomorphism classes are precisely the orbits under the action of the group $G := \prod_{i=0}^{\ell} \mathrm{GL}(r_i)$ induced by conjugation.

Definition 2.1. The *quiver flag variety* associated to the pair $(Q, \underline{r})$ is the GIT quotient

$$Y := \mathrm{Rep}(Q, \underline{r}) /\!\!/_{\chi} G$$

for the special choice of linearisation $\chi := (-\sum_{i=1}^{\ell} r_i, 1, \ldots, 1) \in G^\vee$.

Craw [Cra11] showed that the variety Y is non-empty if for all $i \in Q_0$ we have $r_i \leq s_i := \sum_{\{a \in Q_1 | \mathrm{h}(a) = i\}} r_{\mathrm{t}(a)}$, and in this case proved that Y has many nice properties: it is a smooth, projective Mori Dream Space, and has an iterative structure as a height ℓ tower of Grassmann-bundles; this motivates the terminology *quiver flag varieties*. Additionally, these varieties are the framed quiver moduli of Nakajima [Nak96]. Examples of quiver flag varieties include Grassmannians, (partial) flag varieties of type-A and (towers of) certain projective space bundles.

Many properties of an algebraic variety X can be studied via its bounded derived category of coherent sheaves, $D^b(\mathrm{Coh}(X))$. Typically this category is difficult to work with, but the key results of Baer [Bae88] and Bondal [Bon90] prove that if X carries a *tilting bundle* E (see Definition 2.6) and $\mathrm{Rmod}(A)$

is the category of finitely generated right modules over the endomorphism algebra $A = \mathrm{End}_{\mathcal{O}_X}(E)$, then there is an equivalence of triangulated categories $D^b(\mathrm{Coh}(X)) \cong D^b(\mathrm{Rmod}(A))$. Hence, there is a strong motivation to find tilting bundles for varieties. Beilinson [Bei78] gave a tilting bundle for projective space and Kapranov [Kap84] gave one for the Grassmannian. Craw [Cra11] then generalised these results to give a tilting bundle for any quiver flag variety Y: let $\mathcal{W}_1, \dots, \mathcal{W}_\ell$ be the globally generated vector bundles given by the pullbacks of the tautological quotient bundles on each variety in the tower structure of Y; let $\mathrm{Young}(n, r)$ denote the set of Young diagrams with at most n columns and r rows, and for a Young diagram λ let $\mathbb{S}^\lambda \mathcal{W}_i$ be the image of the Schur functor on $\mathcal{W}_i^{\otimes|\lambda|}$.

Theorem 2.9 ([Cra11, Theorem 4.5])**.** *The vector bundle on Y given by*

$$E := \bigoplus_{1 \leq i \leq \ell,\ \lambda^{(i)} \in \mathrm{Young}(s_i - r_i, r_i)} \mathbb{S}^{\lambda^{(1)}} \mathcal{W}_1 \otimes \cdots \otimes \mathbb{S}^{\lambda^{(\ell)}} \mathcal{W}_\ell$$

is a tilting bundle. In particular, the bounded derived category of coherent sheaves on Y is equivalent to the bounded derived category of finite-dimensional modules over the endomorphism algebra $\mathrm{End}_{\mathcal{O}_Y}(E)$.

In general, let A be a finite dimensional associative $\Bbbk$-algebra and let $\mathbf{v}$ be an indivisible dimension vector. King [Kin94] defined a certain θ-stability criterion for A-modules and showed it is equivalent to Mumford's; see [MFK94, Section 2]. He was therefore able to construct the fine moduli space of θ-stable A-modules with dimension $\mathbf{v}$, denoted $\mathcal{M}(A, \mathbf{v}, \theta)$, as a GIT quotient. Now, generalising work of Craw-Smith [CS08] and Craw-Winn [CW13], given a scheme X with a collection of globally generated vector bundles $E_1, \dots, E_n$, Craw, Ito and Karmazyn [CIK18] construct the universal morphism $f_E : X \to \mathcal{M}(A, \mathbf{v}, \theta)$, where A is the endomorphism algebra of the bundle $E = \bigoplus_{i=0}^n E_i$ with $E_0 = \mathcal{O}_X$. This generalises the classical morphism from a scheme with a basepoint-free line bundle into its linear series. Hence, they call $\mathcal{M}(A, \mathbf{v}, \theta)$ the *multigraded linear series* and define

$$\mathcal{M}(E) := \mathcal{M}(A, \mathbf{v}, \theta).$$

By considering the case that X is a quiver flag variety Y and E is the tilting bundle from Theorem 2.9, using [CIK18, Theorem 2.6, Remark 2.8] we are able to deduce the following.

Theorem 2.11. *The universal property of $\mathcal{M}(E)$ gives a morphism*

$$f_E \colon Y \longrightarrow \mathcal{M}(E) \tag{1.1}$$

which is a closed immersion.

Hence we may embed Y, itself a moduli space, into $\mathcal{M}(E)$, another ambient moduli space.

It is natural to ask when f_E is an isomorphism, thereby providing a reconstruction of the quiver flag variety from a tilting bundle. This thesis provides two classes of examples: when Y is toric, and when $Y = \mathrm{Gr}(n, 2)$, the Grassmannian of 2-dimensional quotients of $\Bbbk^n$. The main tool is to define a quiver Q', which we call the *tilting quiver*, such that there is a surjective $\Bbbk$-algebra homomorphism $\Phi \colon \Bbbk Q' \twoheadrightarrow A$ defined by mapping concatenations of arrows to compositions of maps. Since $\Bbbk Q'/\ker(\Phi) \cong A$, we may regard points of $\mathcal{M}(E)$ as θ-stable representations of Q' with dimension vector $\mathbf{v}$ subject to the relations induced by $\ker(\Phi)$.

Reconstructing toric quiver flag varieties from a tilting bundle

The following result is from the paper [CG18], of which the author of this thesis is a co-author. Please see the declarations on the preliminary pages.

Fix a quiver Q with vertex set $\{0, \ldots, \ell\}$ satisfying the conditions prior to Definition 2.1. When the dimension vector $\underline{r} = (1, \ldots, 1)$, the group G is an algebraic torus and we therefore call Y a *toric quiver flag variety*. In this case, E is a direct sum of line bundles and the Grassmann-bundle tower structure becomes a tower of projective bundles. The main result is as follows.

Theorem 3.2. *Let Y be a toric quiver flag variety. Then the morphism $f_E \colon Y \to \mathcal{M}(E)$ from (1.1) is an isomorphism.*

A special case of Theorem 3.2 is that when $Y = \mathbb{P}^n$ we recover the result of Beilinson [Bei78] where $\mathbb{P}^n$ is reconstructed from the tilting bundle $\bigoplus_{0 \leq i \leq n} \mathcal{O}_{\mathbb{P}^n}(i)$; see Example 3.4. This theorem therefore provides further evidence that toric quiver flag varieties provide good multigraded analogues of projective space.

To prove Theorem 3.2 we give an alternative description of $\mathcal{M}(E)$ using the results of Craw and Smith [CS08]. In the toric setting, the tilting quiver Q' is given by the bound quiver of sections of E. We prove that the vertices of Q'

correspond to the integer points of a certain ℓ-dimensional cuboid in $\mathbb{Z}^\ell$; that the original quiver Q forms a full sub-quiver of Q'; and that the arrow set Q'_1 is given by translating the arrows of Q to everywhere they fit inside that cuboid. To borrow from Example 3.7, the toric quiver flag variety with original quiver (a) below has tilting quiver (b), and e_1, e_2, e_3 denotes the standard basis of $\mathbb{Z}^3$.

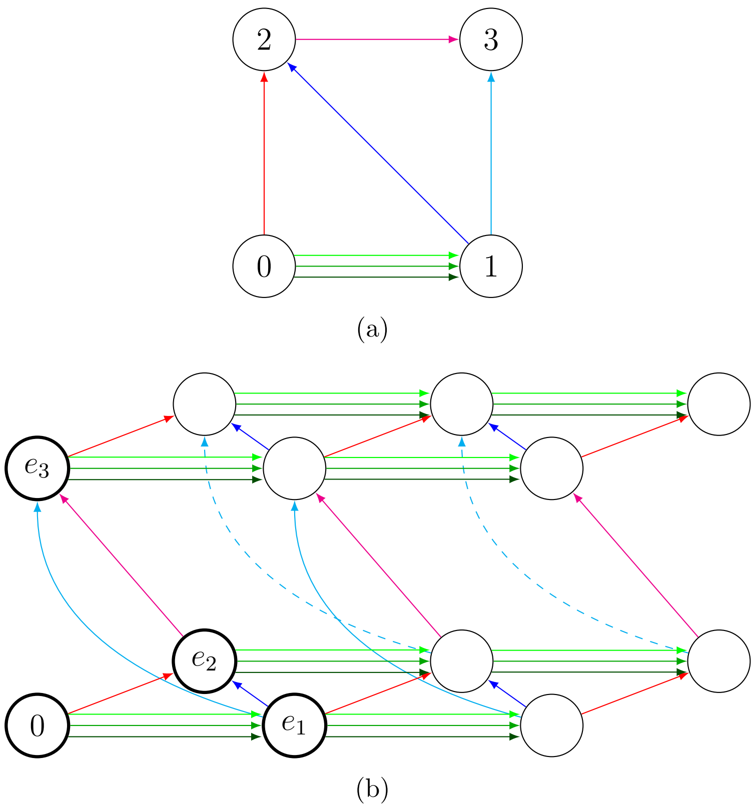


FIGURE 1.1: Example of a quiver Q (a) with tilting quiver Q' (b).

Each arrow of Q' carries a label corresponding to a torus-invariant divisor D_ρ, where $\rho \in \Sigma(1)$ is a ray of the toric fan for Y; this is represented by the different coloured arrows in the above figure. These labels determine the relations $I_R \subset \Bbbk Q'$ on Q'. Let $B_{Q'}$ be the irrelevant ideal that cuts out the θ-unstable locus of $\mathbb{A}_\Bbbk^{Q'_1}$. Then [CS08, Proposition 3.8] implies that $\mathcal{M}(E)$ is equal to the geometric quotient of $\mathbb{V}(I_R) \setminus \mathbb{V}(B_{Q'})$ by the action of G. Using [CS08, Proposition 4.3], we can also describe the image of Y under f_E as a geometric quotient. The proof of Theorem 3.2 then follows by showing that these geometric quotients coincide.

A presentation of Kapranov's tilting algebra for $\mathrm{Gr}(n, 2)$

For the remainder of this thesis we analyse the case that $Y = \mathrm{Gr}(n, 2) := \mathrm{Gr}(V, 2)$, the Grassmannian of 2-dimensional quotients of $V = \Bbbk^n$, although we do give some minor results for $\mathrm{Gr}(n,r)$ in general where stated. Chapter 4 begins with some background material on Schur powers, Littlewood-Richardson numbers and skew-Schur powers.

The tilting bundle E from Theorem 2.9 becomes

$$E = \bigoplus_{\lambda \in \mathrm{Young}(n-2,2)} \mathbb{S}^\lambda \mathcal{W},$$

where $\mathcal{W}$ is the rank 2 tautological quotient bundle of Y. This tilting bundle was originally given by Kapranov [Kap84], and as such we refer to $A = \mathrm{End}_{\mathcal{O}_Y}(E)$ as *Kapranov's tilting algebra.* Like in the toric case, the first step towards proving that $Y \cong \mathcal{M}(E)$ is to write down a presentation of A as a quiver with relations. In particular, we want to define the tilting quiver Q' such that there is a surjective $\Bbbk$-algebra homomorphism $\Phi\colon \Bbbk Q' \to A$. The ideal $\ker(\Phi)$ then defines the relations on the quiver.

Important note: Such a presentation was given by Buchweitz, Leuschke and Van den Bergh in [BLV16]. While they give the tilting quiver with spaces of relations for $\mathrm{Gr}(n,r)$ in general, their indirect approach to computing $\mathrm{End}_{\mathcal{O}_Y}(E)$ relies on heavy machinery and furthermore they do not give explicit generators for the kernel. Since we need these relations to prove that $Y \cong \mathcal{M}(E)$ in Chapter 6, in this thesis we give a new proof which produces an entirely self-contained description of A for the $\mathrm{Gr}(n,2)$ case and an explicit list of elements that generate the ideal $\ker(\Phi) \subset \Bbbk Q'$. We consider the advantages and disadvantages of this approach compared to [BLV16] in more depth in Section 5.3. For now, we continue to give an overview of the approach used in this thesis.

We begin by simplifying a result of Kapranov in the general case. For two Young diagrams λ, μ with λ contained in μ (write $\lambda \leq \mu$), denote by $\mathbb{S}^{\mu/\lambda}V$ the skew-Schur power of V corresponding to the skew diagram μ/λ (see Section 4.3).

Proposition 4.24. *Let $\mathcal{W}$ be the rank r tautological quotient bundle of* $\mathrm{Gr}(V,r)$ *and let $\lambda \leq \mu \in \mathrm{Young}(n-r,r)$. Then*

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W}) \cong \mathbb{S}^{\mu/\lambda}V.$$

Let $e_1, \ldots, e_r$ be the standard basis of $\mathbb{Z}^r$. Then an immediate corollary of Proposition 4.24 is that for all $\lambda, \mu \in \mathrm{Young}(n-r,r) \subset \mathbb{Z}^r$ with $\mu = \lambda + e_i$ for $1 \leq i \leq r$, we have $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W}) \cong V$; see Corollary 4.25.

Now fix $r = 2$. Writing $\mathcal{B} = \{u_1, \ldots, u_n\}$ for a basis of V, we get a basis of each of the homomorphism spaces $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^{\lambda+e_i} \mathcal{W})$ which we denote by

$f^\lambda_{u_1}, \dots, f^\lambda_{u_n}$ if $i = 1$ or $g^\lambda_{u_1}, \dots, g^\lambda_{u_n}$ if $i = 2$; we call these *f-type* and *g-type* maps. We then write down these maps explicitly and use them to prove the following key surjectivity result. Given $\lambda < \mu \in \mathrm{Young}(n-2,2)$, define $m_1 = \mu_1 - \lambda_1$ and $m_2 = \mu_2 - \lambda_2$, and for $0 \leq k \leq m_1 + m_2$ define the sequence of partitions $\tau_k = \lambda + ke_1$ if $0 \leq k \leq m_1$ and $\tau_k = \lambda + m_1 e_1 + (k - m_1)e_2$ if $m_1 \leq k \leq m_1 + m_2$.

Proposition 4.30. *Let $Y = \mathrm{Gr}(n, 2)$ and let $\lambda < \mu \in \mathrm{Young}(n-2, 2)$. Let τ_k be the sequence of partitions defined above. Then the composition map*

$$\Theta_{\lambda,\mu} \colon \bigotimes_{k=1}^{m_1+m_2} \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{\tau_{k-1}}\mathcal{W}, \mathbb{S}^{\tau_k}\mathcal{W}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{\lambda}\mathcal{W}, \mathbb{S}^{\mu}\mathcal{W})$$

is surjective.

Proposition 4.30 implies that any homomorphism in A may be decomposed as a linear combination of f-type and g-type maps. In light of this, we define the tilting quiver Q' to have vertex set corresponding to the irreducible summands of E, namely $\mathbb{S}^\lambda\mathcal{W}$ for each $\lambda \in \mathrm{Young}(n-2, 2)$, and arrow set corresponding to the collection of f-type and g-type maps between each summand. This produces a staircase-like diagram for Q'; see Figure 4.2. Then we can define a $\Bbbk$-algebra homomorphism

$$\Phi \colon \Bbbk Q' \longrightarrow A$$

by mapping each arrow to the appropriate f-type or g-type map, extending this by mapping paths to compositions of these maps, and finally extending linearly to all of $\Bbbk Q'$. We then prove the following.

Theorem 4.34. *Let $Y = \mathrm{Gr}(n, 2)$. Then the $\Bbbk$-algebra homomorphism $\Phi \colon \Bbbk Q' \to A$ is surjective.*

Hence we have $\Bbbk Q'/\ker(\Phi) \cong A$, which concludes Chapter 4. The goal of Chapter 5 is to describe the ideal $\ker(\Phi)$ explicitly, which determines the relations on Q'. By observing that elements of $\ker(\Phi)$ are relations between paths with the same head and tail and length at least two, it is enough to find a basis $K_{\lambda,\mu}$ for the kernels of the induced maps

$$\Phi_{\lambda,\mu} \colon e_\mu \Bbbk Q' e_\lambda \longrightarrow\!\!\!\!\rightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{\lambda}\mathcal{W}, \mathbb{S}^{\mu}\mathcal{W}),$$

where e_μ, e_λ are idempotents corresponding to the length zero paths at the vertices $\mu, \lambda \in Q'_0$ respectively. Define $P := \{(\lambda, \mu) \in {Q'_0}^2 \mid \ \lambda < \mu, |\mu| \geq |\lambda| + 2\}$. Then we have

$$\ker(\Phi) = \left(\bigcup_{(\lambda,\mu)\in P} K_{\lambda,\mu} \right).$$

In Section 5.1 we consider the subset $P_2 := \{(\lambda, \mu) \in {Q'_0}^2 \mid \lambda < \mu, |\mu| = |\lambda|+2\} \subset P$ and write down each set $K_{\lambda,\mu}$ for $(\lambda, \mu) \in P_2$ explicitly. Then we define the ideal

$$I\colon = \left(\bigcup_{(\lambda,\mu)\in P_2} K_{\lambda,\mu} \right).$$

It is clear that $I \subseteq \ker(\Phi)$, and by considering the remaining pairs $(\lambda, \mu) \in P \setminus P_2$ in Section 5.2 we prove that $I = \ker(\Phi)$. Denote the arrows of Q' by their images under Φ (the f-type and g-type maps). The main result is as follows.

Theorem 5.10. *Let $Y = \mathrm{Gr}(n, 2)$, let E be the tilting bundle (4.1) and let $A = \mathrm{End}_{\mathcal{O}_Y}(E)$. Let Q' be the quiver defined in Definition 4.31. Then the $\Bbbk$-algebra A is isomorphic to $\Bbbk Q'/I$ for the ideal*

$$I = \left(\bigcup_{(\lambda,\mu)\in P_2} K_{\lambda,\mu} \right),$$

where

(i) *if* $\mu = (\lambda_1 + 2, \lambda_2)$, $K_{\lambda,\mu} = \left\{ f_{u_j} f^{\lambda}_{u_i} - f_{u_i} f^{\lambda}_{u_j} \mid 1 \leq i, j \leq n \right\}$.

(ii) *if* $\mu = (\lambda_1, \lambda_2 + 2)$, $K_{\lambda,\mu} = \left\{ g_{u_j} g^{\lambda}_{u_i} - g_{u_i} g^{\lambda}_{u_j} \mid 1 \leq i, j \leq n \right\}$.

(iii) *if* $\lambda_1 = \lambda_2$ *and* $\mu = (\lambda_1 + 1, \lambda_1 + 1)$, $K_{\lambda,\mu} = \left\{ g_{u_j} f^{\lambda}_{u_i} + g_{u_i} f^{\lambda}_{u_j} \mid 1 \leq i, j \leq n \right\}$.

(iv) *if* $\lambda_1 > \lambda_2$ *and* $\mu = (\lambda_1 + 1, \lambda_2 + 1)$,

$$K_{\lambda,\mu} = \left\{ (\lambda_1 - \lambda_2)\, g_{u_j} f^{\lambda}_{u_i} - (\lambda_1 - \lambda_2 + 1)\, f_{u_i} g^{\lambda}_{u_j} + f_{u_j} g^{\lambda}_{u_i} \mid 1 \leq i, j \leq n \right\}.$$

We conclude Chapter 5 with an example. For $Y = \mathrm{Gr}(5, 2)$ we write down the tilting quiver and the relations described by I explicitly. We observe that the tilting quiver for $\mathrm{Gr}(4, 2)$ forms a full sub-quiver and the relations form a sublist of those for $\mathrm{Gr}(5, 2)$; this data recovers the example given by Buchweitz, Leuschke and Van den Bergh in [BLV15, Example 8.4].

Reconstructing $\mathrm{Gr}(n, 2)$ from a tilting bundle

For $Y = \mathrm{Gr}(n, 2)$, Theorem 5.10 gives us an explicit presentation of the endomorphism algebra $A \cong \Bbbk Q' / \ker(\Phi)$. Hence points of $\mathcal{M}(E) = \mathcal{M}(A, \mathbf{v}, \theta)$ are θ-stable representations of Q' with dimension vector $\mathbf{v}$ which satisfy the relations induced by $\ker(\Phi)$. Using this presentation we prove the following.

Theorem 6.1. *Let Y be the Grassmannian $\mathrm{Gr}(n, 2)$. Then the morphism $f_E \colon Y \to \mathcal{M}(E)$ is an isomorphism.*

By considering a particular sub-bundle E' of the tautological bundle on the moduli space $\mathcal{M}(E)$, we use the same multigraded linear series technology as in Theorem 2.11 to write down a morphism $f_{E'} \colon \mathcal{M}(E) \to \mathcal{M}(E')$ where our choice of E' yields $\mathcal{M}(E') \cong Y$. The approach is then to demonstrate that $f_{E'}$ is an inverse morphism for f_E, and this requires a technical result using induction as follows.

Begin by considering $Y = \mathrm{Gr}(4, 2)$. Using the presentation above and considering [Cra11, Section 2], we can write down points of $w \in \mathcal{M}(E)$ using two overlapping systems of matrices: one matrix per arrow of Q' satisfying the relations in $\ker(\Phi)$ given by Theorem 5.10, the orders of which are determined by $\mathbf{v}$, and one matrix per vertex of Q' given by concatenating the matrices corresponding to the arrows with head at that vertex, which by θ-stability must be full rank.

Now fix $w \in \mathcal{M}(E)$. Using the constraints on the systems of matrices defining w, namely θ-stability and the relations induced by $\ker(\Phi)$, we prove w is equivalent to a point (modulo the group action) where every entry of each matrix is a polynomial in the entries of the single matrix corresponding to the arrows of Q, the original quiver for Y which is a full sub-quiver of Q'. Once we have done this for $\mathrm{Gr}(4, 2)$, we use induction to prove the same property holds for $\mathrm{Gr}(n, 2)$. It is then possible to show that $f_{E'}$ is the inverse morphism for f_E.

Future directions

We conclude with a discussion about the potential to generalise the methods of this thesis in order to prove the following.

Conjecture 7.1. *For any $1 \leq r < n$ let $Y = \mathrm{Gr}(n, r)$. Then the morphism $f_E : Y \to \mathcal{M}(E)$ is an isomorphism.*

While Buchweitz, Leuschke and Van den Bergh provide the tilting quiver for any $\mathrm{Gr}(n, r)$, considerable combinatorial work must be undertaken in order to write down explicit generators for $\ker(\Phi)$. The difficulty is two-fold: first, write down a 'Pieri system', i.e. a collection of maps $\mathbb{S}^\lambda \mathcal{W} \to \mathbb{S}^{\lambda+e_i}\mathcal{W}$ for $\lambda \in \text{Young}(n-r, r)$ and $1 \le i \le r$ (when $i = 1, 2$ these are the f-type and g-type maps described above). Secondly, use this system to describe the relations of the titling algebra in the style of Chapter 5. We discuss this in more detail in Section 5.3 and Section 7.1.

Without completing the above, it is hard to say anything definitive about generalising the proof of Theorem 6.1 using the methods in Chapter 6. However, different strategies altogether may provide further insight, in particular the work of Bergman and Proudfoot [BP08] which identifies any quiver flag variety Y with a connected component of $\mathcal{M}(E)$. Given the evidence in this thesis, we therefore conjecture the following.

Conjecture 7.2. *Let Y be any quiver flag variety and E the tilting bundle from Theorem 2.9. Then the morphism $f_E : Y \to \mathcal{M}(E)$ is an isomorphism.*

Chapter 2

Background

We begin by recalling some basic definitions and the construction of quiver flag varieties. Then we present a tilting bundle for these varieties and define the corresponding tilting algebra. From this we create a second moduli space, the 'multigraded linear series', and write down a closed immersion from the original quiver flag variety to the multigraded linear series.

Throughout, let $\Bbbk$ be an algebraically closed field of characteristic zero.

2.1 Quiver flag varieties

The background material presented in this section mostly follows [Cra11, Sections 2, 3].

A *quiver* $Q = (Q_0, Q_1)$ is a directed graph with vertex set Q_0 and arrow set Q_1. For each arrow $a \in Q_1$ we denote by $\mathrm{h}(a), \mathrm{t}(a) \in Q_0$ the vertices at the head and tail of a respectively. We say that Q is *finite* if Q_0 and Q_1 are finite; *connected* if the underlying graph is connected; *acyclic* if there exists a labelling of vertices $Q_0 = \{0, 1, \ldots, \ell\} \subset \mathbb{N}$ such that for all $a \in Q_1$ we have $\mathrm{t}(a) < \mathrm{h}(a)$. Unless stated otherwise, we hereafter suppose that Q is finite, connected, acyclic and has a unique source vertex 0, i.e. 0 is the only vertex $i \in Q_0$ such that there are no arrows $a \in Q_1$ with $\mathrm{h}(a) = i$.

A *representation* W of Q consists of a vector space W_i for all $i \in Q_0$ and a linear map $w_a \colon W_{\mathrm{t}(a)} \to W_{\mathrm{h}(a)}$ for all $a \in Q_1$. Denote $r_i := \dim(W_i)$ and define the dimension vector of W to be $\underline{r} := (r_0, \ldots, r_\ell) \in \mathbb{N}^{\ell+1}$.

Henceforth fix a dimension vector $\underline{r} = (r_i) \in \mathbb{N}^{\ell+1}$ satisfying $r_0 = 1$ and denote the space of representations of Q with dimension vector $\underline{r}$ by

$$\operatorname{Rep}(Q, \underline{r}) := \bigoplus_{a \in Q_1} \operatorname{Hom}(\Bbbk^{r_{\mathrm{t}(a)}}, \Bbbk^{r_{\mathrm{h}(a)}}). \tag{2.1}$$

Isomorphism classes of representations with dimension vector $\underline{r}$ are precisely the orbits under the action of the group

$$G := \prod_{i=0}^{\ell} \mathrm{GL}(r_i) \tag{2.2}$$

induced by conjugation, i.e. change of basis.

Definition 2.1. The *quiver flag variety* associated to the pair $(Q, \underline{r})$ is the GIT quotient

$$Y := \operatorname{Rep}(Q, \underline{r}) /\!\!/_{\chi} G$$

for the special choice of linearisation $\chi := (-\sum_{i=1}^{\ell} r_i, 1, \dots, 1) \in G^{\vee}$.

Remark 2.2. It will sometimes be more convenient to consider a coarser decomposition of $\operatorname{Rep}(Q, \underline{r})$ than (2.1). For each $1 \le i \le \ell$ define the subspace $\operatorname{Rep}(Q, \underline{r})_i := \bigoplus_{\{a \in Q_1 : \mathrm{h}(a) = i\}} \operatorname{Hom}(\Bbbk^{r_{\mathrm{t}(a)}}, \Bbbk^{r_i})$, giving

$$\operatorname{Rep}(Q, \underline{r}) = \bigoplus_{1 \le i \le \ell} \operatorname{Rep}(Q, \underline{r})_i. \tag{2.3}$$

We may write points of $\operatorname{Rep}(Q, \underline{r})$ as a tuple of matrices (w_i), $1 \le i \le \ell$, where each w_i has r_i rows.

By [Cra11, Lemma 2.1], a point (w_i) is χ-stable if and only if it is χ-semistable if and only if each matrix w_i satisfies $\operatorname{rank}(w_i) = r_i$. Thus, Craw observed that the quiver flag variety Y is non-empty if and only if the inequality

$$r_i \le s_i := \sum_{\{a \in Q_1 | \mathrm{h}(a) = i\}} r_{\mathrm{t}(a)} \tag{2.4}$$

holds for all $i > 0$. Moreover, by [Cra11, Propositions 2.2, 3.1], Y is equal to the fine moduli space $\mathcal{M}_{\chi}(Q, \underline{r})$ of χ-stable representations of Q with dimension vector $\underline{r}$, and is also a smooth Mori Dream Space of dimension $\sum_{i=1}^{\ell} r_i(s_i - r_i)$. For more on Mori Dream Spaces, see [HK00].

We now state the main structure theorem of quiver flag varieties.

Theorem 2.3 ([Cra11, Theorem 3.3]). *For any quiver flag variety Y, there is a tower of Grassmann-bundles*

$$Y := Y_\ell \longrightarrow Y_{\ell-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 = \operatorname{Spec} \Bbbk, \tag{2.5}$$

where at each stage, Y_i is isomorphic to the Grassmannian of rank r_i quotients of a fixed locally-free sheaf of rank s_i on Y_{i-1}.

We therefore see that quiver flag varieties have an iterative structure as a tower of Grassmann-bundles. Hereafter we assume that the inequality (2.4) is strict for each $i > 0$ to avoid degeneracy in the tower.

Following [Cra11, Equation 2.4], we see that quiver flag varieties naturally carry a collection of vector bundles $\mathcal{W}_1, \ldots, \mathcal{W}_\ell$ that determine many of their algebraic invariants. Indeed, for $i > 0$, the Grassmann-bundle Y_i over Y_{i-1} carries a tautological quotient bundle $\mathcal{V}_i$ of rank r_i, and we write $\mathcal{W}_i := \pi_i^*(\mathcal{V}_i)$ for the bundle of rank r_i on Y obtained as the pullback under the morphism $\pi_i \colon Y \to Y_i$ in the tower. Define $\mathcal{W}_0 := \mathcal{O}_Y$ and let $\Bbbk Q$ be the path algebra of Q. Then there is also a universal $\Bbbk$-algebra homomorphism $\Bbbk Q \to \operatorname{End}(\bigoplus_{i \in Q_0} \mathcal{W}_i)$ obtained associating compositions of morphisms $\mathcal{W}_{\mathrm{t}(a)} \to \mathcal{W}_{\mathrm{h}(a)}$ to paths of arrows $a \in Q_1$.

Proposition 2.4 ([Cra11, Corollary 3.5, Lemma 3.7]). *Let Y be a quiver flag variety with non-trivial tautological bundles $\mathcal{W}_1, \ldots, \mathcal{W}_\ell$.*

(i) *The vector bundles $\mathcal{W}_1, \ldots, \mathcal{W}_\ell$ are globally generated.*

(ii) *The line bundles* $\det(\mathcal{W}_1), \ldots, \det(\mathcal{W}_\ell)$ *are globally generated and provide an integral basis for* $\operatorname{Pic}(Y)$.

(iii) *The universal $\Bbbk$-algebra homomorphism $\Bbbk Q \to \operatorname{End}(\bigoplus_{i \in Q_0} \mathcal{W}_i)$ induces an isomorphism of vector spaces $e_i \Bbbk Q e_0 \cong H^0(Y, \mathcal{W}_i)$ for each $i \in Q_0$, where e_i are the orthogonal idempotents of $\Bbbk Q$.*

Examples 2.5. (i) [CS08, Example 3.6] Consider the quiver in Figure 2.1 with dimension vector $(1, 1, 1)$. Using the tower structure given in (2.5), we see that Y is a projective bundle over $\mathbb{P}^1$. Moreover, using [Cra11, Theorem 3.3] it is possible to calculate the locally-free sheaf of rank $s_2 = 2$ in Theorem 2.3 to be $\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(1)$. Therefore, we have $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(1)) = \mathbb{F}_1$, the first Hirzebruch surface. For more on Hirzebruch surfaces, see [CLS11, Example 3.1.16].

(ii) [Cra11, Example 2.4] Consider the quiver in Figure 2.2 with dimension vector $(1, r)$ and let $|Q_1| := n > r$ so that (2.4) is satisfied. Then the tower from (2.5) is height one and so Y is simply $\operatorname{Gr}(n, r)$, the Grassmannian of r-dimensional

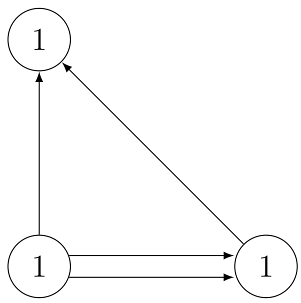


FIGURE 2.1: The quiver for $Y = \mathbb{F}_1$, the first Hirzebruch surface.

quotients of $\Bbbk^n$. Alternatively, observe that points of Y as given by the decomposition (2.3) are $r \times n$ matrices; the discussion following Remark 2.2 implies that a point is χ-stable if and only if it is full rank, and furthermore after identifying matrices that are equivalent under the group action (change of basis), points of Y therefore correspond one-to-one with surjective linear maps $\Bbbk^n \to \Bbbk^r$, or in other words, r-dimensional quotients of $\Bbbk^n$.

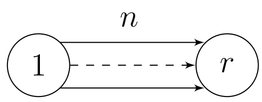


FIGURE 2.2: The quiver Q with n arrows such that $Y = \mathrm{Gr}(n, r)$.

(iii) The quiver with n arrows $0 \to 1$ and one arrow $i \to i+1$ for $1 \le i \le \ell - 1$ with dimension vector satisfying $n > r_1 > \cdots > r_\ell$ makes Y into a (partial) flag variety of type-A; see [Cra11, Example 2.6].

2.2 Tilting bundles

We now move on to the definition of a tilting bundle. For a smooth projective variety X, write $\mathrm{Coh}(X)$ for the abelian category of coherent sheaves on X and $D^b(\mathrm{Coh}(X))$ for the bounded derived category of coherent sheaves on X.

Definition 2.6. A coherent sheaf E on X is *tilting* if:

(i) the algebra $\mathrm{End}_{\mathcal{O}_X}(E)$ has finite global dimension, i.e. the maximal projective dimension of any module over $\mathrm{End}_{\mathcal{O}_X}(E)$ is finite.

(ii) $\mathrm{Ext}^k_{\mathcal{O}_X}(E, E) = 0$ for all $k > 0$.

(iii) E classically generates $D^b(\mathrm{Coh}(X))$, i.e. we have $\langle E \rangle = D^b(\mathrm{Coh}(X))$ where $\langle E \rangle$ is the smallest triangulated subcategory of $D^b(\mathrm{Coh}(X))$ containing E and all of its direct summands.

For a tilting sheaf E, we write $A := \mathrm{End}_{\mathcal{O}_X}(E)$ for its *tilting algebra*.

Remark 2.7. It was noted by Hille and Van den Bergh [HV07] that when X is smooth, conditions (ii) and (iii) together imply (i) in the definition.

The motivation for finding tilting bundles stems from the following important theorem, which gives a nice description of the bounded derived category of coherent sheaves for varieties with a tilting sheaf.

Theorem 2.8 ([Bae88][Bon90])**.** *Let X be a smooth projective variety with tilting sheaf E and tilting algebra A, and write* $\mathrm{Rmod}(A)$ *for the category of finitely generated right A-modules. Then the functor*

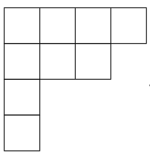
$$\mathrm{Hom}_{\mathcal{O}_X}(E, -)\colon \mathrm{Coh}(X) \longrightarrow \mathrm{Rmod}(A)$$

induces an equivalence of triangulated categories

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{O}_X}(E, -)\colon D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{Rmod}(A)).$$

When a tilting sheaf is also a vector bundle we call it a *tilting bundle*. The results of Beilinson [Bei78] and Kapranov [Kap88] provide tilting bundles for projective space and Grassmannians respectively. In order to describe the generalisation to quiver flag varieties, we first establish our conventions for Young diagrams.

Let r be a positive integer and $\lambda \in \mathbb{N}^r$ a weakly decreasing finite sequence of non-negative integers $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$. We call such λ a *partition* with r *parts*, even if λ ends in a trail of zeroes. We can represent partitions pictorially using *Young diagrams*; these are finite collections of boxes arranged into left-justified rows in descending order. For example, the Young diagram representing $\lambda = (4, 3, 1, 1)$ is



.

Denote the number of boxes in a Young diagram by $|\lambda| := \sum_{i=1}^{r} \lambda_i$. As an abuse of notation we hereafter consider partitions and Young diagrams as one and the same. Denote by $\mathrm{Young}(n, r)$ the set of Young diagrams with at most n columns and r rows. In other words,

$$\mathrm{Young}(n, r) := \{\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{N}^r \mid n \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0\}\,.$$

Recall that for any vector bundle $\mathcal{V}$ of rank r and for $\lambda \in \text{Young}(n, r)$, we obtain a vector bundle $\mathbb{S}^\lambda \mathcal{V}$ whose fibre over each point is the irreducible $\text{GL}(r)$-module of highest weight λ. We may equivalently view $\mathbb{S}^\lambda \mathcal{V}$ as the image of the *Schur functor* on $\mathcal{V}^{\otimes |\lambda|}$; more on this in Chapter 4.

The following theorem provides a tilting bundle for any quiver flag variety Y with tautological bundles $\mathcal{W}_1, \dots, \mathcal{W}_\ell$.

Theorem 2.9 ([Cra11, Theorem 4.5]). *The vector bundle on Y given by*

$$E := \bigoplus_{1 \le i \le \ell,\ \lambda^{(i)} \in \text{Young}(s_i - r_i, r_i)} \mathbb{S}^{\lambda^{(1)}} \mathcal{W}_1 \otimes \cdots \otimes \mathbb{S}^{\lambda^{(\ell)}} \mathcal{W}_\ell \tag{2.6}$$

is a tilting bundle. In particular, the bounded derived category of coherent sheaves on Y is equivalent to the bounded derived category of finitely generated right modules over $A = \text{End}_{\mathcal{O}_Y}(E)$.

Remark 2.10. This result answered affirmatively the question of Nakajima [Nak96, Problem 3.10].

2.3 Multigraded linear series

Consider a quiver flag variety Y with tilting bundle E given by (2.6) and let $E_0, \dots, E_n$ be the indecomposable summands of E with $E_0 = \mathcal{O}_Y$. Denote by $\mathbf{v} := (v_j) \in \mathbb{N}^{n+1}$ the dimension vector satisfying $v_j := \text{rank}(E_j)$ for all $0 \le j \le n$. Following [CIK18, Section 2] and the summary given in [CG18, Section 2], we now briefly describe the construction of a fine moduli space $\mathcal{M}(A, \mathbf{v}, \theta)$ called the *multigraded linear series*.

Multigraded linear series are examples of moduli spaces originally constructed by King [Kin94]. To introduce our choice of stability condition θ, first set $\theta' = (-\sum_{i=1}^n v_i, 1, 1, \dots, 1) \in \text{Hom}(\mathbb{Z}^{n+1}, \mathbb{Q})$. An A-module $M = \bigoplus_{0 \le j \le n} M_j$ of dimension vector $\mathbf{v}$ is θ'-stable if and only if M is generated as an A-module by any nonzero element of M_0; any such stability condition is called *0-generated*. Since $\mathbf{v}$ is indivisible, King [Kin94, Proposition 5.3] constructs the fine moduli space $\mathcal{M}(A, \mathbf{v}, \theta')$ of isomorphism classes of θ'-stable A-modules of dimension vector $\mathbf{v}$ as a GIT quotient. In particular, $\mathcal{M}(A, \mathbf{v}, \theta')$ comes with an ample bundle $\mathcal{O}(1)$. Let $k \ge 1$ denote the smallest positive integer such that $\mathcal{O}(k)$ is very ample. Then $\theta := k\theta'$ is also a 0-generated stability condition, and we write

$$\mathcal{M}(E) := \mathcal{M}(A, \mathbf{v}, \theta)$$

for the fine moduli space of θ-stable A-modules of dimension vector $\mathbf{v}$. The universal property of $\mathcal{M}(E)$ determines a morphism

$$f_E\colon Y \longrightarrow \mathcal{M}(E) \tag{2.7}$$

and moreover we have the following.

Theorem 2.11. *The morphism $f_E\colon Y \longrightarrow \mathcal{M}(E)$ is a closed immersion.*

Proof. Each $\mathcal{W}_i$ is globally generated by Proposition 2.4, and hence so is every indecomposable summand E_i of E from (2.6). Therefore $\det(E_i)$ is also globally generated for each E_i; see [Sno86, Section 4]. By [Cra11, Lemma 3.7] the line bundle $L := \bigotimes_{i=1}^{\ell} \det(\mathcal{W}_i)$ is ample. Hence, the line bundle $\bigotimes_{0\leq j\leq n} \det(E_j)$, which is a tensor product of L and other globally generated bundles, is also ample by [Har77, Exercise II.7.5(d)]. The result now follows from [CIK18, Theorem 2.6]. □

Remark 2.12. Work of Bergman-Proudfoot compares any smooth projective variety admitting a tilting bundle to a fine moduli space of modules over the endomorphism algebra. In fact, [BP08, Theorem 2.4] implies that f_E identifies Y with a connected component of $\mathcal{M}(E)$, because Y is smooth, E is a tilting bundle, and our stability condition θ is 'great' (see the discussion prior to Proposition 2.2 in [BP08]).

Theorem 2.11 implies that $\mathcal{M}(E)$ may be realised as an ambient space for Y; this generalises the classical morphism to the linear series of a basepoint-free line bundle. It is natural to ask when f_E is an isomorphism, thereby providing a reconstruction of the quiver flag variety from a tilting bundle. This thesis provides two classes of examples: when Y is toric (Chapter 3), and when $Y = \mathrm{Gr}(n, 2)$, the Grassmannian of 2-dimensional quotients of $\Bbbk^n$ (Chapter 6). The main tool is to define a quiver Q', which we call the *tilting quiver*, such that there is a surjective $\Bbbk$-algebra homomorphism

$$\Phi\colon \Bbbk Q' \longrightarrow\!\!\!\!\!\rightarrow A$$

defined by mapping paths to compositions of maps. Since $\Bbbk Q'/\ker(\Phi) \cong A$, we may regard points of $\mathcal{M}(E)$, or more specifically θ-stable A-modules with dimension vector $\mathbf{v}$, as θ-stable representations of Q' with dimension vector $\mathbf{v}$ subject to the relations induced by $\ker(\Phi)$.

Chapter 3

Reconstructing toric quiver flag varieties from a tilting bundle

The content of this chapter is taken from the paper [CG18], of which the author of this thesis is a co-author. Please see the declarations on the preliminary pages.

3.1 Statement of the main result

We continue to use the notation introduced in Chapter 2. Let Q be a finite, connected, acyclic quiver with unique source and vertex set $Q_0 = \{0, \ldots, \ell\}$. This chapter considers quiver flag varieties with dimension vector $\underline{r} = (1, \ldots, 1)$; in this case, the group G from (2.2) is an algebraic torus and so the quiver flag variety Y is a toric variety. As such, we make the following definition.

Definition 3.1. A *toric quiver flag variety* is a quiver flag variety Y with dimension vector $\underline{r} = (1, \ldots, 1)$.

The toric fan Σ can be described directly in this case (see [CS08, p1517]), and therefore Y is a tower of projective space bundles via Theorem 2.3. Moreover, the tilting bundle from (2.6) is simply the direct sum of line bundles

$$E = \bigoplus_{1 \leq i \leq \ell,\ 0 \leq m_i < s_i} \mathcal{W}_1^{\otimes m_1} \otimes \cdots \otimes \mathcal{W}_\ell^{\otimes m_\ell} \tag{3.1}$$

on Y, where s_i is defined in (2.4). The main result is as follows.

Theorem 3.2. *Let Y be a toric quiver flag variety. Then the morphism $f_E \colon Y \to \mathcal{M}(E)$ from (2.7) is an isomorphism.*

As a result, toric quiver flag varieties provide a new class of examples where the programme of Bergman-Proudfoot [BP08] can be carried out in full, enabling one to reconstruct the variety from the tilting bundle. The special case where Y is isomorphic to projective space recovers the well-known result that $\mathbb{P}^n$ can be reconstructed from the tilting bundle $\bigoplus_{0\leq i\leq n}\mathcal{O}_{\mathbb{P}^n}(i)$ of Beilinson [Bei78]; see Example 3.4. Theorem 3.2 therefore provides further evidence that toric quiver flag varieties provide good multigraded analogues of projective space.

3.2 The reduction step

The method of proof for Theorem 3.2 is as follows. Set $n+1 := \prod_{1\leq i\leq \ell} s_i$ and list the indecomposable summands from (3.1) as $E_0,\dots,E_n$ with $E_0 \cong \mathcal{O}_Y$. The fact that $\bigotimes_{0\leq j\leq n}\det(E_j) = \bigotimes_{0\leq j\leq n} E_j$ is very ample and f_E is a closed immersion puts us in the situation studied by Craw–Smith [CS08], where it is possible to give an explicit GIT quotient description for both the moduli space $\mathcal{M}(E)$ and the image of the universal morphism f_E. Theorem 3.2 will follow once we prove that these two GIT quotients coincide.

To describe $\mathcal{M}(E)$ as a GIT quotient, we first present the algebra $A = \operatorname{End}_{\mathcal{O}_Y}(E)$ using the *bound quiver of sections* (Q',R) as follows. The quiver Q' has vertex set $Q'_0 = \{0,1,\dots,n\}$ and an arrow from vertex i to j for each *irreducible*, torus-invariant section of $E_j \otimes E_i^{-1}$, i.e. the corresponding homomorphism from E_i to E_j does not factor through some E_k with $k \neq i,j$. To each arrow $a \in Q'_1$ we associate the corresponding torus-invariant 'labelling divisor' $\operatorname{div}(a) \in \mathbb{N}^{\Sigma(1)}$, where $\Sigma(1)$ denotes the set of rays of the fan of Y. The two-sided ideal

$$R := (p - q \in \Bbbk Q' \mid p,q \text{ share the same head, tail and labelling divisor})$$

in $\Bbbk Q'$ satisfies $A \cong \Bbbk Q'/R$ (see [CS08, Proposition 3.3]). Denote the coordinate ring of $\mathbb{A}_{\Bbbk}^{Q'_1}$ by $\Bbbk[y_a]$, where a ranges over Q'_1. The ideal R in the non-commutative ring $\Bbbk Q'$ determines an ideal in $\Bbbk[y_a]$ given by

$$I_R := \left(\prod_{a\in\operatorname{supp}(p)} y_a - \prod_{a\in\operatorname{supp}(q)} y_a \in \Bbbk[y_a] \;\middle|\; \begin{array}{c} p,q \text{ share the same head,} \\ \text{tail and labelling divisor} \end{array} \right), \tag{3.2}$$

where the support of a path $\operatorname{supp}(p)$ is simply the set of arrows that make up the path. This ideal is homogeneous with respect to the action of $T := \prod_{0\leq j\leq n}\mathrm{GL}(1)$

by conjugation. It now follows directly from the definition of King [Kin94] that

$$\mathcal{M}(A,\mathbf{v},\theta) := \mathbb{V}(I_R)/\!\!/_{\theta} T := \operatorname{Proj} \bigoplus_{k\geq 0} \big(\Bbbk[y_a]/I_R\big)_{k\theta}, \tag{3.3}$$

where $\big(\Bbbk[y_a]/I_R\big)_{k\theta}$ denotes the $k\theta$-graded piece. In fact, [CS08, Proposition 3.8] implies that $\mathcal{M}(E) = \mathcal{M}(A,\mathbf{v},\theta)$ is the geometric quotient of $\mathbb{V}(I_R) \setminus \mathbb{V}(B_{Q'})$ by the action of T, where

$$B_{Q'} := \bigcap_{j=1}^{n} \big(y_a \in \Bbbk[y_a] \bigm| \mathrm{h}(a) = j\big) \tag{3.4}$$

is the irrelevant ideal in $\Bbbk[y_a]$ that cuts out the θ-unstable locus in $\mathbb{A}_{\Bbbk}^{Q'_1}$.

Our task is to compare (3.3) with the GIT quotient description of the image of f_E. For this, define a map $\pi\colon \mathbb{Z}^{Q'_1} \to \mathbb{Z}^{Q'_0} \oplus \mathbb{Z}^{\Sigma(1)}$ by setting $\pi(\chi_a) = (\chi_{\mathrm{h}(a)} - \chi_{\mathrm{t}(a)}, \operatorname{div}(a))$, where χ_a for $a \in Q'_1$ and χ_i for $i \in Q'_0$ denote the characteristic functions. The T-homogeneous ideal

$$I_{Q'} := \big(y^u - y^v \in \Bbbk[y_a] \mid u - v \in \ker(\pi)\big) \tag{3.5}$$

contains I_R from (3.2), and [CS08, Proposition 4.3] establishes that the image of the universal morphism f_E is isomorphic to the geometric quotient of $\mathbb{V}(I_{Q'}) \setminus \mathbb{V}(B_{Q'})$ by the action of T.

Proposition 3.3. *Suppose that the T-orbit of every closed point of $\mathbb{V}(I_R)\setminus\mathbb{V}(B_{Q'})$ contains a closed point of $\mathbb{V}(I_{Q'}) \setminus \mathbb{V}(B_{Q'})$. Then Theorem 3.2 holds.*

Proof. The inclusion $\mathbb{V}(I_{Q'}) \subseteq \mathbb{V}(I_R)$ always holds, and the assumption ensures that $\mathbb{V}(I_R)/\!\!/_{\theta} T \subseteq \mathbb{V}(I_{Q'})/\!\!/_{\theta} T$, so the closed immersion f_E is surjective. □

In Section 3.4 we prove that the assumption of Proposition 3.3 holds for every toric quiver flag variety Y. To illustrate the strategy, we recall the following well-known construction of $\mathbb{P}^n$ using Beilinson's tilting bundle.

Example 3.4. For the acyclic quiver Q with vertex set $Q_0 = \{0,1\}$ and $n+1$ arrows from 0 to 1, the toric quiver flag variety Y is isomorphic to $\mathbb{P}^n$ and the quiver of sections Q' for the tilting bundle $\bigoplus_{0\leq i\leq n} \mathcal{O}_{\mathbb{P}^n}(i)$ is shown in Figure 3.1; note that Q is a sub-quiver of Q'. For each $1 \leq m \leq n$ and each ray $\rho \in \Sigma(1)$ in the fan of $\mathbb{P}^n$ defining a torus-invariant divisor D_ρ, let a_ρ^m denote the arrow with head at m and labelling divisor $\operatorname{div}(a_\rho^m) = D_\rho$. Writing $y_\rho^m \in \Bbbk[y_a]$ for the

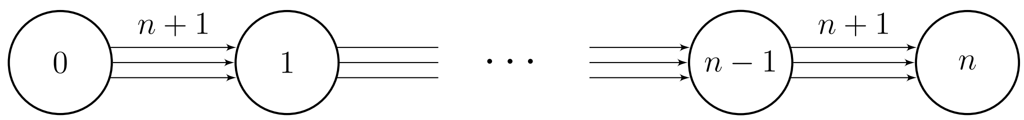


FIGURE 3.1: The tilting quiver for $\mathbb{P}^n$.

variable associated to the arrow a_ρ^m, we have

$$I_R = \left(y_\sigma^{m+1} y_\rho^m - y_\rho^{m+1} y_\sigma^m \in \Bbbk[y_a] \;\middle|\; 1 \leq m \leq n-1;\ \rho, \sigma \in \Sigma(1)\right). \tag{3.6}$$

We claim that a point $(w_\rho^m) \in \mathbb{V}(I_R) \setminus \mathbb{V}(B_{Q'}) \subset \mathbb{A}_{\Bbbk}^{n(n+1)}$ lies in the same T-orbit as the point (v_ρ^m) with components $v_\rho^m := w_\rho^1$ for all $1 \leq m \leq n$ and $\rho \in \Sigma(1)$. Clearly $(v_\rho^m) \in \mathbb{V}(I_{Q'}) \setminus \mathbb{V}(B_{Q'})$, so the claim and Proposition 3.3 show that Theorem 3.2 holds for $\mathbb{P}^n$.

To prove the claim, note that since $(w_\rho^m) \notin \mathbb{V}(B_{Q'})$, the T-action allows us to assume that for all $1 \leq m \leq n$ there exists $\rho(m) \in \Sigma(1)$ such that $w_{\rho(m)}^m = 1$. Then $v_{\rho(1)}^1 = 1$, and (3.6) implies that $w_{\rho(1)}^2 v_\rho^1 = w_\rho^2$ for all $\rho \in \Sigma(1)$. The case $\rho = \rho(2)$ gives $w_{\rho(1)}^2 = (v_{\rho(2)}^1)^{-1} = (w_{\rho(2)}^1)^{-1}$, so

$$w_\rho^2 = v_\rho^1 (w_{\rho(2)}^1)^{-1} = w_\rho^1 (w_{\rho(2)}^1)^{-1} \quad \text{for all } \rho \in \Sigma(1).$$

Let the one-dimensional subgroup $\Bbbk^\times \subset T$ scale by $w_{\rho(2)}^1$ at vertex 2 to obtain a point in the same T-orbit as (w_ρ^m) whose components agree with those of (v_ρ^m) for $m = 1, 2$. Repeating at each successive vertex shows that (v_ρ^m) and (w_ρ^m) lie in the same T-orbit as claimed.

3.3 The tilting quiver

Before establishing that the assumption of Proposition 3.3 holds for every toric quiver flag variety, we describe the *tilting quiver* Q' in detail (see Example 3.7).

For the vertex set Q'_0, recall that the line bundles $\mathcal{W}_1, \ldots, \mathcal{W}_\ell$ provide an integral basis for $\operatorname{Pic}(Y) \cong \mathbb{Z}^\ell$. Since Q'_0 is defined by the summands $\mathcal{W}_1^{\otimes m_1} \otimes \cdots \otimes \mathcal{W}_\ell^{\otimes m_\ell}$ of the tilting bundle E from (3.1), it is convenient to realise Q'_0 as the set of lattice points of a cuboid in $\mathbb{Z}^\ell \otimes_{\mathbb{Z}} \mathbb{R}$ of dimension ℓ with side lengths $s_1 - 1, \ldots, s_\ell - 1$. We label the vertex for $\mathcal{W}_1^{\otimes m_1} \otimes \cdots \otimes \mathcal{W}_\ell^{\otimes m_\ell}$ by the corresponding lattice point $(m_1, \ldots, m_\ell) \in \mathbb{Z}^\ell$, giving

$$Q'_0 = \left\{ (m_1, \ldots, m_\ell) \in \mathbb{Z}^\ell \mid 0 \leq m_i < s_i \right\}.$$

We introduce a total order on Q_0': for $k = (k_1, \dots, k_\ell), m = (m_1, \dots, m_\ell) \in Q_0'$, write $k < m$ if $k_i < m_i$ for the largest index i satisfying $k_i \neq m_i$.

For the arrow set Q_1', first note that the arrows in Q correspond precisely to the torus-invariant prime divisors in Y because Q is the quiver of sections of $\{\mathcal{O}_Y, \mathcal{W}_1, \dots, \mathcal{W}_\ell\}$, [CS08, Remark 3.9]. For $\rho \in \Sigma(1)$ we write $a_\rho \in Q_1$ for the arrow corresponding to the divisor of zeros D_ρ of a torus-invariant section of $\mathcal{W}_{\mathrm{h}(a_\rho)} \otimes \mathcal{W}_{\mathrm{t}(a_\rho)}^{-1}$. Each a_ρ may be regarded as an arrow in Q', so we may identify Q with a complete sub-quiver of Q' that we call the *base quiver* in Q'. More generally, translating each a_ρ around the cuboid described in the preceding paragraph (so that the head and tail lie in Q_0') produces arrows in Q' that we denote $a_\rho^m \in Q_1'$ for $m = \mathrm{h}(a_\rho^m)$ and $D_\rho = \mathrm{div}(a_\rho^m)$. In fact, we have the following:

Lemma 3.5. *Every arrow $a \in Q_1'$ is of the form $a = a_\rho^m$, where $m = \mathrm{h}(a)$ and $D_\rho = \mathrm{div}(a)$.*

Proof. For $a \in Q_1'$, write $\mathrm{h}(a) = m = (m_1, \dots, m_\ell)$ and $\mathrm{t}(a) = m' = (m_1', \dots, m_\ell')$, so $\mathrm{div}(a)$ is the divisor of zeros of a section of $\bigotimes_{1 \leq i \leq \ell} \mathcal{W}_i^{\otimes(m_i - m_i')}$. In terms of prime divisors, we have

$$\mathrm{div}(a) = \sum_{\rho \in \Sigma(1)} \lambda_\rho D_\rho \quad \text{for } \lambda_\rho \in \mathbb{N}.$$

Let $1 \leq k \leq \ell$ be the largest value such that $\lambda_\rho \neq 0$ for some $\rho \in \Sigma(1)$ satisfying $k = \mathrm{h}(a_\rho) \in Q_0$. Note that $0 \leq m_k' < m_k$, and moreover, $j := \mathrm{t}(a_\rho) < k$. Since $\mathrm{div}(a)$ is irreducible, translating a_ρ so that the tail is at vertex m' forces the head to lie outside the cuboid, giving $m_j' = 0$ or $m_k' = s_k - 1$; similarly, translating a_ρ so that the head is at m forces the tail to lie outside the cuboid, giving $m_j = s_j - 1$ or $m_k = 0$. Since $0 \leq m_k' < m_k$, both $m_j' = 0$ and $m_j = s_j - 1$ must hold, so $m_j' < m_j$. As a result, there must exist $\sigma \in \Sigma(1)$ satisfying $\lambda_\sigma \neq 0$ for $j = \mathrm{h}(a_\sigma)$. If we set $i := \mathrm{t}(a_\sigma)$ and repeat the argument above, we deduce that $m_i' < m_i$. Continuing in this way, we eventually find $\tau \in \Sigma(1)$ such that $\lambda_\tau \neq 0$ with $\mathrm{h}(a_\tau) = 1$ and $\mathrm{t}(a_\tau) = 0$. But then $0 = m_1' < m_1 = s_1 - 1$, so we can place a translation of a_τ with head at m and tail in the cuboid (or tail at m' and head in the cuboid). This shows $\mathrm{div}(a)$ is reducible, a contradiction. □

Remark 3.6. Since Q is the quiver of sections of $\{\mathcal{O}_Y, \mathcal{W}_1, \dots, \mathcal{W}_\ell\}$, the vertices of the base quiver are the vertices $e_0, e_1, \dots, e_\ell \in Q_0' \subset \mathbb{Z}^\ell$, where e_i denotes the i^{th} standard basis vector for $i > 0$, and where $e_0 := (0, \dots, 0)$.

The next example illustrates how the base quiver sits inside Q'.

Example 3.7. The quiver Q shown in Figure 3.2(a) defines the toric quiver flag variety $Y = \mathbb{P}_Z(\mathcal{O}(1,0) \oplus \mathcal{O}(0,1))$ where $Z = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$; the colours of the arrows indicate the distinct labelling divisors. We have $s_1 = 3$ and $s_2 = s_3 = 2$, so the tilting quiver Q' has 12 vertices shown in Figure 3.2(b) using the ordering described above. Note that the base quiver is the complete sub-quiver of Q' whose

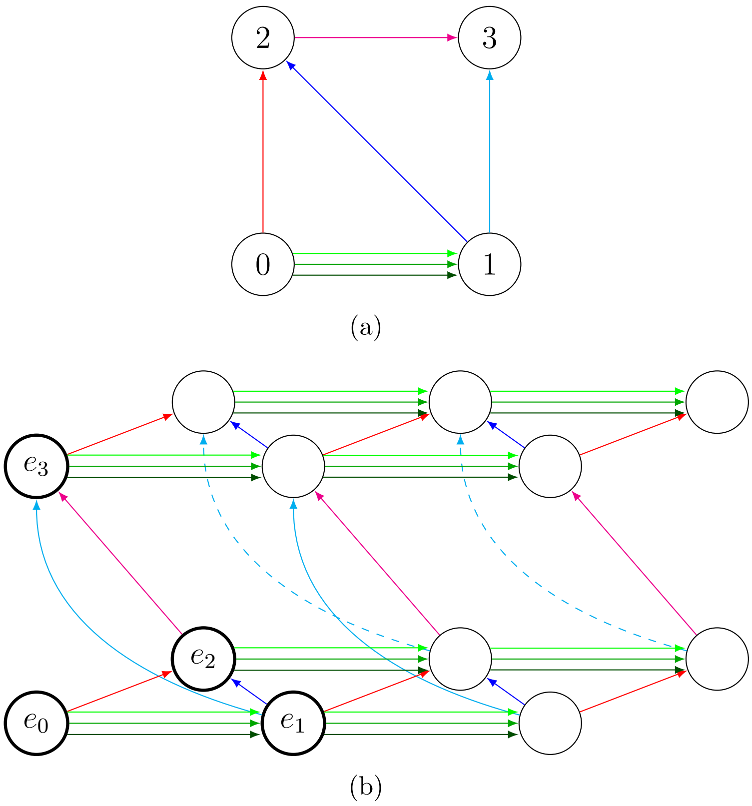


FIGURE 3.2: Quivers for Y: (a) original quiver Q; (b) tilting quiver Q'.

vertices are shown in bold in Figure 3.2(b). The colour of each arrow of Q' is determined by its unique translate arrow from the base quiver.

3.4 Proof of Theorem 3.2

In light of Lemma 3.5, each point of $\mathbb{A}_{\Bbbk}^{Q_1'}$ is a tuple (w_ρ^m) where $w_\rho^m \in \Bbbk$ for $\rho \in \Sigma(1)$ and for all relevant $m \in Q_0'$. Motivated by Example 3.4, we associate to $(w_\rho^m) \in \mathbb{A}_{\Bbbk}^{Q_1'}$ an auxiliary point $(v_\rho^m) \in \mathbb{V}(I_{Q'}) \subseteq \mathbb{A}_{\Bbbk}^{Q_1'}$ whose components satisfy

$$v_\rho^m := w_\rho \text{ for } \rho \in \Sigma(1) \text{ and all relevant } m \in Q_0', \tag{3.7}$$

where for $\rho \in \Sigma(1)$ we write $w_\rho \in \Bbbk$ for the component of the point (w_ρ^m) corresponding to the unique arrow a_ρ in the base quiver satisfying $\operatorname{div}(a_\rho) = D_\rho$.

Lemma 3.8. *If* $(w_\rho^m) \notin \mathbb{V}(B_{Q'})$*, then* $(v_\rho^m) \notin \mathbb{V}(B_{Q'})$*.*

Proof. Fix $m = (m_1, \ldots, m_\ell) \in Q_0'$ and let $1 \leq j \leq \ell$ be minimal such that $m_j \neq 0$. Then for all ρ satisfying $\mathrm{h}(a_\rho) = j \in Q_0$, the arrow a_ρ^m obtained by translating a_ρ until the head lies at m is an arrow of Q'. At least one of the values $\{w_\rho \mid \mathrm{h}(a_\rho) = j\}$ is nonzero by assumption, and hence for this value of ρ we have $v_\rho^m = w_\rho \neq 0$ as required. □

We now establish notation for the proof of Theorem 3.2. For any vertex $k = (k_1, \ldots, k_\ell) \in Q_0'$, let $(Q'(k), R(k))$ denote the bound quiver of sections of the line bundles $\mathcal{W}_1^{\otimes m_1} \otimes \cdots \otimes \mathcal{W}_\ell^{\otimes m_\ell}$ on Y with $(m_1, \ldots, m_\ell) \leq k$. Explicitly, $Q'(k)$ is the complete sub-quiver of Q' with vertex set $Q'(k)_0 := \{m \in Q_0' \mid m \leq k\}$, and the ideal of relations $R(k) := \Bbbk Q'(k) \cap R$ satisfies

$$\frac{\Bbbk Q'(k)}{R(k)} \cong \operatorname{End}\left(\bigoplus_{(m_1,\ldots,m_\ell)\leq k} \mathcal{W}_1^{\otimes m_1} \otimes \cdots \otimes \mathcal{W}_\ell^{\otimes m_\ell} \right).$$

As in Section 3.2, the coordinate ring $\Bbbk[y_\rho^m \mid \rho \in \Sigma(1), m \leq k]$ of the affine space $\mathbb{A}_\Bbbk^{Q'(k)_1}$ contains ideals $I_{R(k)}, B_{Q'(k)}$ and $I_{Q'(k)}$ defined as in equations (3.2), (3.4) and (3.5) respectively, each of which is homogeneous with respect to the action of $T(k) := \prod_{0 \leq i \leq k} \mathrm{GL}(1)$ by conjugation. The projection onto the coordinates indexed by arrows a_ρ^m satisfying $m \leq k$, denoted

$$\pi_k \colon \mathbb{A}_\Bbbk^{Q_1'} \longrightarrow \mathbb{A}_\Bbbk^{Q'(k)_1}, \tag{3.8}$$

is equivariant with respect to the actions of T and $T(k)$. Notice that $\pi_k(\mathbb{V}(I_R)) \subseteq \mathbb{V}(I_{R(k)})$, $\pi_k(\mathbb{V}(B_{Q'})) \subseteq \mathbb{V}(B_{Q'(k)})$ and $\pi_k(\mathbb{V}(I_{Q'})) \subseteq \mathbb{V}(I_{Q'(k)})$.

Proof of Theorem 3.2. Fix a point $w = (w_\rho^m) \in \mathbb{V}(I_R) \setminus \mathbb{V}(B_{Q'})$ and the corresponding point $v = (v_\rho^m) \in \mathbb{V}(I_{Q'}) \setminus \mathbb{V}(B_{Q'})$ whose components are defined in equation (3.7). Since $w \notin \mathbb{V}(B_{Q'})$, the action of T enables us to assume that for all $m \in Q_0'$ there exists $\rho(m) \in \Sigma(1)$ such that $w_{\rho(m)}^m = 1$. In particular, $v_{\rho(m)}^m = 1$ for all relevant $m \in Q_0'$. Now, for $0 \leq k \leq (s_1 - 1, \ldots, s_\ell - 1)$, the morphism π_k from (3.8) sends the points w and v to

$$\pi_k(w) \in \mathbb{V}(I_{R(k)}) \setminus \mathbb{V}(B_{Q'(k)}) \quad \text{and} \quad \pi_k(v) \in \mathbb{V}(I_{Q'(k)}) \setminus \mathbb{V}(B_{Q'(k)})$$

respectively. We claim that $\pi_k(v)$ lies in the $T(k)$-orbit of $\pi_k(w)$. Given the claim, the special case $k = (s_1 - 1, \ldots, s_\ell - 1)$ shows that the point v lies in the T-orbit of the point w, so Theorem 3.2 follows immediately from Proposition 3.3.

We prove the claim by induction on the vertex $k = (k_1, \ldots, k_\ell)$ using the total order on Q'_0 from Section 3.3. The case $k = e_0$ is immediate, and for $(1, 0, \ldots, 0) \leq k \leq (s_1 - 1, 0, \ldots, 0)$ the claim follows from Example 3.4; hereafter we assume that $\ell \geq 2$. Suppose the claim holds for all $m < k$, so we may assume that $w^m_\rho = w_\rho$ for all $m < k$. It is enough to show for all $\rho \in \Sigma(1)$, that $w_{\rho(k)} \neq 0$ and

$$w^k_\rho = w_\rho (w_{\rho(k)})^{-1}, \tag{3.9}$$

because then we may let the one-dimensional subgroup $\Bbbk^\times \subset T(k)$ scale by $w_{\rho(k)}$ at vertex k. Before establishing the claim (3.9), we introduce some notation that we use in the proof.

Notation 3.9. 1. Recall from Section 3.3 that vertices of the tilting quiver Q' are elements $k = (k_1, \ldots, k_\ell)$ in the lattice $\mathbb{Z}^\ell$, so $k_i \in \mathbb{Z}$ for $1 \leq i \leq \ell$. Note also (see Remark 3.6) that the standard basis vectors $e_1, \ldots, e_\ell$ of $\mathbb{Z}^\ell$ denote certain vertices of Q'. This notation is standard and we hope that no confusion arises in what follows.

2. It is convenient to distinguish certain elements of Q_0 and $\mathbb{Z}^\ell$.

   - First we distinguish certain elements of the vertex set $Q_0 = \{0, 1, \ldots, \ell\}$ of the original quiver. For the ray $\rho(k)$ appearing in (3.9), define $0 \leq \alpha < \beta \leq \ell$ by

   $$\alpha := \mathrm{t}(a_{\rho(k)}) \quad \text{and} \quad \beta := \mathrm{h}(a_{\rho(k)}),$$

   where $a_{\rho(k)}$ is the arrow in the original quiver Q satisfying $\mathrm{div}(a_{\rho(k)}) = D_{\rho(k)}$. Also, let $1 \leq \delta \leq \ell$ be minimal such that the induction vertex $k = (k_1, \ldots, k_\ell)$ satisfies $k_\delta \neq 0$, and define $0 \leq \gamma < \delta$ by setting

   $$\gamma := \mathrm{t}(a_{\rho(e_\delta)}).$$

   Minimality of δ implies that either $\gamma = 0$ or $k_\gamma = 0$ and, moreover, that $\delta \leq \beta$.

   - Next we introduce certain elements of $\mathbb{Z}^\ell$. For any ray $\rho \in \Sigma(1)$, define

   $$\underline{d}(\rho) := e_{\mathrm{h}(a_\rho)} - e_{\mathrm{t}(a_\rho)} \in \mathbb{Z}^\ell,$$

   where a_ρ is the arrow in the original quiver satisfying $\mathrm{div}(a_\rho) = D_\rho$ (recall that $e_0 := 0$). In particular, by the previous bullet point we have

$$\underline{d}(\rho(k)) = e_\beta - e_\alpha \quad \text{and} \quad \underline{d}(\rho(e_\delta)) = e_\delta - e_\gamma.$$

We now return to the proof of the claim (3.9), treating the cases $\delta < \beta$ and $\delta = \beta$ separately.

Case 1: Suppose first that $\delta < \beta$. In this case we proceed in three steps:

Step 1: Show that equation (3.9) holds for $\rho = \rho(e_\delta)$ when $\gamma = \alpha = 0$ or $\gamma \neq \alpha$. We use generators of the ideal $I_{R(k)}$ corresponding to pairs of paths in $Q'(k)$ with head at k. Consider paths of length two as in Figure 3.3, where for now we substitute $\rho(k)$ and $\rho(e_\delta)$ in place of ρ_1 and ρ_2. In this case, we claim

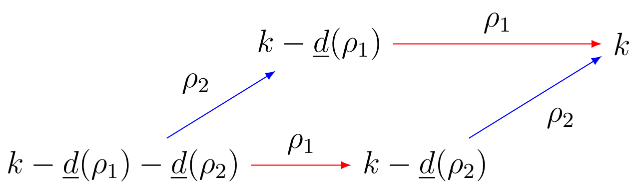


Figure 3.3

that each vertex in Figure 3.3 lies in the quiver $Q'(k)$. Indeed, $a^k_{\rho(k)} \in Q'(k)_1$, so its head k and tail $k - e_\beta + e_\alpha$ lie in $Q'(k)_0$; this implies $k_\beta > 0$ and either $\alpha = 0$ or $k_\alpha < s_\alpha - 1$. Also, $k_\delta > 0$ and either $\gamma = 0$ or $k_\gamma = 0$, so $k - \underline{d}(\rho(e_\delta))$ is equal to $k - e_\delta + e_\gamma$, which lies in the quiver $Q'(k)$. For the fourth vertex in Figure 3.3, either:

(i) $\gamma = \alpha = 0$, giving $e_\gamma = e_\alpha = 0$, and the inequalities $k_\beta, k_\delta > 0$ imply that the fourth vertex $k - e_\beta - e_\delta$ lies in $Q'(k)_0$ as claimed; or

(ii) $\gamma \neq \alpha$, and since $\gamma < \delta < \beta$, the fourth vertex $k - e_\beta + e_\alpha - e_\delta + e_\gamma$ lies in $Q'(k)_0$ because $k_\beta, k_\delta > 0$, either $\alpha = 0$ or $k_\alpha < s_\alpha - 1$ and either $\gamma = 0$ or $k_\gamma = 0$.

Figure 3.3 therefore determines a binomial in $I_{R(k)}$ which implies that

$$w^{k-\underline{d}(\rho(k))}_{\rho(e_\delta)} w^k_{\rho(k)} = w^{k-\underline{d}(\rho(e_\delta))}_{\rho(k)} w^k_{\rho(e_\delta)}.$$

Our induction assumption gives $w^m_\rho = w_\rho$ for all $m < k$, and since $w_{\rho(e_\delta)} = 1 = w^k_{\rho(k)}$, we have $1 = w_{\rho(k)} w^k_{\rho(e_\delta)}$. In particular, $w_{\rho(k)} \neq 0$ and

$$w^k_{\rho(e_\delta)} = (w_{\rho(k)})^{-1}$$

which establishes equation (3.9) for $\rho = \rho(e_\delta)$ when $\gamma = \alpha = 0$ or $\gamma \neq \alpha$.

Step 2: Show that equation (3.9) holds for $\rho = \rho(e_\delta)$ when $\gamma = \alpha \neq 0$. Since $k_\alpha = k_\gamma = 0$, the method from Step 1 applies verbatim unless $s_\gamma = 2$. In this case, define $0 \leq \varepsilon < \gamma$ by

$$\varepsilon := \mathrm{t}(a_{\rho(e_\gamma)}),$$

giving $\underline{d}(\rho(e_\gamma)) = e_\gamma - e_\varepsilon$. Consider paths of length three as in Figure 3.4, where for now we substitute $\rho(k)$, $\rho(e_\delta)$ and $\rho(e_\gamma)$ in place of ρ_1, ρ_2 and ρ_3. Again, we

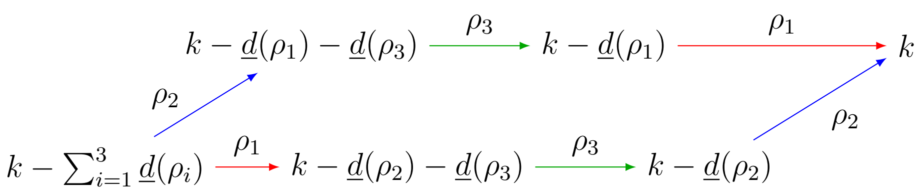


Figure 3.4

claim that each vertex in Figure 3.4 lies in the quiver $Q'(k)$; the proof is similar to that from Step 1 (here, minimality of δ implies $\varepsilon = 0$ or $k_\varepsilon = 0$, and we use the inequalities $\varepsilon < \gamma < \delta < \beta$). Thus we obtain a binomial in $I_{R(k)}$ which, applying the inductive assumption $w_\rho^m = w_\rho$ for all $m < k$, gives

$$w_{\rho(e_\delta)} w_{\rho(e_\gamma)} w^k_{\rho(k)} = w_{\rho(k)} w_{\rho(e_\gamma)} w^k_{\rho(e_\delta)}.$$

Since $w_{\rho(e_\delta)} = w_{\rho(e_\gamma)} = w^k_{\rho(k)} = 1$, we have $w_{\rho(k)} \neq 0$ and $w^k_{\rho(e_\delta)} = (w_{\rho(k)})^{-1}$ which implies that equation (3.9) holds for $\rho = \rho(e_\delta)$.

Step 3: Show that equation (3.9) holds for all $\rho \in \Sigma(1)$. Consider any arrow a^k_ρ in Q' with head at k. The vertices

$$\lambda := \mathrm{t}(a_\rho) \quad \text{and} \quad \mu := \mathrm{h}(a_\rho)$$

satisfy $\underline{d}(\rho) = e_\mu - e_\lambda$ with $0 \leq \lambda < \mu \leq \ell$. We proceed using the approach from Steps 1-2:

(i) If $\mu \neq \beta$, then we substitute ρ and $\rho(k)$ in place of ρ_1 and ρ_2 in Figure 3.3 as in Step 1, unless $\lambda = \alpha \neq 0$ and $s_\alpha = 2$ in which case we substitute $\rho(e_\alpha)$ in place of ρ_3 in Figure 3.4 as in Step 2. In either case, we obtain an equation relating components of w_k which, after applying the inductive hypothesis if necessary, becomes

$$w_{\rho(k)} w^k_\rho = w_\rho w^k_{\rho(k)}.$$

Steps 1 and 2 established $w_{\rho(k)} \neq 0$, and $w^k_{\rho(k)} = 1$, so equation (3.9) holds.

(ii) Otherwise, $\mu = \beta$. Substitute $\rho(e_\delta)$ and ρ in place of ρ_1 and ρ_2 in Figure 3.3 as in Step 1, unless $\lambda = \gamma \neq 0$ and $s_\gamma = 2$ in which case we substitute $\rho(e_\gamma)$ in place of ρ_3 in Figure 3.4 as in Step 2. As in part (i) above, we obtain an equation which simplifies to

$$w_\rho^k = w_\rho w_{\rho(e_\delta)}^k. \tag{3.10}$$

Steps 1 and 2 established $w_{\rho(e_\delta)}^k = (w_{\rho(k)})^{-1}$, so equation (3.9) follows.

This completes the proof of equation (3.9) in Case 1.

Case 2: Suppose instead that $\delta = \beta$. If $k_\delta > 1$ then the proof is identical to Case 1. If on the other hand $k_\delta = 1$, then the vertex $k - \underline{d}(\rho(e_\delta)) - \underline{d}(\rho(k)) = k - 2e_\delta + e_\gamma + e_\alpha$ that plays a key role in Case 1 does not lie in $Q'(k)_0$. In the special case that $k = e_\delta$, making k a vertex of the base quiver, then we have $w_\rho^k = w_\rho$ for all relevant $\rho \in \Sigma(1)$ and there is nothing to prove. If $k \neq e_\delta$, we introduce another useful vertex of the original quiver: let ξ be minimal such that $\delta < \xi \leq \ell$ and $k_\xi \neq 0$, and define $0 \leq \eta < \xi$ by setting

$$\eta := \mathrm{t}(a_{\rho(e_\xi)})$$

giving $\underline{d}(\rho(e_\xi)) = e_\xi - e_\eta$. We treat the cases $\eta \neq \delta$ and $\eta = \delta$ separately.

Subcase 2A: If $\eta \neq \delta (= \beta)$, then either $\eta = 0$ or $k_\eta = 0$, so $k - \underline{d}(\rho(e_\xi)) = k - e_\xi + e_\eta$ is a vertex of $Q'(k)_0$. We may now proceed just as in Case 1 except that $\rho(e_\xi)$ replaces $\rho(e_\delta)$ throughout (so ξ and η replace δ and γ respectively).

Subcase 2B: Suppose instead that $\eta = \delta (= \beta)$. We've already reduced to the case $k_\delta = 1$. If $s_\delta > 2$ then once again, $k - \underline{d}(\rho(e_\xi)) = k - e_\xi + e_\delta$ is a vertex of $Q'(k)_0$ and we proceed as in Case 1 with $\rho(e_\xi)$ replacing $\rho(e_\delta)$ throughout. If $s_\delta = 2$, then we proceed as follows:

Step 1: Show that $w_{\rho(k)} \neq 0$. If $\gamma \neq \alpha$ or $\gamma = \alpha = 0$, then we use Figure 3.4 with $\rho_1 = \rho(k)$, $\rho_2 = \rho(e_\delta)$ and $\rho_3 = \rho(e_\xi)$ to obtain the equation

$$1 = w_{\rho(e_\delta)} w_{\rho(e_\xi)} w_{\rho(k)}^k = w_{\rho(k)} w_{\rho(e_\xi)} w_{\rho(e_\delta)}^k$$

which gives $w_{\rho(k)} \neq 0$. Otherwise, $\gamma = \alpha \neq 0$, giving $\underline{d}(\rho(k)) = e_\delta - e_\gamma = \underline{d}(\rho(e_\delta))$. It may be that $\rho(k) = \rho(e_\delta)$, in which case $w_{\rho(k)} = w_{\rho(e_\delta)} = 1$ and hence $w_{\rho(k)} \neq 0$ as required. If $\rho(k) \neq \rho(e_\delta)$, then consider the pair of paths of length four as in Figure 3.5, where we substitute $\rho(k)$, $\rho(e_\delta)$, $\rho(e_\xi)$ and $\rho(e_\gamma)$ in place of $\rho_1, \dots, \rho_4$ (in fact, both paths pass through the same set of vertices in this case).

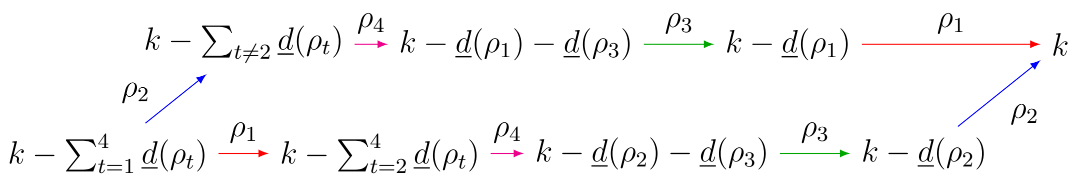


FIGURE 3.5

We obtain the equation

$$1 = w_{\rho(e_\delta)} w_{\rho(e_\gamma)} w_{\rho(e_\xi)} w^k_{\rho(k)} = w_{\rho(k)} w_{\rho(e_\gamma)} w_{\rho(e_\xi)} w^k_{\rho(e_\delta)}$$

which gives $w_{\rho(k)} \neq 0$ and completes STEP 1.

STEP 2: Show that equation (3.9) holds for all $\rho \in \Sigma(1)$. For any $a^k_\rho \in Q'_1$, the vertices

$$\lambda := \mathrm{t}(a_\rho) \quad \text{and} \quad \mu := \mathrm{h}(a_\rho)$$

satisfy $\underline{d}(\rho) = e_\mu - e_\lambda$ with $0 \leq \lambda < \mu \leq \ell$.

(i) If $\mu > \delta$, use Figure 3.3 with $\rho_1 = \rho(k)$ and $\rho_2 = \rho$, unless $\lambda = \alpha \neq 0$ and $s_\alpha = 2$ in which case use Figure 3.4 with the addition of $\rho_3 = \rho(e_\alpha)$. Either way, we obtain the equation $w_{\rho(k)} w^k_\rho = w_\rho w^k_{\rho(k)}$ which, since $w_{\rho(k)} \neq 0$ by STEP 1, gives (3.9).

(ii) If $\mu = \delta$, use Figure 3.4 with $\rho_1 = \rho(k)$, $\rho_2 = \rho$ and $\rho_3 = \rho(e_\xi)$ unless $\lambda = \alpha \neq 0$ and $s_\alpha = 2$ in which case use Figure 3.5 with the addition of $\rho_4 = \rho(e_\alpha)$. Either way, we obtain $w_{\rho(k)} w^k_\rho = w_\rho w^k_{\rho(k)}$ which, since $w_{\rho(k)} \neq 0$ by STEP 1, gives (3.9).

This concludes the proof in CASE 2, and completes the proof of Theorem 3.2. □

Remark 3.10. Our approach relies on the explicit description of the image of the morphism f_E in Theorem 3.2 as the GIT quotient $\mathbb{V}(I_{Q'}) /\!\!/_\theta T$, see [CS08, Theorem 1.1]. We do not at present have a similar description in the non-toric setting.

Example 3.11. We conclude with an example to illustrate the proof of Theorem 3.2. Let Q and Q' be the quivers in Figure 3.2, so $\ell = 3$. Suppose $k = (0,1,1) \in Q'_0$, so $\delta = 2$. The three arrows with head at k have tails at $(1,1,0)$ (light blue), $(0,0,1)$ (red) and $(1,0,1)$ (blue),and we label the corresponding rays ρ_1, ρ_2 and ρ_3 respectively. We now illustrate in two different situations why $w_{\rho(k)} \neq 0$ and why the equation $w^k_\rho = w_\rho (w_{\rho(k)})^{-1}$ from (3.9) holds for $\rho = \rho_1, \rho_2, \rho_3$.

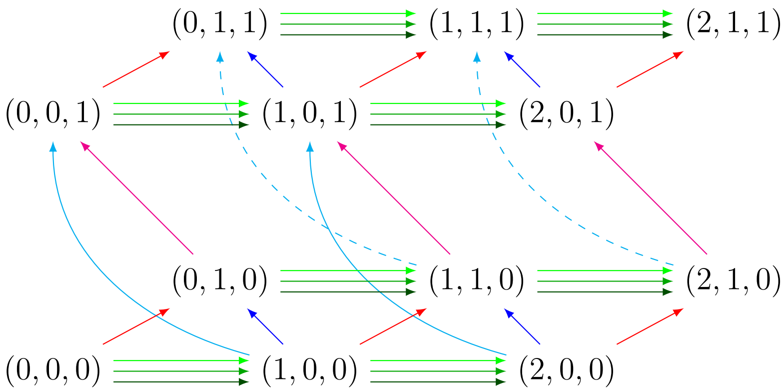
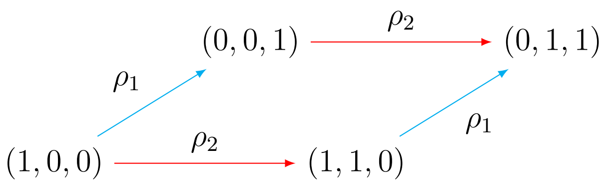


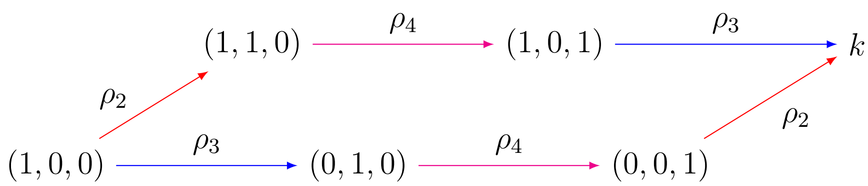
FIGURE 3.6: The tilting quiver for Q from Figure 3.2.

1. Suppose that $\rho(k) = \rho_1$. Then $\beta = 3$ and $\alpha = 1$ (see Figure 3.2(a)), and $w^k_{\rho_1} = 1$. Suppose $\rho(e_\delta) = \rho(e_2) = \rho_2$ so that $\gamma = 0$ and $w_{\rho_2} = 1$. This is an example of CASE 1 as $\delta < \beta$, and since $\gamma = 0$ we require only STEP 1. In this case Figure 3.3 becomes



and the relation gives the equation $w_{\rho_2} w^k_{\rho_1} = w_{\rho_1} w^k_{\rho_2}$. Moreover, $w_{\rho_2} = 1 = w^k_{\rho_1}$ implies $w_{\rho_1} \neq 0$ and $w^k_{\rho_2} = (w_{\rho_1})^{-1}$ which establishes (3.9) for $\rho = \rho_1, \rho_2$. The remaining arrow $a^k_{\rho_3}$ with head at k requires STEP 3, and in this case for $\rho = \rho_3$ we have $\mu = 2$ and $\lambda = 1$. Since $\mu \neq \beta$ and $s_\alpha = s_1 \neq 2$, we require STEP 3(i) to deduce $w_{\rho_3} w^k_{\rho_1} = w_{\rho_1} w^k_{\rho_3}$. This implies $w^k_{\rho_3} = w_{\rho_3} (w_{\rho_1})^{-1}$, establishing (3.9) for $\rho = \rho_3$.

2. Suppose $\rho(k) = \rho_3$, so $\beta = 2$, $\alpha = 1$ and $w^k_{\rho_3} = 1$. Suppose that $\rho(e_2) = \rho_2$, so $\gamma = 0$ and $w_{\rho_2} = 1$. Since $\delta = \beta$ and $k_\delta = k_2 = 1$, this is an example of CASE 2. Since $k \neq e_2$, we compute $\xi = 3$. Write ρ_4 for the label of the pink arrow with head at $(0, 0, 1)$ and tail at $(0, 1, 0)$, and suppose $\rho(e_3) = \rho_4$. Then $\eta = \mathrm{t}(a_{\rho_4}) = 2$ and $w_{\rho_4} = 1$. Since $\eta = \delta$ and $s_\delta = 2$, we require SUBCASE 2B. Following STEP 1, since $\gamma = 0$ we use Figure 3.4 as shown below. This yields the equation $w_{\rho_2} w_{\rho_4} w^k_{\rho_3} = w_{\rho_3} w_{\rho_4} w^k_{\rho_2}$ which simplifies to $1 = w_{\rho_3} w^k_{\rho_2}$, giving $w_{\rho_3} \neq 0$ as required. STEP 2 of SUBCASE 2B establishes (3.9) for $\rho = \rho_1, \rho_2, \rho_3$: we already know this for $\rho = \rho_3$ by assumption; the case $\rho = \rho_2$ is provided by STEP 1 since $w^k_{\rho_2} = (w_{\rho_3})^{-1}$; and the case $\rho = \rho_1$ is a simple application of STEP 2(i), where we apply Figure 3.3 to



the rectangle with vertices $(2,0,0),(1,1,0),(1,0,1),k$ and arrows labelled ρ_1 and ρ_3.

Chapter 4

The tilting quiver for $\mathrm{Gr}(n, 2)$

The remainder of this thesis considers the case where Y is the Grassmannian of 2-dimensional quotients of a n-dimensional vector space, though we begin in this chapter with some minor results for the general case when $Y = \mathrm{Gr}(n,r)$ for any $n > r \geq 2$.

Let r, n be positive integers satisfying $1 \leq r < n$ and hereafter fix $V = \Bbbk^n$. Let Q be the quiver with two vertices $0, 1$ and n arrows from 0 to 1 as in Figure 4.1. With dimension vector $\underline{r} = (1, r)$ and $\chi = (-1, 1)$, the quiver flag variety $Y = \mathrm{Rep}(Q, \underline{r}) /\!\!/_{\chi} G$ is isomorphic to $\mathrm{Gr}(n,r) := \mathrm{Gr}(V, r)$, the Grassmannian of r-dimensional quotients of V, as shown in Example 2.5(ii). Note that since Chapter 3 covers the case where Y is projective space, and because $\mathrm{Gr}(n,r) \cong \mathrm{Gr}(n, n-r)$, we hereafter assume that $n \geq 4$ and $1 < r \leq n/2$.

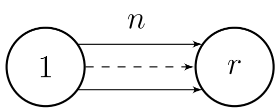


FIGURE 4.1: The quiver Q with n arrows such that $Y = \mathrm{Gr}(n,r)$.

In this case, the tower of Grassmann-bundles from Theorem 2.3 is height one and so we define $\mathcal{W} := \mathcal{W}_1$, the rank r tautological quotient bundle on Y. Recall that

$$s_1 = \sum_{\{a \in Q_1 \mid \mathrm{h}(a) = 1\}} r_{\mathrm{t}(a)} = n,$$

so the tilting bundle from (2.6) is given by

$$E = \bigoplus_{\lambda \in \mathrm{Young}(n-r, r)} \mathbb{S}^{\lambda} \mathcal{W}. \tag{4.1}$$

This tilting bundle was first introduced by Kapranov in [Kap84], and as such we refer to $A = \mathrm{End}_{\mathcal{O}_Y}(E)$ as *Kapranov's tilting algebra.*

As in the toric case, we aim to present A using Q', a quiver with relations that we call the tilting quiver. Since the indecomposable summands of E need not be line bundles, the process of determining Q' is much more involved than in the toric case. In this chapter we will describe the structure of the tilting quiver Q' for $Y = \mathrm{Gr}(n, 2)$, which has vertex set given by the irreducible summands of E and arrow set corresponding to the homomorphisms between these summands, and use this to define a surjective $\Bbbk$-algebra homomorphism

$$\Phi \colon \Bbbk Q' \twoheadrightarrow A.$$

4.1 The Schur functor

In this section we use [Ful97, p.104-7] to construct $\mathbb{S}^\lambda \mathcal{W}$, the vector bundle whose fibre over each point is the irreducible $\mathrm{GL}(r)$-module of highest weight λ, as a quotient of $\mathcal{W}^{\otimes |\lambda|}$. We will do this using the Schur functor.

Let V be a finite-dimensional $\Bbbk$-vector space and λ be a partition. Whereas the Cartesian product of n copies of V is denoted $V^n = V \times \cdots \times V$, we write $V^{\times \lambda}$ for the Cartesian product of $|\lambda|$ copies of V labelled by the boxes of the Young diagram of λ. We will consider maps $\varphi \colon V^{\times\lambda} \to U$, where U is a $\Bbbk$-vector space, satisfying three properties:

(i) φ is multilinear.

(ii) φ is alternating in the entries of each column of λ.

(iii) $\varphi(v) = \sum \varphi(w)$, where the sum is taken over all w obtained from v by an *exchange* (see below) between two fixed columns and a fixed subset of boxes in the right hand column.

The process of an exchange is defined as follows. Given a Young diagram λ, fix two columns c_1, c_2 with c_1 to the left of c_2 with $n_1 \geq n_2 \geq 1$ the number of boxes in each column respectively. Let d be a set of $n_3 \leq n_2$ boxes in c_2. Swap the n_3 boxes of d with any n_3 boxes of c_1 whilst maintaining the vertical order of each subset of boxes; this is called an *exchange*. The sum $\varphi(v) = \sum \varphi(w)$ is taken over all such exchanges, i.e. each combination of choosing n_3 boxes of c_1 after fixing c_1, c_2 and d. Given v, c_1, c_2, d, there are $\binom{n_1}{n_3}$ many w in the sum $\varphi(v) = \sum \varphi(w)$.

Example 4.1. Suppose $\lambda = (3,3,2)$ and fix the first and last columns of λ. Suppose that the fixed subset of boxes in the right hand column is simply the top box. Then the sum $\varphi(v) = \sum \varphi(w)$ becomes

$$\varphi\left(\begin{array}{|c|c|c|}\hline x_1 & y_1 & z_1 \\ \hline x_2 & y_2 & z_2 \\ \hline x_3 & y_3 & \\ \cline{1-2}\end{array}\right) = \varphi\left(\begin{array}{|c|c|c|}\hline z_1 & y_1 & x_1 \\ \hline x_2 & y_2 & z_2 \\ \hline x_3 & y_3 & \\ \cline{1-2}\end{array}\right) + \varphi\left(\begin{array}{|c|c|c|}\hline x_1 & y_1 & x_2 \\ \hline z_1 & y_2 & z_2 \\ \hline x_3 & y_3 & \\ \cline{1-2}\end{array}\right) + \varphi\left(\begin{array}{|c|c|c|}\hline x_1 & y_1 & x_3 \\ \hline x_2 & y_2 & z_2 \\ \hline z_1 & y_3 & \\ \cline{1-2}\end{array}\right).$$

Now suppose we fix the same columns but select both boxes in the right hand column as the fixed subset. Being careful to maintain the vertical order of the boxes, the sum $\varphi(v) = \sum \varphi(w)$ becomes

$$\varphi\left(\begin{array}{|c|c|c|}\hline x_1 & y_1 & z_1 \\ \hline x_2 & y_2 & z_2 \\ \hline x_3 & y_3 & \\ \cline{1-2}\end{array}\right) = \varphi\left(\begin{array}{|c|c|c|}\hline z_1 & y_1 & x_1 \\ \hline z_2 & y_2 & x_2 \\ \hline x_3 & y_3 & \\ \cline{1-2}\end{array}\right) + \varphi\left(\begin{array}{|c|c|c|}\hline z_1 & y_1 & x_1 \\ \hline x_2 & y_2 & x_3 \\ \hline z_2 & y_3 & \\ \cline{1-2}\end{array}\right) + \varphi\left(\begin{array}{|c|c|c|}\hline x_1 & y_1 & x_2 \\ \hline z_1 & y_2 & x_3 \\ \hline z_2 & y_3 & \\ \cline{1-2}\end{array}\right).$$

Definition 4.2 ([Ful97, p.106-7])**.** Let V be a finite-dimensional $\Bbbk$-vector space and $\lambda = (\lambda_1, \dots, \lambda_r)$ a partition. The *Schur power* $\mathbb{S}^\lambda V$ is the universal target vector space for maps φ as described above. Explicitly, $\mathbb{S}^\lambda V$ is the quotient space of $V^{\otimes|\lambda|}$ given by

$$\mathbb{S}^\lambda V = \frac{(\bigwedge^r V)^{\otimes \lambda_r} \otimes \left(\bigwedge^{r-1} V\right)^{\otimes(\lambda_{r-1}-\lambda_r)} \otimes \cdots \otimes \left(\bigwedge^2 V\right)^{\otimes(\lambda_2-\lambda_3)} \otimes V^{\otimes(\lambda_1-\lambda_2)}}{E_\lambda}$$

where E_λ is the subspace generated by all possible exchanges on the Young diagram λ as described above. We call the elements of E_λ *exchange relations*. The map $V^{\times\lambda} \to \mathbb{S}^\lambda V$ is given by taking the wedge product of entries in each column from top to bottom and then tensoring these together.

Theorem 4.3 ([Ful97, Theorem 2, p.114])**.** *Let V be a finite-dimensional $\Bbbk$-vector space and λ be a partition with at most $\dim(V)$ parts. Then $\mathbb{S}^\lambda V$ is the irreducible polynomial representation of* $\mathrm{GL}(V)$ *with highest weight λ. Moreover, these are all of the irreducible polynomial representations of* $\mathrm{GL}(V)$*.*

Definition 4.4. The functor of finite-dimensional $\Bbbk$-vector spaces $\mathbb{S}^\lambda \colon \mathrm{Vect} \to \mathrm{Vect}$, $V \mapsto \mathbb{S}^\lambda V$, is called the *Schur functor*. While we have defined this functor on Vect, it may also be defined on many other categories, in particular vector bundles and G-modules.

Examples 4.5. (i) If $\lambda = (1, \dots, 1)$ with $|\lambda| = k \geq 1$ then the Young diagram of λ has only one column and so E_λ is trivial, therefore $\mathbb{S}^\lambda V = \bigwedge^k V$.

(ii) Suppose $\lambda = (3,3,2)$ as in Example 4.1. Then $\mathbb{S}^\lambda V = \dfrac{(\bigwedge^3 V)^{\otimes 2} \otimes \bigwedge^2 V}{E_\lambda}$ and the map $V^{\times\lambda} \to \mathbb{S}^\lambda V$ is given by

$$\begin{array}{|c|c|c|} \hline x_1 & y_1 & z_1 \\ \hline x_2 & y_2 & z_2 \\ \hline x_3 & y_3 \\ \cline{1-2} \end{array} \mapsto x_1 \wedge x_2 \wedge x_3 \otimes y_1 \wedge y_2 \wedge y_3 \otimes z_1 \wedge z_2.$$

(iii) If $\lambda = (k)$ then $\mathbb{S}^{(k)}V = V^{\otimes k}/E_{(k)}$. However, every column of λ contains only one box and so the exchange relations are reduced to simply permuting the boxes. This implies $\mathbb{S}^{(k)}V = \operatorname{Sym}^k V$.

Remark 4.6. As a result of Example 4.5(iii), it will be more convenient to identify the $V^{\otimes(\lambda_1-\lambda_2)}$ term in the definition of $\mathbb{S}^\lambda V$ with $\operatorname{Sym}^{(\lambda_1-\lambda_2)} V$ after taking the quotient by E_λ. Henceforth we write

$$\mathbb{S}^\lambda V = \frac{(\bigwedge^r V)^{\otimes\lambda_r} \otimes \left(\bigwedge^{r-1} V\right)^{\otimes(\lambda_{r-1}-\lambda_r)} \otimes \cdots \otimes \left(\bigwedge^2 V\right)^{\otimes(\lambda_2-\lambda_3)} \otimes \operatorname{Sym}^{(\lambda_1-\lambda_2)} V}{E_\lambda}. \tag{4.2}$$

4.2 Littlewood-Richardson numbers

By Theorem 4.3, $\mathbb{S}^\lambda V$ is irreducible and so products such as $\mathbb{S}^\lambda V \otimes \mathbb{S}^\mu V$ are in general reducible. The *Pieri rule* gives us the irreducible decomposition of a Schur power multiplied by a symmetric power or alternating power.

Proposition 4.7 (Pieri rule [FH91, Eqn 6.8-9])**.** *Let λ be a partition and $m \in \mathbb{N}$.*

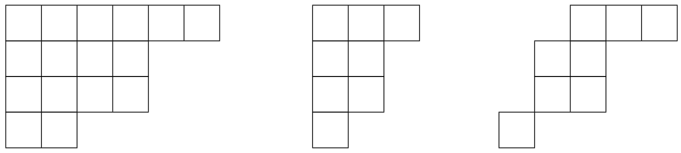
(i) *We have*
$$\mathbb{S}^\lambda V \otimes \operatorname{Sym}^m V \cong \bigoplus_\gamma \mathbb{S}^\gamma V$$
where γ ranges over all partitions formed by adding m boxes to λ with no two new boxes in the same column.

(ii) *We have*
$$\mathbb{S}^\lambda V \otimes \bigwedge^m V \cong \bigoplus_\gamma \mathbb{S}^\gamma V$$
where γ ranges over all partitions formed by adding m boxes to λ with no two new boxes in the same row.

In this section we will see how to decompose the product of any two Schur powers using the *Littlewood-Richardson rule*. To do this we define Littlewood-Richardson numbers and present the rule following mostly [Ful97, Chapter 5].

Let λ, μ be partitions with r parts and suppose μ *contains* λ, i.e. $\lambda_i \leq \mu_i$ for all $i = 1, \ldots, r$, and we write $\lambda \leq \mu$. Note that $\lambda \leq \mu \implies |\lambda| \leq |\mu|$ but the converse is not true in general. We write $\lambda < \mu$ for strict containment.

The *skew diagram* μ/λ is given by the Young diagram μ with λ removed from the top left corner. For example, if $\lambda = (3, 2, 2, 1)$ and $\mu = (6, 4, 4, 2)$ then μ/λ is as follows:



μ $\qquad$ λ $\qquad$ μ/λ

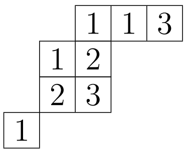
Observe that any Young diagram μ is also a skew diagram μ/λ where $\lambda = (0)$.

A *filling* of a skew diagram is the insertion of a positive integer into each box. A *(semi-standard) skew tableau* is a skew diagram with a filling such that:

(i) each row is weakly increasing.

(ii) each column is strictly increasing.

We say that a skew tableau μ/λ has *content* $\gamma = (\gamma_1, \ldots, \gamma_k) \in \mathbb{N}^k$ if μ/λ contains γ_1 1's, γ_2 2's, and so on up to γ_k k's. For example, taking λ and μ as above, one possible skew tableau with shape μ/λ and content $\gamma = (4, 2, 2)$ is



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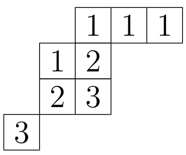
The sequence of integers given by concatenating the rows of a skew tableau from top to bottom and in reverse order is called the *reverse word* of the tableau, e.g. the reverse word of the tableau above is $3, 1, 1, 2, 1, 3, 2, 1$. We say that a word is a *lattice word* if the content of every initial sequence is a partition, i.e. for every initial sequence of the word there must contain at least as many 1's as 2's, at least as many 2's as 3's, and so on. The reverse word above is not a lattice word, for example, as the first number is a 3 so there are more 3's than 1's in the first letter of the reverse word. However, if we swap the 3 in the top right box with the 1 in the bottom left box as follows,



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we now have a tableau whose reverse word $1, 1, 1, 2, 1, 3, 2, 3$ is a lattice word. This leads to the following definition.

Definition 4.8. A *Littlewood-Richardson tableau* is a skew tableau whose reverse word is a lattice word.

Remark 4.9. Due to the lattice word condition, the rightmost box of the top row of a Littlewood-Richardson tableau must contain a 1. Since we require rows to be weakly increasing, this implies that the entire top row must contain only 1's. This observation extends to the fact that any integer k may not appear above the k-th row in the skew diagram.

Definition 4.10. Let λ, μ, γ be partitions such that $\lambda \leq \mu$ and $|\lambda| + |\gamma| = |\mu|$. The *Littlewood-Richardson number* $c^{\mu}_{\lambda,\gamma} \in \mathbb{N}$ is equal to the number of Littlewood-Richardson tableaux of shape μ/λ and content γ.

Proposition 4.11 ([Ful97, §5.2 Corollary. 2])**.** *Let λ, μ, γ be partitions such that $|\lambda| + |\gamma| = |\mu|$. Then $c^{\mu}_{\lambda,\gamma} = c^{\mu}_{\gamma,\lambda}$. Moreover, $c^{\mu}_{\lambda,\gamma} = 0$ if either λ or γ is not contained in μ.*

Remark 4.12. We observe some useful facts for some basic Littlewood-Richardson numbers.

(i) If λ, γ, μ have at most two parts then $c^{\mu}_{\lambda,\gamma}$ is equal to either 0 or 1. This is because only 1's may be placed in the top row of μ/λ, and all the 2's must be right aligned in the bottom row. The remaining 1's must then be placed left of the 2's.

(ii) When the skew diagram μ/λ consists of a single row or column of size k, the reverse lattice word condition implies that $c^{\mu}_{\lambda,\gamma} = 1$ when $\gamma = (k)$ or $\gamma = (1, \ldots, 1) \in \mathbb{Z}^k$ respectively, otherwise $c^{\mu}_{\lambda,\gamma} = 0$.

It turns out that Littlewood-Richardson numbers determine the multiplicity of summands in the decomposition of a tensor product of Schur powers into irreducibles.

Proposition 4.13 ([FH91, §6.1 Eqn. 6.7: Littlewood-Richardson rule])**.** *Let V be an r-dimensional vector space and let λ, γ be partitions with at most r parts. Then*

$$\mathbb{S}^{\lambda} V \otimes \mathbb{S}^{\gamma} V \cong \bigoplus_{\mu} (\mathbb{S}^{\mu} V)^{\oplus c^{\mu}_{\lambda,\gamma}},$$

where μ ranges over all partitions satisfying $|\mu| = |\lambda| + |\gamma|$.

Remark 4.14. In Proposition 4.13 we needn't insist μ has at most r parts because when $\dim(V) = r$, we have $\mathbb{S}^\mu V = 0$ whenever $\mu_{r+1} > 0$, see [FH91, §6.1, p.76].

4.3 The skew-Schur functor

We now construct the *skew-Schur functor*, which generalises the Schur functor defined in Section 4.1. The following strategy can be found in [FH91, Ex 6.19].

Notation 4.15. Hereafter we fix a basis $\mathcal{B} = \{u_1, \dots, u_n\}$ of V. When choosing an arbitrary collection of these vectors, possibly with multiplicity, we will use the letters $v_i \in \mathcal{B}$. We do this to avoid writing u_{i_j} for elements of $\mathcal{B}$, for example, as later we will need space for multiple other subscripts.

Fix partitions $\lambda < \mu$ with at most r parts and set $d = |\mu| - |\lambda|$. For any filling of the skew diagram μ/λ with entries in $\{1, \dots, n\}$, there is a corresponding basis vector of $V^{\otimes d}$ given by reading the content of the boxes from left to right, top to bottom. For example, if $\mu/\lambda = (3, 2)/(1, 0)$ and $n = 4$ then

$$\begin{array}{ccc} & \boxed{3} & \boxed{1} \\ \boxed{4} & \boxed{3} & \end{array} \longleftrightarrow u_3 \otimes u_1 \otimes u_4 \otimes u_3 \in V^{\otimes 4}.$$

Consider the action of S_d, the permutation group on $\{1, \dots, d\}$, on $V^{\otimes d}$ by permuting the indices, i.e. for $\sigma \in S_d$ and $v_i \in \mathcal{B}$ we have $(v_1 \otimes \cdots \otimes v_d) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}$. Define the subgroups

$$\begin{aligned} P_{\text{row}} &= \{\sigma \in S_d \mid \sigma \text{ preserves the content of each row of } \mu/\lambda\}, \\ P_{\text{col}} &= \{\sigma \in S_d \mid \sigma \text{ preserves the content of each column of } \mu/\lambda\}. \end{aligned}$$

Now consider the group algebra $\Bbbk S_d$ with generators e_σ and define the elements

$$\begin{aligned} a_{\mu/\lambda} &= \sum_{\sigma \in P_{\text{row}}} e_\sigma, \\ b_{\mu/\lambda} &= \sum_{\sigma \in P_{\text{col}}} \operatorname{sgn}(\sigma) e_\sigma. \end{aligned}$$

These define endomorphisms on $V^{\otimes d}$ by setting $e_\sigma(v) = v \cdot \sigma$, and we have

$$\begin{aligned} \operatorname{im}(a_{\mu/\lambda}) &\cong \operatorname{Sym}^{\mu_1 - \lambda_1} V \otimes \cdots \otimes \operatorname{Sym}^{\mu_r - \lambda_r} V, \\ \operatorname{im}(b_{\mu/\lambda}) &\cong \bigwedge^{\mu'_1 - \lambda'_1} V \otimes \cdots \otimes \bigwedge^{\mu'_k - \lambda'_k} V, \end{aligned}$$

where λ', μ' are the conjugate partitions to λ, μ (list the heights of the columns left to right instead of the lengths of the rows top to bottom).

Definition 4.16. The *Young symmetrizer* with respect to the skew diagram μ/λ is defined by

$$c_{\mu/\lambda} := b_{\mu/\lambda} a_{\mu/\lambda}.$$

The image of $c_{\mu/\lambda}$ on a vector in $V^{\otimes d}$ is given by summing over the symmetrization of rows followed by the anti-symmetrization of columns. This defines an endomorphism on $V^{\otimes d}$ and we call its image the *skew-Schur power*, denoted $\mathbb{S}^{\mu/\lambda}V$, i.e.

$$\mathbb{S}^{\mu/\lambda}V := \operatorname{im}(c_{\mu/\lambda}).$$

As in Definition 4.4, the skew-Schur functor $\mathbb{S}^{\mu/\lambda}$ may be defined on many other categories.

Proposition 4.17 ([FH91, Ex 6.19])**.** *Let $\lambda < \mu$ with $d = |\mu| - |\lambda|$ and let $\mathcal{B} = \{u_1, \dots, u_n\}$ be a basis of V. Then for each semi-standard skew tableau with shape μ/λ filled with integers from $\{1, \dots, n\}$, the images of the corresponding basis vectors in $V^{\otimes d}$ under $c_{\mu/\lambda}$ form a basis of $\mathbb{S}^{\mu/\lambda}V$.*

Example 4.18. Let $\lambda = (1,0)$, $\mu = (2,2)$ and $\mathcal{B} = \{u_1, u_2, u_3\}$ be a basis of V. We have $d = |\mu| - |\lambda| = 3$. Consider the semi-standard skew tableau $\begin{array}{cc} & \boxed{1} \\ \boxed{2} & \boxed{3} \end{array}$, which corresponds to the basis vector $u_1 \otimes u_2 \otimes u_3 \in V^{\otimes 3}$. Symmetrizing the rows, we have $a_{\mu/\lambda} = \begin{array}{cc} & \boxed{1} \\ \boxed{2} & \boxed{3} \end{array} + \begin{array}{cc} & \boxed{1} \\ \boxed{3} & \boxed{2} \end{array}$, and now anti-symmetrizing the columns gives

$$c_{\mu/\lambda}\left(\begin{array}{cc} & \boxed{1} \\ \boxed{2} & \boxed{3} \end{array}\right) = \begin{array}{cc} & \boxed{1} \\ \boxed{2} & \boxed{3} \end{array} - \begin{array}{cc} & \boxed{3} \\ \boxed{2} & \boxed{1} \end{array} + \begin{array}{cc} & \boxed{1} \\ \boxed{3} & \boxed{2} \end{array} - \begin{array}{cc} & \boxed{2} \\ \boxed{3} & \boxed{1} \end{array}.$$

Thus, $c_{\mu/\gamma}(u_1 \otimes u_2 \otimes u_3) = u_1 \otimes u_2 \otimes u_3 - u_3 \otimes u_2 \otimes u_1 + u_1 \otimes u_3 \otimes u_2 - u_2 \otimes u_3 \otimes u_1$. In general therefore, $\mathbb{S}^{(2,2)/(1,0)}V$ is the subspace of $V^{\otimes 3}$ spanned by vectors of the form

$$u_{i_1} \otimes u_{i_2} \otimes u_{i_3} - u_{i_3} \otimes u_{i_2} \otimes u_{i_1} + u_{i_1} \otimes u_{i_3} \otimes u_{i_2} - u_{i_2} \otimes u_{i_3} \otimes u_{i_1}$$

where $\begin{array}{cc} & \boxed{i_1} \\ \boxed{i_2} & \boxed{i_3} \end{array}$ is a semi-standard skew tableau with each $i_j \in \{1,2,3\}$.

Remark 4.19. When $\lambda = (0)$ the skew-Schur functor coincides with the Schur functor. In this case, we write c_μ (rather than $c_{\mu/\lambda}$) for the corresponding Young symmetrizer. An important application of Young symmetrizers of the form c_μ is that it allows us to describe $\mathbb{S}^\mu V$ from (4.2) as a subspace of $V^{\otimes|\mu|}$ rather than as a quotient.

As a consequence of Theorem 4.3, skew-Schur powers are not irreducible in general. Their decomposition is given by the following proposition.

Proposition 4.20 ([FH91, §6.1 Ex. 6.19]). *Let V be an r-dimensional vector space and let λ, μ be partitions with at most r parts satisfying $\lambda \leq \mu$. Then $\mathbb{S}^{\mu/\lambda}V$ is a polynomial representation of $\mathrm{GL}(V)$ with irreducible decomposition*

$$\mathbb{S}^{\mu/\lambda}V \cong \bigoplus_{\gamma} (\mathbb{S}^\gamma V)^{\oplus c^\mu_{\lambda,\gamma}},$$

where γ ranges over all partitions satisfying $|\gamma| = |\mu| - |\lambda|$.

4.4 Generators for Kapranov's tilting algebra

For $Y = \mathrm{Gr}(n, r) = \mathrm{Gr}(V, r)$, recall the tilting bundle E from (4.1). Kapranov's tilting algebra $A = \mathrm{End}_{\mathcal{O}_Y}(E)$ may be decomposed as the collection of spaces $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda\mathcal{W}, \mathbb{S}^\mu\mathcal{W})$ for all pairs $\lambda, \mu \in \mathrm{Young}(n-r, r)$. In this section we give a presentation of these spaces and describe them explicitly in some simple cases. The key tool for this will be a result from Kapranov's presentation of $D^b(\mathrm{Coh}(Y))$. Unless stated otherwise, we hereafter identify $\mathbb{S}^\lambda\mathcal{W}$ with the quotient of $\mathcal{W}^{\otimes|\lambda|}$ given by the expression in Remark 4.6.

Theorem 4.21 ([Kap84, p.189 3.0]). *Let $\mathcal{W}$ be the rank r tautological quotient bundle of $\mathrm{Gr}(V, r)$ and let $\lambda, \mu \in \mathrm{Young}(n-r, r)$. Then*

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda\mathcal{W}, \mathbb{S}^\mu\mathcal{W}) \cong \bigoplus_{\gamma} \mathbb{S}^\gamma V, \tag{4.3}$$

where γ ranges over the positive summands in the decomposition of $\mathbb{S}^{(-\lambda_r,\ldots,-\lambda_1)}\mathcal{W} \otimes \mathbb{S}^{(\mu_1,\ldots,\mu_r)}\mathcal{W}$ into irreducibles.

We will analyse this result more closely, in particular concentrating on finding precisely the multiplicities of each γ in (4.3). Firstly, for partitions with negative entries we use the identity [Kap84, Eqn 0.1],

$$\mathbb{S}^{(-\lambda_r,\ldots,-\lambda_1)}\mathcal{W} \cong \mathbb{S}^\lambda(\mathcal{W}^\vee) \cong (\mathbb{S}^\lambda\mathcal{W})^\vee, \tag{4.4}$$

where $\mathcal{W}^\vee$ denotes the dual bundle of $\mathcal{W}$. Moreover, Kapranov explains that Schur powers of bundles with negative entries may be dealt with by multiplying and then dividing by a line bundle ([Kap84, 2.1, p.187]). For $m \in \mathbb{Z}$, this implies the identity

$$\mathbb{S}^{\lambda}\mathcal{W} \cong \det(\mathcal{W})^{-m} \otimes \mathbb{S}^{(\lambda_1+m,\dots,\lambda_r+m)}\mathcal{W}. \tag{4.5}$$

Lemma 4.22 ([Kap88, Lem 3.2a]). *Let $\mathcal{W}$ be the rank r tautological quotient bundle of $Y = \mathrm{Gr}(V, r)$ and let $\lambda \in \mathbb{Z}^r$ be weakly decreasing. Then*

$$H^0(Y, \mathbb{S}^{\lambda}\mathcal{W}) \cong \begin{cases} \mathbb{S}^{\lambda}V & \text{if } \lambda_i \geq 0 \ \forall i, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.23 ([Kap88, 3.5, p.490]). *Let λ, μ be partitions with at most r parts. Then*

(i) $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{\lambda}\mathcal{W}, \mathbb{S}^{\lambda}\mathcal{W}) \cong \Bbbk$.

(ii) $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{\lambda}\mathcal{W}, \mathbb{S}^{\mu}\mathcal{W}) \neq 0$ *only if* $\lambda_i \leq \mu_i$ *for all* i.

We are now able to prove a more concise version of Theorem 4.21.

Proposition 4.24. *Let $\mathcal{W}$ be the rank r tautological quotient bundle of $\mathrm{Gr}(V, r)$ and let $\lambda \leq \mu \in \mathrm{Young}(n-r, r)$. Then*

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{\lambda}\mathcal{W}, \mathbb{S}^{\mu}\mathcal{W}) \cong \mathbb{S}^{\mu/\lambda}V. \tag{4.6}$$

Proof. As a consequence of [Har77, Proposition 3.6.7], since $\mathbb{S}^{\lambda}\mathcal{W}$ is a vector bundle we have the isomorphism

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{\lambda}\mathcal{W}, \mathbb{S}^{\mu}\mathcal{W}) \cong H^0(Y, (\mathbb{S}^{\lambda}\mathcal{W})^{\vee} \otimes \mathbb{S}^{\mu}\mathcal{W}).$$

By combining (4.4) and (4.5) with $m = \lambda_1$, we have

$$\begin{aligned}(\mathbb{S}^{\lambda}\mathcal{W})^{\vee} \otimes \mathbb{S}^{\mu}\mathcal{W} &\cong \mathbb{S}^{(-\lambda_r,\dots,-\lambda_1)}\mathcal{W} \otimes \mathbb{S}^{(\mu_1,\dots,\mu_r)}\mathcal{W} \\ &\cong \det(\mathcal{W})^{-\lambda_1} \otimes (\mathbb{S}^{\widetilde{\lambda}}\mathcal{W} \otimes \mathbb{S}^{\mu}\mathcal{W})\end{aligned}$$

where $\widetilde{\lambda} := (\lambda_1 - \lambda_r, \lambda_1 - \lambda_{r-1}, \dots, \lambda_1 - \lambda_2, 0)$. The decomposition of $\mathbb{S}^{\widetilde{\lambda}}\mathcal{W} \otimes \mathbb{S}^{\mu}\mathcal{W}$ into irreducibles ranges over partitions of size $|\widetilde{\lambda}| + |\mu|$, but then multiplying back by $\det(\mathcal{W})^{-\lambda_1}$ results in partitions γ of size $|\mu| - |\lambda|$, many of which contain negative entries. However, when taking global sections these vanish by Lemma 4.22, and so we are left with those γ containing only non-negative entries and satisfying $|\gamma| = |\mu| - |\lambda|$.

It remains to find the multiplicities of each summand. The multiplicity of $\mathbb{S}^\gamma \mathcal{W}$ in $(\mathbb{S}^\lambda \mathcal{W})^\vee \otimes \mathbb{S}^\mu \mathcal{W}$ is given by

$$\begin{aligned}\dim(\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\gamma \mathcal{W}, (\mathbb{S}^\lambda \mathcal{W})^\vee \otimes \mathbb{S}^\mu \mathcal{W})) &= \dim(\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\gamma \mathcal{W} \otimes \mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W})) \\ &= \dim(\mathrm{Hom}_{\mathcal{O}_Y}(\oplus_{\mu'}(\mathbb{S}^{\mu'} \mathcal{W})^{\oplus c^{\mu'}_{\lambda,\gamma}}, \mathbb{S}^\mu \mathcal{W})) \\ &= \dim(\oplus_{\mu'} \mathrm{Hom}_{\mathcal{O}_Y}((\mathbb{S}^{\mu'} \mathcal{W})^{\oplus c^{\mu'}_{\lambda,\gamma}}, \mathbb{S}^\mu \mathcal{W})) \\ &= \dim(\oplus_{\mu'} \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{\mu'} \mathcal{W}, \mathbb{S}^\mu \mathcal{W})^{\oplus c^{\mu'}_{\lambda,\gamma}}) \\ &= \dim(\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{\mu} \mathcal{W}, \mathbb{S}^\mu \mathcal{W})^{\oplus c^{\mu}_{\lambda,\gamma}}) \\ &= c^\mu_{\lambda,\gamma}\end{aligned}$$

where μ' ranges over $|\mu'| = |\gamma| + |\lambda| = |\mu|$ and the fifth and sixth equalities follow from Lemma 4.23 (ii) and (i) respectively. Thus, we have shown that $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W}) = \bigoplus_\gamma (\mathbb{S}^\gamma V)^{\oplus c^\mu_{\lambda,\gamma}}$, and the identity from Proposition 4.20 completes the proof. □

Corollary 4.25. *Let $\lambda \in \mathrm{Young}(n-r, r)$ and let $e_1, \dots, e_r$ denote the standard basis of $\mathbb{Z}^r$. Then for all $1 \le i \le r$ and all $m > 0$ such that $\lambda + me_i \in \mathrm{Young}(n-r, r)$, we have*

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^{\lambda+me_i} \mathcal{W}) \cong \mathrm{Sym}^m V.$$

In particular,

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^{\lambda+e_i} \mathcal{W}) \cong V.$$

Proof. By Proposition 4.24 we have $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^{\lambda+me_i} \mathcal{W}) \cong \bigoplus_\gamma (\mathbb{S}^\gamma V)^{\oplus c^{\lambda+me_i}_{\lambda,\gamma}}$, where γ ranges over all partitions with m boxes. The skew diagram $(\lambda + me_i)/\lambda$ however is just a single row of length m, hence by Remark 4.12(ii) the only non-zero Littlewood-Richardson number in this decomposition occurs when $\gamma = (m)$, in which case $c^{\lambda+me_i}_{\lambda,(m)} = 1$. Therefore $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^{\lambda+me_i} \mathcal{W}) \cong \mathbb{S}^{(m)} V = \mathrm{Sym}^m V$. The second statement follows immediately. □

As a result of Proposition 4.24, $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W})$ depends only on the shape of the skew diagram μ/λ and so we are able to deduce some simple invariance results. We may add redundant rows both above and below λ and μ without changing μ/λ and therefore also without changing $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W})$. Similarly, we may add or remove redundant columns to the left of both λ and μ. We state this more precisely in the following corollary, but first set some notation. Let ν_1, ν_2, ν_3 be partitions such that for $i = 1, 2$, the bottom row of ν_i is at least as long as the top row of ν_{i+1}. Denote by $(\nu_1 : \nu_2 : \nu_3)$ the Young diagram formed by gluing each of these diagrams on top of one another, keeping all rows left-aligned. Note that this is also well-defined if any of the ν_i are empty.

Corollary 4.26. *Let $\lambda, \mu \in \mathrm{Young}(n-r, r)$ with $\lambda \leq \mu$. Let ν be either empty or a Young diagram with bottom row greater than or equal to μ_1, and δ be either empty or a Young diagram such that its top row is less than or equal to λ_r. Let $c \in \mathbb{Z}$ such that $c \geq -\lambda_r$.*

(i) *If both $(\nu : \lambda : \delta), (\nu : \mu : \delta) \in \mathrm{Young}(n-r, r)$, then*

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{(\nu:\lambda:\delta)} \mathcal{W}, \mathbb{S}^{(\nu:\mu:\delta)} \mathcal{W}).$$

(ii) *If both $(\lambda_1 + c, \ldots, \lambda_r + c), (\mu_1 + c, \ldots, \mu_r + c) \in \mathrm{Young}(n-r, r)$, then*

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{(\lambda_1+c,\ldots,\lambda_r+c)} \mathcal{W}, \mathbb{S}^{(\mu_1+c,\ldots,\mu_r+c)} \mathcal{W}).$$

4.5 Homomorphisms of adjacent summands

Hereafter we restrict to the case where $Y = \mathrm{Gr}(n, 2)$, i.e. $r = 2$ and all partitions considered have at most two parts.

Fix a basis $\mathcal{B} = \{u_1, \ldots, u_n\}$ of V. In this section we write down explicitly the homomorphisms between adjacent summands in the tilting quiver, i.e. those defining $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^{\lambda+e_i} \mathcal{W})$ for all pairs $\lambda, \lambda + e_i \in \mathrm{Young}(n-2, 2)$ with $i \in \{1, 2\}$; by Corollary 4.25 these spaces are all isomorphic to V. Using (4.2), the tilting summands $\mathbb{S}^\lambda \mathcal{W}$ are of the form

$$\mathbb{S}^\lambda \mathcal{W} \cong \frac{(\bigwedge^2 \mathcal{W})^{\otimes \lambda_2} \otimes \mathrm{Sym}^{\lambda_1 - \lambda_2} \mathcal{W}}{E_\lambda}, \tag{4.7}$$

where E_λ is the sub-bundle of exchange relations.

Remark 4.27. We have $\bigwedge^2 \mathcal{W} = \det(\mathcal{W}) = \mathcal{O}_Y(1)$, but rather than writing $\mathbb{S}^\lambda \mathcal{W}$ as some twisting of $\mathrm{Sym}^{\lambda_1 - \lambda_2} \mathcal{W}$ we will use the presentation in (4.7) so that we can make use of the exchange relations, described explicitly, in various proofs.

First consider $\lambda = (0, 0)$ and $i = 1$, so $\lambda + e_i = (1, 0)$. Then

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{W}) \cong H^0(Y, \mathcal{W}) \cong V, \tag{4.8}$$

so given $v \in \mathcal{B}$ there is a homomorphism $s_v : \mathcal{O}_Y \to \mathcal{W}$ and a uniquely determined global section $z_v := s_v(1)$. Next suppose $\lambda = (1, 0)$ and $\lambda + e_i = (1, 1)$. We also have $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{W}, \bigwedge^2 \mathcal{W}) \cong V$ by Corollary 4.25, so there is a homomorphism $s'_v \colon \mathcal{W} \to \bigwedge^2 \mathcal{W}$ which we can define using the same section: $s'_v(x) = x \wedge z_v$.

Unfortunately, writing down homomorphisms $\mathbb{S}^\lambda \mathcal{W} \to \mathbb{S}^{\lambda+e_i}\mathcal{W}$ is in general not as straightforward as adding in a new variable z_v where required; we need to consider well-definedness with respect to the exchange relations E_λ (recall Section 4.1 and Example 4.1). In general, write sections of $\mathbb{S}^\lambda \mathcal{W}$ as

$$w^\lambda := x_{1,1} \wedge x_{1,2} \otimes \cdots \otimes x_{\lambda_2,1} \wedge x_{\lambda_2,2} \otimes y_1 \cdots y_{\lambda_1-\lambda_2}. \tag{4.9}$$

Since $r = 2$, the Young diagram for λ has at most two rows, therefore all exchange relations on $\mathbb{S}^\lambda \mathcal{W}$ may be characterised as one of the following two types:

(E1): We may take a single box from any column (remember that the symmetric part $\mathrm{Sym}^{(\lambda_1-\lambda_2)} \mathcal{W}$ counts as $\lambda_1 - \lambda_2$ distinct columns) and perform an exchange with any height two column to the left of it.

(E2): We may take both boxes in a height two column and exchange with a column of height two to the left of it.

As an example of (E1), from (4.9) choose y_1 and the first column (containing $x_{1,1} \wedge x_{1,2}$) to get

$$\begin{aligned} w^\lambda_{E1} &:= y_1 \wedge x_{1,2} \otimes \cdots \otimes x_{\lambda_2,1} \wedge x_{\lambda_2,2} \otimes x_{1,1} y_2 \cdots y_{\lambda_1-\lambda_2} \\ &\quad + x_{1,1} \wedge y_1 \otimes \cdots \otimes x_{\lambda_2,1} \wedge x_{\lambda_2,2} \otimes x_{1,2} y_2 \cdots y_{\lambda_1-\lambda_2}. \end{aligned}$$

Then modulo the exchange relations, we have $w^\lambda = w^\lambda_{E1}$.

As an example of (E2), from (4.9) choose the first column and the one containing $x_{\lambda_2,1} \wedge x_{\lambda_2,2}$ to get

$$w^\lambda_{E2} := x_{\lambda_2,1} \wedge x_{\lambda_2,2} \otimes \cdots \otimes x_{1,1} \wedge x_{1,2} \otimes y_1 \cdots y_{\lambda_1-\lambda_2}.$$

Then modulo the exchange relations, we have $w^\lambda = w^\lambda_{E2}$.

Proposition 4.28. *Let $\lambda, \lambda + e_i \in \mathrm{Young}(n-2, 2)$ where $i \in \{1, 2\}$ and let $w^\lambda \in \mathbb{S}^\lambda \mathcal{W}$ be as in (4.9). Let $v \in \mathcal{B}$ and z_v be as above.*

(i) *If $i = 1$, then the map $f^\lambda_v \colon \mathbb{S}^\lambda \mathcal{W} \to \mathbb{S}^{\lambda+e_1}\mathcal{W}$ given by*

$$f^\lambda_v(w^\lambda) = x_{1,1} \wedge x_{1,2} \otimes \cdots \otimes x_{\lambda_2,1} \wedge x_{\lambda_2,2} \otimes y_1 \cdots y_{\lambda_1-\lambda_2} z_v$$

is a well-defined linear homomorphism.

(ii) *If $i = 2$, then the map $g_v^\lambda \colon \mathbb{S}^\lambda \mathcal{W} \to \mathbb{S}^{\lambda+e_2}\mathcal{W}$ given by*

$$g_v^\lambda(w^\lambda) = \sum_{k=1}^{\lambda_1-\lambda_2} x_{1,1} \wedge x_{1,2} \otimes \cdots \otimes x_{\lambda_2,1} \wedge x_{\lambda_2,2} \otimes y_k \wedge z_v \otimes \prod_{j \neq k} y_j$$

is a well-defined linear homomorphism.

Moreover, $f_{u_1}^\lambda, \ldots, f_{u_n}^\lambda$ or $g_{u_1}^\lambda, \ldots, g_{u_n}^\lambda$ form a basis for $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda\mathcal{W}, \mathbb{S}^{\lambda+e_i}\mathcal{W})$ where $i = 1$ or 2 respectively.

Proof. The maps are clearly linear so we just prove they are well-defined with respect to the exchange relations on $\mathbb{S}^\lambda\mathcal{W}$. Using w^λ from (4.9) and $w_{E1}^\lambda, w_{E2}^\lambda$ defined above, we will show that $f_v^\lambda(w^\lambda) = f_v^\lambda(w_{E1}^\lambda) = f_v^\lambda(w_{E2}^\lambda)$ and $g_v^\lambda(w^\lambda) = g_v^\lambda(w_{E1}^\lambda) = g_v^\lambda(w_{E2}^\lambda)$. Any other choice of exchange is of the form (E1) or (E2) and the proof is identical.

(i) If $i = 1$, the symmetric power in $\mathbb{S}^{\lambda+e_1}\mathcal{W}$ compared to $\mathbb{S}^\lambda\mathcal{W}$ is increased by one while the alternating part is left unchanged; see (4.7). It turns out that simply inserting z_v into the symmetric part is well-defined. Firstly, we have

$$\begin{aligned} f_v^\lambda(w_{E1}^\lambda) &= y_1 \wedge x_{1,2} \otimes \cdots \otimes x_{\lambda_2,1} \wedge x_{\lambda_2,2} \otimes x_{1,1} y_2 \cdots y_{\lambda_1-\lambda_2} z_v \\ &\quad + x_{1,1} \wedge y_1 \otimes \cdots \otimes x_{\lambda_2,1} \wedge x_{\lambda_2,2} \otimes x_{1,2} y_2 \cdots y_{\lambda_1-\lambda_2} z_v \\ &= x_{1,1} \wedge x_{1,2} \otimes \cdots \otimes x_{\lambda_2,1} \wedge x_{\lambda_2,2} \otimes y_1 \cdots y_{\lambda_1-\lambda_2} z_v \\ &= f_v^\lambda(w^\lambda) \end{aligned}$$

where for the second equality we perform the inverse to an (E1) exchange. Secondly, since the exchange defining w_{E2}^λ has no effect on the symmetric part where z_v is added, $f_v^\lambda(w_{E2}^\lambda) = f_v^\lambda(w^\lambda)$ is immediate.

(ii) If $i = 2$, the symmetric power in $\mathbb{S}^{\lambda+e_2}\mathcal{W}$ decreases by one while the alternating power increases by one. The new height two column in $\lambda + e_2$ requires two variables; one of these will be z_v while the other will be a variable removed from the symmetric part. To make this well-defined with respect to the exchange relations, we must sum over every choice of variable we remove from the symmetric part to pair with z_v, which leads to the definition of g_v^λ. For w_{E1}^λ, we have

$$\begin{aligned} g_v(w_{E1}^\lambda) = &\sum_{k=1}^{\lambda_1-\lambda_2} y_1 \wedge x_{1,2} \otimes \cdots \otimes x_{\lambda_2,1} \wedge x_{\lambda_2,2} \otimes y_k \wedge z_v \otimes x_{1,1} \prod_{j\neq k,1} y_j \\ &+ \sum_{k=1}^{\lambda_1-\lambda_2} x_{1,1} \wedge y_1 \otimes \cdots \otimes x_{\lambda_2,1} \wedge x_{\lambda_2,2} \otimes y_k \wedge z_v \otimes x_{1,2} \prod_{j\neq k,1} y_j \\ = &\sum_{k=1}^{\lambda_1-\lambda_2} x_{1,1} \wedge x_{1,2} \otimes \cdots \otimes x_{\lambda_2,1} \wedge x_{\lambda_2,2} \otimes y_k \wedge z_v \otimes \prod_{j\neq k} y_j \\ = &\, g_v^\lambda(w^\lambda), \end{aligned}$$

where for the second equality we perform the inverses to the (E1)-type exchanges that move y_1 into the first column on each pair of terms from the two sums in turn. As for w_{E2}^λ, this is similar to (i) because the exchange in question occurs left of the column where g_v^λ inserts z_v, hence $g_v^\lambda(w_{E2}^\lambda) = g_v^\lambda(w^\lambda)$ is immediate.

Finally, using the basis $\mathcal{B}$ and (4.8), we get a basis $s_{u_\rho}, 1 \leq \rho \leq n$ of $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{W})$ and in turn a collection of linearly independent global sections z_{u_ρ}. Then for any given λ the maps $f_{u_\rho}^\lambda$ (or $g_{u_\rho}^\lambda$) are also linearly independent: for $1 \leq \rho \leq n$ the images of w^λ under each $f_{u_\rho}^\lambda$ (or $g_{u_\rho}^\lambda$) are pairwise distinct, the sections z_{u_ρ} may not be written in terms of one another, and no sequence of exchanges will produce a linear dependence relation since exchanges never introduce new variables, only move around the existing ones. □

The maps constructed in Proposition 4.28 are all of the maps between adjacent tilting summands. We will hereafter refer to them as 'f-type' and 'g-type' maps.

Remark 4.29. As an example of the danger of defining g-type maps by simply adding in z_v without summing over each choice of y_i to pair with it, consider the case when $g_v^{(2,0)} : \mathrm{Sym}^2 \mathcal{W} \to \bigwedge^2 \mathcal{W} \otimes \mathcal{W}$. Suppose we were to define $g_v^{(2,0)}(y_1 y_2) = z_v \wedge y_1 \otimes y_2$. Now $y_1 y_2 = y_2 y_1$ and using this definition of $g_v^{(2,0)}$ we would have

$$\begin{aligned} g_v^{(2,0)}(y_2 y_1) &= z_v \wedge y_2 \otimes y_1 \\ &= y_1 \wedge y_2 \otimes z_v + z_v \wedge y_1 \otimes y_2 \\ &= y_1 \wedge y_2 \otimes z_v + g_v^{(2,0)}(y_1 y_2) \\ &\neq g_v^{(2,0)}(y_1 y_2) \end{aligned}$$

where for the second equality we perform an (E1)-type exchange to move y_1 into the other column.

4.6 Structure of the tilting quiver for $\mathrm{Gr}(n, 2)$

We now construct the tilting quiver Q' for $Y = \mathrm{Gr}(n, 2)$ and use it to define a surjective $\Bbbk$-algebra homomorphism $\Phi\colon \Bbbk Q' \to A$.

The vertex set Q'_0 will be given by the irreducible summands of E, namely $\mathbb{S}^\lambda \mathcal{W}$ for all $\lambda \in \mathrm{Young}(n-2, 2)$. Note that we will sometimes directly refer to vertices by λ rather than $\mathbb{S}^\lambda \mathcal{W}$.

Recall that $A = \mathrm{End}_{\mathcal{O}_Y}(E)$ may be decomposed as the collection of spaces $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W})$ for all pairs $\lambda, \mu \in \mathrm{Young}(n-r, r)$. The arrow set Q'_1 will be given by a minimal set of generators for the spaces satisfying $\lambda < \mu$. By Corollary 4.25, for adjacent vertices we have $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^{\lambda+e_i} \mathcal{W}) \cong V$; depending on i, these spaces are spanned by a collection of f-type or g-type maps defined in Proposition 4.28. Hence, for all pairs $\lambda, \lambda + e_i \in \mathrm{Young}(n-2, 2)$ we will have n arrows in Q'_1 from $\mathbb{S}^\lambda \mathcal{W} \to \mathbb{S}^{\lambda+e_i} \mathcal{W}$ corresponding to the f-type or g-type basis of $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^{\lambda+e_i} \mathcal{W})$ as appropriate.

Claim: For any pair $\lambda < \mu \in \mathrm{Young}(n-2, 2)$, every map in $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W})$ may be written as a linear combination of compositions of f-type and g-type maps.

A proof of this claim implies that the collection of f-type and g-type maps constitutes a minimal set of generators for the spaces $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W})$ with $\lambda < \mu$. Therefore, the arrows between adjacent summands described above form the complete arrow set Q'_1. We actually prove a stronger statement than in the claim, which is that the compositions formed strictly by a sequence of f-type maps followed by a sequence of g-type maps is enough. In other words, given $\lambda < \mu$ define $m_1 = \mu_1 - \lambda_1$ and $m_2 = \mu_2 - \lambda_2$, and for $0 \le k \le m_1 + m_2$ define the sequence of partitions

$$\tau_k := \begin{cases} \lambda + k e_1 & \text{if } 0 \le k \le m_1, \\ \lambda + m_1 e_1 + (k - m_1) e_2 & \text{if } m_1 \le k \le m_1 + m_2. \end{cases} \tag{4.10}$$

Then the claim follows from the following proposition.

Proposition 4.30. *Let $Y = \mathrm{Gr}(n, 2)$ and let $\lambda < \mu \in \mathrm{Young}(n-2, 2)$. Let τ_k be the sequence of partitions defined in (4.10). Then the composition map*

$$\Theta_{\lambda,\mu}\colon \bigotimes_{k=1}^{m_1+m_2} \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{\tau_{k-1}} \mathcal{W}, \mathbb{S}^{\tau_k} \mathcal{W}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W}),$$

where

$$f_{v_1}^{\tau_0} \otimes \cdots \otimes f_{v_{m_1}}^{\tau_{m_1-1}} \otimes g_{v_{m_1+1}}^{\tau_{m_1}} \otimes \cdots \otimes g_{v_{m_1+m_2}}^{\tau_{m_1+m_2-1}} \mapsto g_{v_{m_1+m_2}}^{\tau_{m_1+m_2-1}} \circ \cdots \circ g_{v_{m_1+m_2}}^{\tau_{m_1+m_2-1}} \circ f_{v_{m_1}}^{\tau_{m_1-1}} \circ \cdots \circ f_{v_1}^{\tau_0},$$

is surjective.

The proof is technical and we postpone it until Section 4.7. Assuming Proposition 4.30, we can now introduce the tilting quiver Q' and establish the main result of this chapter.

Definition 4.31. For $Y = \mathrm{Gr}(n, 2)$, define the tilting quiver Q' by

$$Q'_0 = \left\{ \lambda \in \mathbb{Z}^2 \mid n - 2 \geq \lambda_1 \geq \lambda_2 \geq 0 \right\},$$

$$Q'_1 = \left\{ a_\rho^{\lambda,i} \;\middle|\; \begin{array}{l} 1 \leq \rho \leq n \\ i \in \{1,2\},\ \lambda, \lambda + e_i \in Q'_0 \\ \mathrm{t}(a_\rho^{\lambda,i}) = \lambda,\ \mathrm{h}(a_\rho^{\lambda,i}) = \lambda + e_i \end{array} \right\}.$$

See Figure 4.2. Note that when drawing the tilting quiver we will use the presentation of $\mathbb{S}^\lambda \mathcal{W}$ given in (4.7) to label the vertices. Observe that the notation for the arrows is consistent with Chapter 3, where the superscript records the location of the arrow and the subscript records the label corresponding to a basis vector in $\mathcal{B}$.

Remark 4.32. Observe that, as in the toric case, the original quiver Q may be identified with a complete sub-quiver of Q' that we call the *base quiver* in Q'; this is the sub-quiver with vertex set $\{\mathcal{O}_Y, \mathcal{W}\}$, positioned at $(0, 0) \to (1, 0)$. See example 3.7.

For each $\lambda \in Q'_0$ let $e_\lambda \in \Bbbk Q'$ denote the idempotent corresponding to the path of length zero at that vertex. For all $\lambda, \mu \in Q'_0$ satisfying $\lambda < \mu$, with the convention that we traverse paths from right to left (the same way that composition of maps is performed), $e_\mu \Bbbk Q' e_\lambda$ denotes the space of paths $\lambda \to \mu$. Recall the basis $\mathcal{B} = \{u_1, \ldots, u_n\}$ of V.

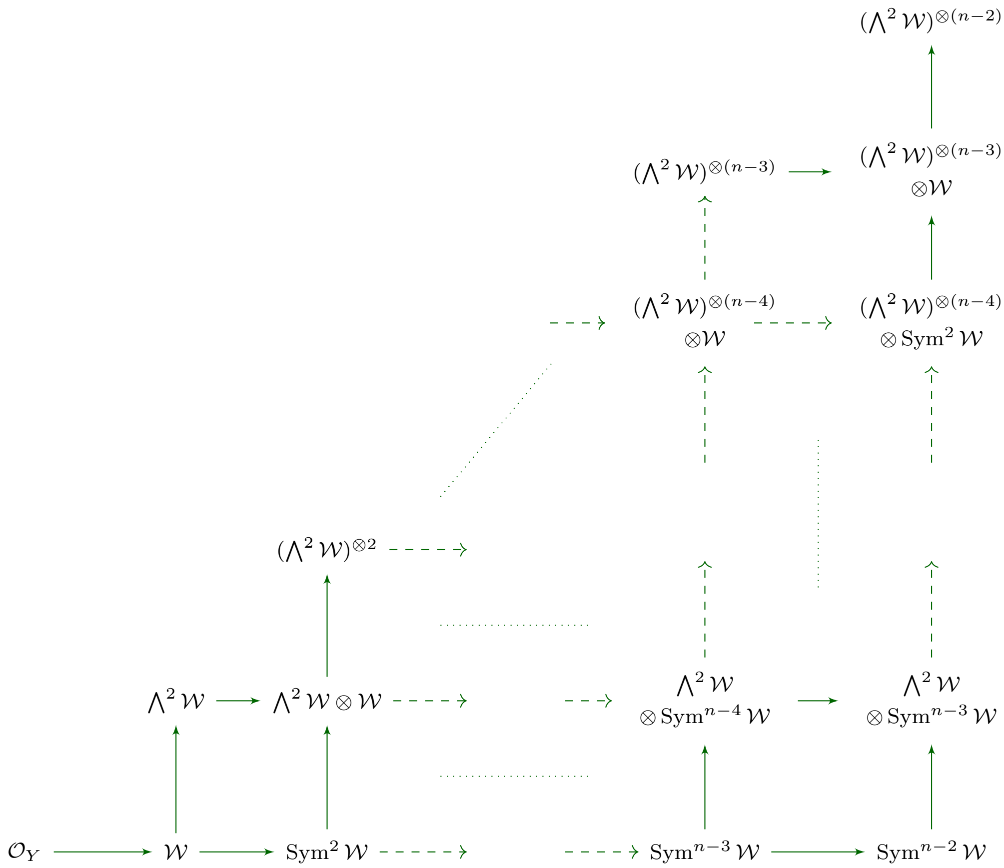

FIGURE 4.2: The tilting quiver Q' for $\mathrm{Gr}(n, 2)$. Each arrow in the figure represents n arrows in the quiver corresponding to $\mathcal{B}$. The sub-bundle of exchange relations E_λ (see (4.7)) is implicit.

Definition 4.33. Let $Y = \mathrm{Gr}(n, 2)$. We now define a $\Bbbk$-algebra homomorphism

$$\Phi : \Bbbk Q' \longrightarrow A. \tag{4.11}$$

By definition of Q'_1 and Corollary 4.25, when $\mu = \lambda + e_i$ for $i \in \{1, 2\}$, we have $e_{\lambda+e_i} \Bbbk Q' e_\lambda \cong V \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^{\lambda+e_i} \mathcal{W})$, and by Proposition 4.28 the latter space has a basis given by $f^\lambda_{u_1}, \ldots, f^\lambda_{u_n}$ if $i = 1$ or $g^\lambda_{u_1}, \ldots, g^\lambda_{u_n}$ if $i = 2$. Thus, for all $\lambda \in Q', 1 \le \rho \le n$ and $i \in \{1, 2\}$ as appropriate, we define:

$$\Phi(e_\lambda) = \mathrm{id}_\lambda \in \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\lambda \mathcal{W}) \cong \Bbbk,$$

$$\Phi(a^{\lambda,i}_\rho) = \begin{cases} f^\lambda_{u_\rho} \in \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^{\lambda+e_1} \mathcal{W}) & \text{if } i = 1, \\ g^\lambda_{u_\rho} \in \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^{\lambda+e_2} \mathcal{W}) & \text{if } i = 2. \end{cases}$$

The images of the horizontal (and vertical) arrows in Q'_1 are therefore exactly the

f-type (and g-type maps) defined in Proposition 4.28. We extend Φ to any path in Q' by mapping the concatenation of arrows $a_\rho^{\lambda,i}$ to the composition of maps in A as appropriate. Finally, we extend Φ linearly over $\Bbbk$ to combination of paths in $\Bbbk Q'$.

Theorem 4.34. *Let $Y = \mathrm{Gr}(n,2)$. The $\Bbbk$-algebra homomorphism $\Phi\colon \Bbbk Q' \to A$ is surjective.*

Proof. Firstly, whenever λ is not contained in μ we have $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda\mathcal{W}, \mathbb{S}^\mu\mathcal{W}) = 0$ by Lemma 4.23(ii), thus surjectivity is trivial in these cases; indeed, we defined no arrows in Q' for such pairs λ, μ. As a consequence, Q' is acyclic and $(0,0)$ is the unique source vertex.

Now suppose $\lambda < \mu \in Q'_0$ and $h \in \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda\mathcal{W}, \mathbb{S}^\mu\mathcal{W}) \subset A$. Proposition 4.30 implies that h may be factorised as a linear combination of compositions of f-type and g-type maps. Therefore the corresponding linear combination of paths given by concatenating arrows of the form $a_\rho^{\lambda,1}, a_\rho^{\lambda,2}$ maps to h under Φ. Hence, Φ is surjective. □

Remark 4.35. Theorem 6.1 provides a new proof of [BLV16, Theorem 6.9] in the case $Y = \mathrm{Gr}(n,2)$. We discuss this result and the methods used in Section 5.3.

4.7 Proof of Proposition 4.30

For convenience we restate Proposition 4.30 here:

Proposition 4.30. *Let $Y = \mathrm{Gr}(n,2)$ and let $\lambda < \mu \in \mathrm{Young}(n-2,2)$. Let τ_k be the sequence of partitions defined in (4.10). Then the composition map*

$$\Theta_{\lambda,\mu}\colon \bigotimes_{k=1}^{m_1+m_2} \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{\tau_{k-1}}\mathcal{W}, \mathbb{S}^{\tau_k}\mathcal{W}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda\mathcal{W}, \mathbb{S}^\mu\mathcal{W}),$$

where

$$f_{v_1}^{\tau_0}\otimes\cdots\otimes f_{v_{m_1}}^{\tau_{m_1-1}}\otimes g_{v_{m_1+1}}^{\tau_{m_1}}\otimes\cdots\otimes g_{v_{m_1+m_2}}^{\tau_{m_1+m_2-1}} \mapsto g_{v_{m_1+m_2}}^{\tau_{m_1+m_2-1}}\circ\cdots\circ g_{v_{m_1+m_2}}^{\tau_{m_1+m_2-1}}\circ f_{v_{m_1}}^{\tau_{m_1-1}}\circ\cdots\circ f_{v_1}^{\tau_0},$$

is surjective.

First of all, due to the invariance result Corollary 4.26(ii), it is enough to consider the special case where $\lambda_2 = 0$; the general case follows immediately since $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{(\lambda_1,\lambda_2)}\mathcal{W}, \mathbb{S}^{(\mu_1,\mu_2)}\mathcal{W}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{(\lambda_1-\lambda_2,0)}\mathcal{W}, \mathbb{S}^{(\mu_1-\lambda_2,\mu_2-\lambda_2)}\mathcal{W})$, and we adjust the sequence ν_k accordingly. We therefore assume $\lambda_2 = 0$ throughout the entirety of the proof, and in turn we have $m_1 = \mu_1 - \lambda_1$, $m_2 = \mu_2$.

We first consider the domain and codomain of $\Theta_{\lambda,\mu}$. By Corollary 4.25, the domain of $\Theta_{\lambda,\mu}$ is isomorphic to $V^{\otimes(m_1+m_2)}$ and by Proposition 4.24, the codomain is isomorphic to $\mathbb{S}^{\mu/\lambda}V$. Both of these spaces have an irreducible decomposition in terms of Schur powers of V, and we see in Section 4.3 that $\mathbb{S}^{\mu/\lambda}V$ is a $\mathrm{GL}(V)$-submodule of $V^{\otimes(m_1+m_2)}$.

Lemma 4.36. *The irreducible decomposition of* $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda\mathcal{W}, \mathbb{S}^\mu\mathcal{W})$ *is given by*

$$\mathbb{S}^{\mu/\lambda}V \cong \bigoplus_{\gamma\in\Gamma_{\mu/\lambda}} \mathbb{S}^\gamma V, \tag{4.12}$$

where

- *if* $\mu_2 \leq \lambda_1$, $\Gamma_{\mu/\lambda}$ *consists of the partitions* $(\max\{m_1, m_2\}, \min\{m_1, m_2\})$, $(\max\{m_1, m_2\} + 1, \min\{m_1, m_2\} - 1), \ldots, (m_1 + m_2, 0)$.
- *if* $\mu_2 > \lambda_1$, $\Gamma_{\mu/\lambda}$ *consists of the partitions* $(\max\{m_1, m_2\}, \min\{m_1, m_2\})$, $(\max\{m_1, m_2\} + 1, \min\{m_1, m_2\} - 1), \ldots, (\mu_1, \mu_2 - \lambda_1)$.

Proof. The main tool for this is Proposition 4.20, which tells us that

$$\mathbb{S}^{\mu/\lambda}V \cong \bigoplus_{\gamma} (\mathbb{S}^\gamma V)^{\oplus c^\mu_{\lambda,\gamma}}$$

where γ ranges over all partitions satisfying $|\gamma| = |\mu| - |\lambda| = m_1 + m_2$. By Proposition 4.11 we have $c^\mu_{\lambda,\gamma} \neq 0 \implies \gamma \leq \mu$, hence we only need to consider γ with at most two parts that satisfy $\gamma_1 \leq \mu_1$ and $\gamma_2 \leq \mu_2$. Additionally, $c^\mu_{\lambda,\gamma}$ is equal to either 0 or 1 by Remark 4.12(i).

The set $\Gamma_{\mu/\lambda}$ is given by the collection of such γ satisfying $c^\mu_{\lambda,\gamma} = 1$, i.e. those such that the skew diagram μ/λ filled with content γ is a Littlewood-Richardson tableau. First suppose that $\mu_2 \leq \lambda_1$, which means the two rows of μ/λ do not overlap. Since there are no columns of height two, the strictly increasing columns condition cannot be broken and so a filling consisting of only 1's, i.e. $\gamma = (m_1 + m_2, 0)$, is permissible. By Remark 4.12(i), 2's may only be placed in the right of the bottom row; therefore, to avoid breaking the reverse lattice word condition, the number of 2's we may use is bounded above by the length of the top row, i.e. $\gamma_2 \leq m_1$. Since we must also have $\gamma_2 \leq \mu_2 = m_2$, we have $\min\{m_1, m_2\} \geq \gamma_2 \geq 0$ as required.

The argument for the case when $\mu_2 > \lambda_1$ carries over from above, except we now also have a positive lower bound for γ_2 because columns of height two exist in μ/λ. We therefore require at least $\mu_2 - \lambda_1$ many 2's to place in the height two columns, and this gives the second case in the statement of the lemma. □

We now outline the strategy of proof for Proposition 4.30. Fix $\lambda < \mu$. We will show that the restriction of $\Theta_{\lambda,\mu}$ to the submodule $\mathbb{S}^{\mu/\lambda}V$ of $V^{\otimes(m_1+m_2)}$ is an isomorphism. It therefore follows that $\Theta_{\lambda,\mu}$ is surjective.

To do this we take advantage of Schur's lemma (see [FH91, Lemma 1.7]): if $\varphi\colon W_1 \to W_2$ is a G-module homomorphism of irreducible G-modules, then φ is either an isomorphism or the zero map. Recall $\Gamma_{\mu/\lambda}$ from Lemma 4.36. We have that $\mathbb{S}^\gamma V$ is irreducible over $G = \mathrm{GL}(V)$ for all $\gamma \in \Gamma_{\mu/\lambda}$ and $\Theta_{\lambda,\mu}$ is a $\mathrm{GL}(V)$-module homomorphism. Since every summand in (4.12) appears with multiplicity one, Schur's lemma implies that it is enough to show $\Theta_{\lambda,\mu}$ is non-zero when restricted to $\mathbb{S}^\gamma V \subset V^{\otimes(m_1+m_2)}$ for each $\gamma \in \Gamma_{\mu/\lambda}$. In summary, the proof of Proposition 4.30 is completed by the following lemma.

Lemma 4.37. *Let $\gamma \in \Gamma_{\mu/\lambda}$. Then $\Theta_{\lambda,\mu}\big|_{\mathbb{S}^\gamma V} \neq 0$.*

The strategy for the proof of Lemma 4.37 is as follows. For each $\gamma \in \Gamma_{\mu/\lambda}$ we first write down an element h_γ in $V^{\otimes(m_1+m_2)}$, the domain of $\Theta_{\lambda,\mu}$. We then get an element $c_\gamma(h_\gamma)$ of $\mathbb{S}^\gamma V$ by applying the Young symmetrizer $c_\gamma\colon V^{\otimes m_1+m_2} \twoheadrightarrow \mathbb{S}^\gamma V$; see Example 4.18. Then, by formulating the evaluation of a section w^λ under the map $\Theta_{\lambda,\mu}(c_\gamma(h_\gamma))$, we show this is non-zero to complete the proof. In order to implement this strategy we need some new notation.

Notation 4.38. (i) Fix a basis $\mathcal{B} = \{u_1, \dots, u_n\}$ of V and let $\lambda < \mu \in \mathrm{Young}(n-2, 2)$. Recall $m_1 = \mu_1 - \lambda_1$ and $m_2 = \mu_2$ (we are assuming $\lambda_2 = 0$), so that $\mu = \lambda + m_1e_1 + m_2e_2$. In this section we always consider compositions of maps given by the image of $\Theta_{\lambda,\mu}$, i.e. those of the form

$$g_{v_{m_1+m_2}}^{\tau_{m_1+m_2-1}} \circ \cdots \circ g_{v_{m_1+m_2}}^{\tau_{m_1+m_2-1}} \circ f_{v_{m_1}}^{\tau_{m_1-1}} \circ \cdots \circ f_{v_1}^{\tau_0}$$

where each $v_i \in \mathcal{B}$. Because the domain of each map in this composition may be derived from whether the previous map is f-type or g-type, we will suppress the superscript of all maps but the first and instead write the above as

$$g_{v_{m_1+m_2}} \circ \cdots \circ g_{v_{m_1+1}} \circ f_{v_{m_1}} \circ \cdots \circ f_{v_1}^{\lambda}.$$

(ii) We require the notion of *multisets*; these are sets with possible multiplicities of elements, and we distinguish multisets from sets by using square brackets [ ]. The cardinality of a multiset counts these multiplicities, e.g. $[1, 1, 2]$ has cardinality 3. Denote *ordered multisets* using $[\,]^o$, and given a multiset M define $\mathrm{O}_k(M)$ to be the collection of all ordered sub-multisets of M with cardinality k. If the cardinality of M is $m \geq 0$, then the size of the collection $\mathrm{O}_k(M)$ is $m!/(m-k)!$. For example, if $M = [1, 1, 2]$ then $\mathrm{O}_2(M)$ is the collection $[1, 1]^o, [1, 1]^o, [1, 2]^o, [1, 2]^o, [2, 1]^o, [2, 1]^o$.

For every $\gamma \in \Gamma_{\mu/\lambda}$, we now define our candidates $h_\gamma \in V^{\otimes(m_1+m_2)}$ that we use in the proof of Lemma 4.37. Recall Section 4.3: the Young symmetrizer c_γ is defined by $b_\gamma a_\gamma$, so $c_\gamma(h_\gamma)$ will be a double sum taken over all ways of first symmetrizing the rows of γ, followed by anti-symmetrizing the columns. We will simplify matters by defining h_γ such that a_γ is trivial. The simplest such map is the basis vector of $V^{\otimes(m_1+m_2)}$ corresponding to the skew tableau of shape γ with the top row filled with 1's and the bottom row filled with 2's, i.e.

$$\overbrace{\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1\\ \hline 2&2&2&2 \\ \cline{1-4}\end{array}}^{\gamma_1}\ \ \underbrace{\qquad\qquad}_{\gamma_2} \longleftrightarrow \underbrace{\underbrace{f^\lambda_{u_1} \otimes \cdots \otimes f_{u_1}}_{m_1} \otimes g_{u_1} \otimes \cdots \otimes g_{u_1}}_{\gamma_1} \otimes \underbrace{g_{u_2} \otimes \cdots \otimes g_{u_2}}_{\gamma_2} =: h_\gamma.$$

(m_2 counts the g-type maps $g_{u_1} \otimes \cdots \otimes g_{u_1} \otimes g_{u_2} \otimes \cdots \otimes g_{u_2}$.)

Thus, there are m_1 f-type maps followed by m_2 g-type maps, the first γ_1 of which are defined using the basis vector u_1 and the last γ_2 defined using the basis vector u_2. By the conditions on $\gamma \in \Gamma_{\mu/\lambda}$, we always have $\gamma_2 \leq \min\{m_1, m_2\}$.

We now write down $c_\gamma(h_\gamma) = b_\gamma a_\gamma(h_\gamma)$. By construction, a_γ acts trivially on h_γ so it remains to sum over all ways of anti-symmetrizing the height two columns of γ. Therefore

$$c_\gamma(h_\gamma) = b_\gamma(h_\gamma) = \sum_{\sigma \in P_{\mathrm{col}}} \mathrm{sgn}(\sigma) h_\gamma \cdot \sigma,$$

which, since γ has γ_2 columns of height two, is a sum of 2^{γ_2} terms. Each σ defines a unique ordered multiset

$$\underline{\xi_\sigma} = [v_1, \ldots, v_{\gamma_2}]^o,$$

where each v_i is equal to either u_1 or u_2 as given by the top row of the height two columns in $h_\gamma \cdot \sigma$. Define α_σ and β_σ to be the number of v_1's and v_2's appearing in $\underline{\xi_\sigma}$ respectively; then $\alpha_\sigma + \beta_\sigma = \gamma_2$ and we have $\mathrm{sgn}(\sigma) = (-1)^{\beta_\sigma}$. Given $\underline{\xi_\sigma}$ define v'_i for $1 \leq i \leq \gamma_2$ to be the vectors on the bottom row of the height two columns in $h \cdot \sigma$, i.e. if $v_i = u_1$ then $v'_i = u_2$ and vice versa. This yields $c_\gamma(h_\gamma)$ equal to

$$\sum_{\sigma \in P_{\mathrm{col}}} (-1)^{\beta_\sigma} f^\lambda_{v_1} \otimes \cdots \otimes f_{v_{\gamma_2}} \otimes f_{u_1} \otimes \cdots \otimes f_{u_1} \otimes g_{u_1} \otimes \cdots \otimes g_{u_1} \otimes g_{v'_1} \otimes \cdots \otimes g_{v'_{\gamma_2}}$$

and therefore $\Theta_{\lambda,\mu}(c_\gamma(h_\gamma))$ is given by

$$\sum_{\sigma\in P_{\mathrm{col}}}(-1)^{\beta_\sigma} g_{v'_{\gamma_2}}\circ\cdots\circ g_{v'_1}\circ g_{u_1}\circ\cdots\circ g_{u_1}\circ f_{u_1}\circ\cdots\circ f_{u_1}\circ f_{v_{\gamma_2}}\circ\cdots\circ f^{\lambda}_{v_1}. \tag{4.13}$$

Notation 4.39. Before detailing the evaluation of a section $w^\lambda = y_1\cdots y_{\lambda_1}\in \mathbb{S}^\lambda\mathcal{W}\cong \mathrm{Sym}^{\lambda_1}\mathcal{W}$ under the above sum, we will first describe the evaluation of w^λ under a single composition $g_{v_{m_1+m_2}}\circ\cdots\circ g_{v_{m_1+1}}\circ f_{v_{m_1}}\circ\cdots\circ f^{\lambda}_{v_1}\in \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda\mathcal{W},\mathbb{S}^\mu\mathcal{W})$ where each $v_i\in\mathcal{B}$; this will also be useful in Chapter 5. Using Proposition 4.28, evaluating the composition of the f-type maps is easy: we simply have

$$f_{v_{m_1}}\circ\cdots\circ f^{\lambda}_{v_1}(w^\lambda) = y_1\cdots y_{\lambda_1} z_{v_1}\cdots z_{v_{m_1}}.$$

The evaluation of a g-type map is the sum over each way of pairing a variable in the symmetric part with the new variable introduced by the map, hence the evaluation of a succession of g-type maps is given by summing over all ordered ways of doing this. Thus, define the multiset $M=[y_1,\ldots,y_{\lambda_1},z_{v_1},\ldots,z_{v_{m_1}}]$, and recall that $O_{m_2}(M)$ is the collection of ordered sub-multisets of M with cardinality m_2. Then the image of w^λ under the composition above is given by

$$w^\lambda \mapsto \sum_{\substack{X=[x_1,\ldots,x_{m_2}]^o\\ \in \mathrm{O}_{m_2}(M)}} x_1\wedge z_{v_{m_1+1}}\otimes\cdots\otimes x_{m_2}\wedge z_{v_{m_1+m_2}}\otimes\prod_{z\in M\setminus X} z. \tag{4.14}$$

Proof of Lemma 4.37. Now recall (4.13); the goal is to find a section w^λ such that $\Theta_{\lambda,\mu}(c_\gamma(h_\gamma))(w^\lambda)\neq 0$. Define δ to be the number of g-type maps in each term of $c_\gamma(h_\gamma)$ with fixed defining basis vector u_1, i.e. $\delta = m_2-\gamma_2$. Set

$$w^\lambda := z_{u_1}^{\lambda_1-\delta} z_{u_2}^{\delta}.$$

Note that this choice of section is possible because $\gamma\in\Gamma_{\mu/\lambda}\implies \gamma_2\geq \mu_2-\lambda_1 = m_2-\lambda_1 \implies \lambda_1\geq m_2-\gamma_2=\delta$.

Since $\Theta_{\lambda,\mu}$ is linear we will analyse the evaluation of w^λ under each term $\Theta_{\lambda,\mu}(h_\gamma\cdot\sigma)$ in the sum (4.13) separately; we will deal with the sign $(-1)^{\beta_\sigma}$ at the end. Thus, fix $\sigma\in P_{\mathrm{col}}$ and consider $\underline{\xi_\sigma},\alpha_\sigma,\beta_\sigma$ as defined above. Following Notation 4.39, the composition of the f-type maps in $\Theta_{\lambda,\mu}(h_\gamma\cdot\sigma)$ contributes $z_{u_1}^{m_1-\beta_\sigma}$ and $z_{u_2}^{\beta_\sigma}$ to w^λ, bringing the total exponent of z_{u_1} to $\lambda_1-\delta+m_1-\beta_\sigma = \mu_1-\mu_2+\alpha_\sigma$ and the total exponent of z_{u_2} to $\delta+\beta_\sigma=\mu_2-\gamma_2+\beta_\sigma$. Hence, the multiset M in (4.14) is given by

$$M = [\underbrace{z_{u_1}, \dots, z_{u_1}}_{\mu_1-\mu_2+\alpha_\sigma}, \underbrace{z_{u_2}, \dots, z_{u_2}}_{\mu_2-\gamma_2+\beta_\sigma}].$$

After composing the remaining g-type maps, $\Theta_{\lambda,\mu}(h_\gamma \cdot \sigma)(w^\lambda)$ is equal to

$$\sum_{\substack{X=[x_1,\dots,x_{m_2}]^o \\ \in \mathrm{O}_{m_2}(M)}} x_1 \wedge z_{u_1} \otimes \cdots \otimes x_{m_2-\gamma_2} \wedge z_{u_1} \otimes x_{m_2-\gamma_2+1} \wedge z_{v'_1} \otimes \cdots \otimes x_{m_2} \wedge z_{v'_{\gamma_2}} \otimes \prod_{z \in M \setminus X} z. \tag{4.15}$$

Every variable in this sum is either z_{u_1} or z_{u_2}, thus the only $X \in \mathrm{O}_{m_2}(M)$ that produce a non-zero term are those with $x_1, \dots, x_{m_2-\gamma_2}$ equal to z_{u_2} and for $m_2 - \gamma_2 + 1 \leq j \leq m_2$, $x_j = z_{u_1}$ if $v'_{j-m_2+\gamma_2} = z_{u_2}$ and vice versa. Hence every ordered sub-multiset X that produces a non-zero term is identical, and consists of α_σ many z_{u_1}'s and $\mu_2 - \gamma_2 + \beta_\sigma$ many z_{u_2}'s. Define the total number of such X to be η_σ; this is given by the number of ordered ways of choosing α_σ many z_{u_1}'s from M, multiplied by the number of ordered ways of choosing $\mu_2 - \gamma_2 + \beta_\sigma$ many z_{u_2}'s from M, i.e.

$$\eta_\sigma = \frac{(\mu_2 - \gamma_2 + \beta_\sigma)!(\mu_1 - \mu_2 + \alpha_\sigma)!}{(\mu_1 - \mu_2)!}.$$

The sum (4.15) therefore simplifies to

$$\eta_\sigma z_{u_2} \wedge z_{u_1} \otimes \cdots \otimes z_{u_2} \wedge z_{u_1} \otimes z_{v_1} \wedge z_{v'_1} \otimes \cdots \otimes z_{v_{\gamma_2}} \wedge z_{v'_{\gamma_2}} \otimes z_{u_1}^{\mu_1-\mu_2}.$$

We now use anti-symmetrization in the columns with content $z_{v_i} \wedge z_{v'_i}$ to ensure that the first entry is z_{u_2} while the second is z_{u_1}. There are α_σ many columns that are not in this order, since this is the number of $z_{v'_i}$ terms that are equal to z_{u_2}. Hence, rearranging each column so that the content reads $z_{u_2} \wedge z_{u_1}$ means we must multiply by $(-1)^{\alpha_\sigma}$. Therefore we simplify the above once more, yielding

$$\Theta_{\lambda,\mu}(h_\gamma \cdot \sigma)(w^\lambda) = (-1)^{\alpha_\sigma} \eta_\sigma z_{u_2} \wedge z_{u_1} \otimes \cdots \otimes z_{u_2} \wedge z_{u_1} \otimes z_{u_1}^{\mu_1-\mu_2}.$$

In conclusion, using (4.13) we have

$$\begin{aligned}\Theta_{\lambda,\mu}(c_\gamma(h_\gamma))(w^\lambda) &= \sum_{\sigma\in P_{\text{col}}}(-1)^{\beta_\sigma}(-1)^{\alpha_\sigma}\eta_\sigma z_{u_2}\wedge z_{u_1}\otimes\cdots\otimes z_{u_2}\wedge z_{u_1}\otimes z_{u_1}^{\mu_1-\mu_2}\\ &= (-1)^{\gamma_2}\sum_{\sigma\in P_{\text{col}}}\eta_\sigma z_{u_2}\wedge z_{u_1}\otimes\cdots\otimes z_{u_2}\wedge z_{u_1}\otimes z_{u_1}^{\mu_1-\mu_2}\\ &\neq 0,\end{aligned}$$

since $z_{u_2}\wedge z_{u_1}\otimes\cdots\otimes z_{u_2}\wedge z_{u_1}\otimes z_{u_1}^{\mu_1-\mu_2}\neq 0$, $(-1)^{\gamma_2}$ is constant, and $\eta_\sigma > 0$ for all σ. This completes the proof of Lemma 4.37, and hence Proposition 4.30. □

Chapter 5

The ideal of relations for the tilting quiver of $\mathrm{Gr}(n, 2)$

Let $Y = \mathrm{Gr}(n, 2)$. In this chapter we identify the kernel of the $\Bbbk$-algebra homomorphism $\Phi : \Bbbk Q' \twoheadrightarrow A$ from Theorem 4.34. The ideal $\ker(\Phi)$ then induces relations on the tilting quiver Q'. Having completed our presentation of the tilting algebra for $Y = \mathrm{Gr}(n, 2)$, we compare this with the work of Buchweitz, Leuschke and Van den Bergh in Section 5.3.

Throughout, let $\mathcal{B} = \{u_1, \ldots, u_n\}$ be a basis of V and recall Notation 4.15: when choosing an arbitrary collection of these vectors, possibly with multiplicity, we will use the letters $v_i \in \mathcal{B}$. Our convention is to write angle brackets $\langle\ \rangle$ for linear subspaces and round brackets $(\ )$ for ideals.

Strategy for finding $\ker(\Phi)$**:** Recall that for each $\lambda \in Q'_0$, $e_\lambda \in \Bbbk Q'$ denotes the idempotent corresponding to the path of length zero at that vertex. Then for all pairs $\lambda < \mu \in Q'_0$, we denote by $\Phi_{\lambda,\mu}$ the induced $\Bbbk$-linear map obtained by restricting Φ to the subspace spanned by paths with tail at λ and head at μ, i.e.

$$\Phi_{\lambda,\mu} \colon e_\mu \Bbbk Q' e_\lambda \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W}) \cong \mathbb{S}^{\mu/\lambda} V.$$

Note that $\Phi_{\lambda,\mu}$ is surjective by Proposition 4.30. Now, Q' is acyclic and there are no relations involving paths of length one. Indeed, relations only arise between paths that share the same head and tail, and for all $a \in Q'_1$ the only paths p in Q' satisfying $\mathrm{t}(a) = \mathrm{t}(p)$ and $\mathrm{h}(a) = \mathrm{h}(p)$ are the arrows between the same two vertices, and these have linearly independent images under the map Φ. It therefore suffices to find $\ker(\Phi_{\lambda,\mu})$ for every pair (λ, μ) in the set

$$P := \{(\lambda, \mu) \in {Q'_0}^2 \mid \ \lambda < \mu, |\mu| \geq |\lambda| + 2\}.$$

Denote by $K_{\lambda,\mu}$ a set of basis vectors for the subspace $\ker(\Phi_{\lambda,\mu})$. Then $\ker(\Phi)$ is the ideal generated by the union of these bases:

$$\ker(\Phi) = \left(\bigcup_{(\lambda,\mu)\in P} K_{\lambda,\mu} \right). \tag{5.1}$$

We divide this chapter into two main steps. Define

$$P_2 := \left\{ (\lambda,\mu) \in {Q_0'}^2 \mid \lambda < \mu, |\mu| = |\lambda| + 2 \right\} \subset P, \tag{5.2}$$

the pairs of vertices separated by paths of length two. The first step is to find $\ker(\Phi_{\lambda,\mu})$, and therefore $K_{\lambda,\mu}$, for all $(\lambda,\mu) \in P_2$. We identify elements of $\ker(\Phi_{\lambda,\mu})$ by studying various compositions of the f-type and g-type maps defined in Proposition 4.28 evaluated on an arbitrary section w^λ of $\mathbb{S}^\lambda\mathcal{W}$. We use to define the ideal

$$I := \left(\bigcup_{(\lambda,\mu)\in P_2} K_{\lambda,\mu} \right) \subset \Bbbk Q' \tag{5.3}$$

generated by the relations of length two. Then we have $I \subseteq \ker(\Phi)$, and by considering longer paths in $\Bbbk Q'$ the second step is to show that $I = \ker(\Phi)$.

Note that in the case of $Y = \mathrm{Gr}(4,2)$, the ideal I was written down by Buchweitz, Leuschke and Van den Bergh in [BLV15, Example 8.4]. We recover this example in passing in our discussion of $\mathrm{Gr}(5,2)$ in Example 5.11.

5.1 Relations between paths of length two

In this section we identify the vector spaces $\ker(\Phi_{\lambda,\mu})$ for all $(\lambda,\mu) \in P_2$, and then extract bases $K_{\lambda,\mu}$ in order to define the ideal $I \subseteq \ker(\Phi)$ from (5.3). For each $(\lambda,\mu) \in P_2$, after finding a certain collection of relations we will perform a dimension count to prove these relations span $\ker(\Phi_{\lambda,\mu})$. Hence, we first decompose the codomain $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda\mathcal{W}, \mathbb{S}^\mu\mathcal{W}) \cong \mathbb{S}^{\mu/\lambda}V$ into a sum of irreducibles.

It will be useful to recall the construction of $\mathrm{Sym}^2 V$ and $\bigwedge^2 V$ as subspaces embedded in $V^{\otimes 2}$. We have the quotients

$$\mathrm{Sym}^2 V := \frac{V^{\otimes 2}}{\langle v_1 \otimes v_2 - v_2 \otimes v_1 \mid v_i \in \mathcal{B} \rangle}$$

and

$$\bigwedge^2 V := \frac{V^{\otimes 2}}{\langle v_1 \otimes v_2 + v_2 \otimes v_1 \mid v_i \in \mathcal{B} \rangle},$$

and there is a natural isomorphism $V^{\otimes 2} \longrightarrow \operatorname{Sym}^2 V \oplus \bigwedge^2 V$ given by

$$v_1 \otimes v_2 \mapsto \frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1,\ v_1 \otimes v_2 - v_2 \otimes v_1) =: (v_1 v_2, v_1 \wedge v_2), \tag{5.4}$$

which enables us to identify $\operatorname{Sym}^2 V$ and $\bigwedge^2 V$ as subspaces of $V^{\otimes 2}$.

Proposition 5.1. *Let* $(\lambda, \mu) \in P_2$*.*

(i) *If* $\mu = (\lambda_1 + 2, \lambda_2)$ *then* $\operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W}) \cong \operatorname{Sym}^2 V$*.*

(ii) *If* $\mu = (\lambda_1, \lambda_2 + 2)$ *then* $\operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W}) \cong \operatorname{Sym}^2 V$*.*

(iii) *If* $\lambda_1 = \lambda_2$ *and* $\mu = (\lambda_1 + 1, \lambda_1 + 1)$ *then* $\operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W}) \cong \bigwedge^2 V$*.*

(iv) *If* $\lambda_1 > \lambda_2$ *and* $\mu = (\lambda_1 + 1, \lambda_2 + 1)$ *then* $\operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W}) \cong V^{\otimes 2}$*.*

Proof. For (i) and (ii) this is just Corollary 4.25. For (iii) and (iv) fix $\mu = (\lambda_1 + 1, \lambda_2 + 1)$, and using the notation of Lemma 4.36, let $\Gamma_{\mu/\lambda}$ be the set of partitions γ corresponding to the irreducible summands of $\operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W}) \cong \mathbb{S}^{\mu/\lambda} V$. By Lemma 4.36, we have $\gamma \in \Gamma_{\mu/\lambda}$ if and only if $|\gamma| = 2$ and $\lambda_2 + 1 - \lambda_1 \leq \gamma_2 \leq 1$. If $\lambda_1 = \lambda_2$ then the only $\gamma \in \Gamma_{\mu/\lambda}$ is $(1,1)$, giving $\operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W}) \cong \mathbb{S}^{(1,1)} V = \bigwedge^2 V$ as required. If $\lambda_1 > \lambda_2$ then both $(2,0), (1,1) \in \Gamma_{\mu/\lambda}$, hence $\operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W}) \cong \operatorname{Sym}^2 V \oplus \bigwedge^2 V \cong V^{\otimes 2}$. □

Remark 5.2. In the following four subsections we describe $\ker(\Phi_{\lambda,\mu})$ for the four cases of Proposition 5.1 respectively. To do this, we will find relations by evaluating elements of $\operatorname{im}(\Phi_{\lambda,\mu})$ on an arbitrary section w^λ of $\mathbb{S}^\lambda \mathcal{W}$. Such elements are given by compositions of the f-type and g-type maps defined in Proposition 4.28.

In the first three cases there is only one route from λ to μ so the domain of $\Phi_{\lambda,\mu}$ satisfies $e_\mu \Bbbk Q' e_\lambda \cong V^{\otimes 2}$. Since $\Phi_{\lambda,\mu}$ is surjective and $V^{\otimes 2} \cong \operatorname{Sym}^2 V \oplus \bigwedge^2 V$, we just need to find relations that span a space isomorphic to the irreducible summand in the decomposition of $V^{\otimes 2}$ that is complement to the summand given by $\operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W})$ in Proposition 5.1; these relations must then span $\ker(\Phi_{\lambda,\mu})$. Case (iv) is slightly different and we deal with that in Section 5.1.4.

Notation 5.3. (i) Let w^λ be a section of $\mathbb{S}^\lambda \mathcal{W}$ as in (4.9). In each of the following subsections, the $x_{1,1} \wedge x_{1,2} \otimes \cdots \otimes x_{\lambda_2,1} \wedge x_{\lambda_2,2}$ part of w^λ is never altered so we simplify the notation by denoting $\underline{x} := x_{1,1} \wedge x_{1,2} \otimes \cdots \otimes x_{\lambda_2,1} \wedge x_{\lambda_2,2}$. Thus we have

$$w^\lambda = \underline{x} \otimes y_1 \cdots y_{\lambda_1 - \lambda_2}.$$

(ii) When writing down the subspaces $\ker(\Phi_{\lambda,\mu})$, we will abuse notation slightly by writing $f^{\lambda}_{u_\rho}$ and $g^{\lambda}_{u_\rho}$ for the arrows $a^{\lambda,1}_{\rho}$ and $a^{\lambda,2}_{\rho}$ respectively. Henceforth the symbols $f^{\lambda}_{u_\rho}, g^{\lambda}_{u_\rho}$ are juxtaposed to denote paths in $\Bbbk Q'$, but separated by $\circ$ for homomorphisms in A.

(iii) Particularly in figure environments, we will use $\bullet$ as a place-holder for vectors in $\mathcal{B}$.

5.1.1 Paths of two horizontal arrows

Here we consider paths of the form

$$\mathbb{S}^{(\lambda_1,\lambda_2)}\mathcal{W} \longrightarrow \mathbb{S}^{(\lambda_1+1,\lambda_2)}\mathcal{W} \longrightarrow \mathbb{S}^{(\lambda_1+2,\lambda_2)}\mathcal{W},$$

as shown in Figure 5.1. Denote $\nu = (\lambda_1+1,\lambda_2)$.

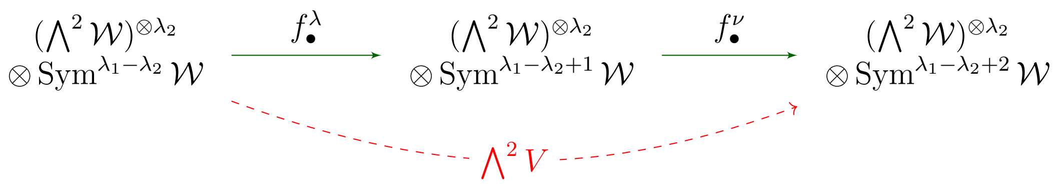

FIGURE 5.1: Paths of two horizontal arrows with relations given by the dashed arrow.

For all $v_1, v_2 \in \mathcal{B}$ we have

$$\begin{aligned} f^{\nu}_{v_2} \circ f^{\lambda}_{v_1}(w^{\lambda}) &= f^{\nu}_{v_2}\left(\underline{x} \otimes y_1 \cdots y_{\lambda_1-\lambda_2} z_{v_1}\right) \\ &= \underline{x} \otimes y_1 \cdots y_{\lambda_1-\lambda_2} z_{v_1} z_{v_2} \\ &= \underline{x} \otimes y_1 \cdots y_{\lambda_1-\lambda_2} z_{v_2} z_{v_1} \\ &= f^{\nu}_{v_1}\left(\underline{x} \otimes y_1 \cdots y_{\lambda_1-\lambda_2} z_{v_2}\right) \\ &= f^{\nu}_{v_1} \circ f^{\lambda}_{v_2}(w^{\lambda}) \end{aligned}$$

and so $f^{\nu}_{v_2} \circ f^{\lambda}_{v_1} - f^{\nu}_{v_1} \circ f^{\lambda}_{v_2} = 0$. Using (5.4) and identifying the tensor product with composition of maps, we may write down an isomorphism

$$\bigwedge^2 V \xrightarrow{\cong} \left\langle f^{\nu}_{v_2} \circ f^{\lambda}_{v_1} - f^{\nu}_{v_1} \circ f^{\lambda}_{v_2} \mid v_1, v_2 \in \mathcal{B} \right\rangle$$

where $v_1 \wedge v_2 \mapsto f^{\nu}_{v_2} \circ f^{\lambda}_{v_1} - f^{\nu}_{v_1} \circ f^{\lambda}_{v_2}$. We have $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{\lambda}\mathcal{W}, \mathbb{S}^{(\lambda_1+2,\lambda_2)}\mathcal{W}) \cong \mathrm{Sym}^2 V$ by Proposition 5.1(i); hence, it follows from Remark 5.2 that this is the entire subspace of relations because the domain of $\Phi_{\lambda,\mu}$ satisfies $e_\mu \Bbbk Q' e_\lambda \cong V^{\otimes 2} \cong$

$\operatorname{Sym}^2 V \oplus \bigwedge^2 V$. Thus we may conclude

$$\ker(\Phi_{\lambda,\mu}) = \left\langle f^{\nu}_{v_2} f^{\lambda}_{v_1} - f^{\nu}_{v_1} f^{\lambda}_{v_2} \mid v_1, v_2 \in \mathcal{B} \right\rangle .$$

Note that in the original notation used in Definition 4.31, this subspace is given by $\ker(\Phi_{\lambda,\mu}) = \left\langle a^{\nu,1}_{\rho_1} a^{\lambda,1}_{\rho_2} - a^{\nu,1}_{\rho_2} a^{\lambda,1}_{\rho_1} \mid 1 \leq \rho_1, \rho_2 \leq n \right\rangle$.

5.1.2 Paths of two vertical arrows

Next we consider paths of the form

$$\mathbb{S}^{(\lambda_1,\lambda_2)}\mathcal{W} \longrightarrow \mathbb{S}^{(\lambda_1,\lambda_2+1)}\mathcal{W} \longrightarrow \mathbb{S}^{(\lambda_1,\lambda_2+2)}\mathcal{W},$$

as shown in Figure 5.2. Denote $\nu = (\lambda_1, \lambda_2 + 1)$.

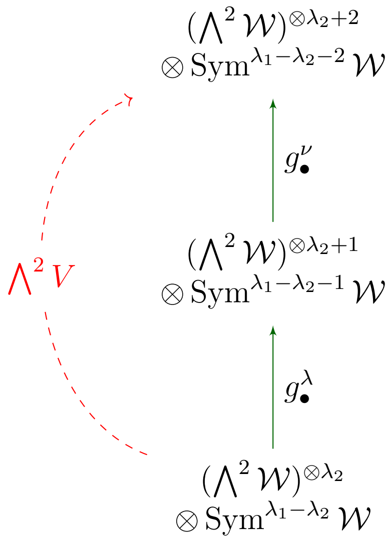

FIGURE 5.2: Paths of two vertical arrows with relations given by the dashed arrow.

For all $v_1, v_2 \in \mathcal{B}$ we have

$$\begin{aligned}
g^{\nu}_{v_2} \circ g^{\lambda}_{v_1}(w^{\lambda}) &= g^{\nu}_{v_2}\left(\sum_{k=1}^{\lambda_1-\lambda_2} \underline{x} \otimes y_k \wedge z_{v_1} \otimes \prod_{j\neq k} y_j\right) \\
&= \sum_{k=1}^{\lambda_1-\lambda_2} g^{\nu}_{v_2}\left(\underline{x} \otimes y_k \wedge z_{v_1} \otimes \prod_{j\neq k} y_j\right) \\
&= \sum_{k=1}^{\lambda_1-\lambda_2}\left(\sum_{i\neq k} \underline{x} \otimes y_k \wedge z_{v_1} \otimes y_i \wedge z_{v_2} \otimes \prod_{j\neq k,i} y_j\right) \\
&= \sum_{i=1}^{\lambda_1-\lambda_2}\left(\sum_{k\neq i} \underline{x} \otimes y_i \wedge z_{v_2} \otimes y_k \wedge z_{v_1} \otimes \prod_{j\neq k,i} y_j\right) \\
&= \sum_{i=1}^{\lambda_1-\lambda_2} g^{\nu}_{v_1}\left(\underline{x} \otimes y_i \wedge z_{v_2} \otimes \prod_{j\neq i} y_j\right) \\
&= g^{\nu}_{v_1} \circ g^{\lambda}_{v_2}(w^{\lambda})
\end{aligned}$$

where in the fourth equality we use an (E2)-type exchange to swap the column containing $y_k \wedge z_{v_1}$ with the column containing $y_i \wedge z_{v_2}$, and the order of summation is swapped. Therefore $g^{\nu}_{v_2} \circ g^{\lambda}_{v_1} - g^{\nu}_{v_1} \circ g^{\lambda}_{v_2} = 0$, so like the previous case, using Proposition 5.1(ii) and Remark 5.2 yields

$$\ker(\Phi_{\lambda,\mu}) = \left\langle g^{\nu}_{v_2} g^{\lambda}_{v_1} - g^{\nu}_{v_1} g^{\lambda}_{v_2} \mid v_1, v_2 \in \mathcal{B} \right\rangle .$$

5.1.3 Paths between vertices on the diagonal

Next we suppose $\lambda_1 = \lambda_2$ and consider paths of the form

$$\mathbb{S}^{(\lambda_1,\lambda_1)}\mathcal{W} \longrightarrow \mathbb{S}^{(\lambda_1+1,\lambda_1)}\mathcal{W} \longrightarrow \mathbb{S}^{(\lambda_1+1,\lambda_1+1)}\mathcal{W},$$

as shown in Figure 5.3. Denote $\nu = (\lambda_1 + 1, \lambda_1)$.

Since $\mathbb{S}^{\lambda}\mathcal{W}$ has no symmetric part $w^{\lambda} = \underline{x}$. Then for all $v_1, v_2 \in \mathcal{B}$ we have

$$\begin{aligned}
g^{\nu}_{v_2} \circ f^{\lambda}_{v_1}(w^{\lambda}) &= g^{\nu}_{v_2}\left(\underline{x} \otimes z_{v_1}\right) \\
&= \underline{x} \otimes z_{v_1} \wedge z_{v_2} \\
&= -\underline{x} \otimes z_{v_2} \wedge z_{v_1} \\
&= -g^{\nu}_{v_1}\left(\underline{x} \otimes z_{v_2}\right) \\
&= -g^{\nu}_{v_1} \circ f^{\lambda}_{v_2}(w^{\lambda})
\end{aligned}$$

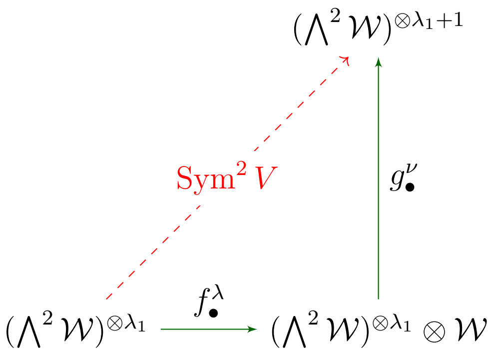

FIGURE 5.3: Paths between vertices on the diagonal with relations given by the dashed arrow.

and so $g_{v_2}^\nu \circ f_{v_1}^\lambda + g_{v_1}^\nu \circ f_{v_2}^\lambda = 0$. Again, using (5.4) and identifying the tensor product with composition of maps, we may write down an isomorphism

$$\mathrm{Sym}^2 V \xrightarrow{\cong} \left\langle g_{v_2}^\nu \circ f_{v_1}^\lambda + g_{v_1}^\nu \circ f_{v_2}^\lambda \mid v_1, v_2 \in \mathcal{B} \right\rangle$$

where $v_1 v_2 \mapsto g_{v_2}^\nu \circ f_{v_1}^\lambda + g_{v_1}^\nu \circ f_{v_2}^\lambda$. We have $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^{(\lambda_1+1,\lambda_1+1)}\mathcal{W}) \cong \bigwedge^2 V$ by Proposition 5.1(iii); hence, it follows from Remark 5.2 that this is the entire subspace of relations because the domain of $\Phi_{\lambda,\mu}$ satisfies $e_\mu \Bbbk Q' e_\lambda \cong V^{\otimes 2} \cong \mathrm{Sym}^2 V \oplus \bigwedge^2 V$. Thus we may conclude

$$\ker(\Phi_{\lambda,\mu}) = \left\langle g_{v_2}^\nu f_{v_1}^\lambda + g_{v_1}^\nu f_{v_2}^\lambda \mid v_1, v_2 \in \mathcal{B} \right\rangle .$$

5.1.4 Paths around a square

Now we suppose $\lambda_1 > \lambda_2$ and $\mu = (\lambda_1 + 1, \lambda_2 + 1)$, and consider paths around a square as in Figure 5.4. In this case there are two routes from λ to μ, so we have $e_\mu \Bbbk Q' e_\lambda \cong V^{\otimes 2} \oplus V^{\otimes 2}$. Akin to Remark 5.2, surjectivity of $\Phi_{\lambda,\mu}$ and counting dimensions implies that $\ker(\Phi_{\lambda,\mu}) \cong V^{\otimes 2}$ since $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W}) \cong V^{\otimes 2}$ by Proposition 5.1(iv). Thus, we are looking for relations that span a space isomorphic to $V^{\otimes 2}$.

Lemma 5.4. *Denote $\nu = (\lambda_1+1, \lambda_2)$ and $\delta = (\lambda_1, \lambda_2+1)$. Then for all $v_1, v_2 \in \mathcal{B}$, we have*

$$(\lambda_1 - \lambda_2)\, g_{v_2}^\nu \circ f_{v_1}^\lambda = (\lambda_1 - \lambda_2 + 1)\, f_{v_1}^\delta \circ g_{v_2}^\lambda - f_{v_2}^\delta \circ g_{v_1}^\lambda . \tag{5.5}$$

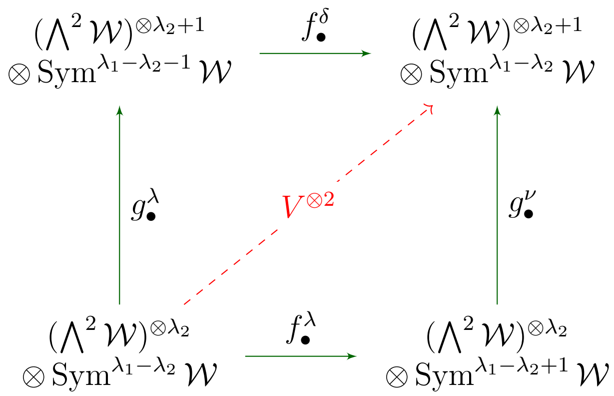

FIGURE 5.4: Paths around a square, with space of relations isomorphic to $V^{\otimes 2}$, generated by (5.5).

Proof. Starting with the right hand side, we have

$$\begin{aligned}
&(\lambda_1-\lambda_2+1)\, f^\delta_{v_1} \circ g^\lambda_{v_2}(w^\lambda) - f^\delta_{v_2} \circ g^\lambda_{v_1}(w^\lambda) \\
&= (\lambda_1-\lambda_2+1)\, f^\delta_{v_1} \left(\sum_{k=1}^{\lambda_1-\lambda_2} \underline{x} \otimes y_k \wedge z_{v_2} \otimes \prod_{j \neq k} y_j \right) - f^\delta_{v_2} \left(\sum_{k=1}^{\lambda_1-\lambda_2} \underline{x} \otimes y_k \wedge z_{v_1} \otimes \prod_{j \neq k} y_j \right) \\
&= (\lambda_1-\lambda_2+1) \sum_{k=1}^{\lambda_1-\lambda_2} \underline{x} \otimes y_k \wedge z_{v_2} \otimes z_{v_1} \prod_{j \neq k} y_j - \sum_{k=1}^{\lambda_1-\lambda_2} \underline{x} \otimes y_k \wedge z_{v_1} \otimes z_{v_2} \prod_{j \neq k} y_j .
\end{aligned}$$

The left hand side becomes

$$\begin{aligned}
&(\lambda_1-\lambda_2)\, g^\nu_{v_2} \circ f^\lambda_{v_1}(w^\lambda) \\
&= (\lambda_1-\lambda_2)\, g^\nu_{v_2}\, (\underline{x} \otimes y_1 \cdots y_{\lambda_1-\lambda_2} z_{v_1}) \\
&= (\lambda_1-\lambda_2) \sum_{k=1}^{\lambda_1-\lambda_2} \left(\underline{x} \otimes y_k \wedge z_{v_2} \otimes z_{v_1} \prod_{j \neq k} y_j \right) + (\lambda_1-\lambda_2)\, \underline{x} \otimes z_{v_1} \wedge z_{v_2} \otimes \prod_{j=1}^{\lambda_1-\lambda_2} y_j .
\end{aligned}$$

Now subtract the left hand side from the right hand side to give

$$\begin{aligned}
&(\lambda_1-\lambda_2+1)\, f^\delta_{v_1} \circ g^\lambda_{v_2} - f^\delta_{v_2} \circ g^\lambda_{v_1} - (\lambda_1-\lambda_2)\, g^\nu_{v_2} \circ f^\lambda_{v_1} \\
&= \sum_{k=1}^{\lambda_1-\lambda_2} \underline{x} \otimes y_k \wedge z_{v_2} \otimes z_{v_1} \prod_{j \neq k} y_j - \sum_{k=1}^{\lambda_1-\lambda_2} \underline{x} \otimes y_k \wedge z_{v_1} \otimes z_{v_2} \prod_{j \neq k} y_j \\
&\quad - (\lambda_1-\lambda_2)\, \underline{x} \otimes z_{v_1} \wedge z_{v_2} \otimes \prod_{j=1}^{\lambda_1-\lambda_2} y_j \quad =: (\dagger).
\end{aligned}$$

We now perform slightly different exchanges with each of the $\lambda_1 - \lambda_2$ copies of

$\underline{x} \otimes z_{v_1} \wedge z_{v_2} \otimes \prod_{j=1}^{\lambda_1-\lambda_2} y_j$ defining the last term of (†). With the first copy, perform an (E1)-type exchange by moving into y_1 the column containing $z_{v_1} \wedge z_{v_2}$, yielding

$$\begin{aligned}\underline{x} \otimes z_{v_1} \wedge z_{v_2} \otimes \prod_{j=1}^{\lambda_1-\lambda_2} y_j &= \underline{x} \otimes y_1 \wedge z_{v_2} \otimes z_{v_1} \prod_{j\neq 1} y_j + \underline{x} \otimes z_{v_1} \wedge y_1 \otimes z_{v_2} \prod_{i\neq 1} y_i \\ &= -\underline{x} \otimes z_{v_2} \wedge y_1 \otimes z_{v_1} \prod_{j\neq 1} y_j + \underline{x} \otimes z_{v_1} \wedge y_1 \otimes z_{v_2} \prod_{j\neq 1} y_j\end{aligned}$$

Now perform a similar exchange with the second copy using y_2, and in general with the k-th copy using y_k. Adding these all together, we get

$$\begin{aligned}&(\lambda_1 - \lambda_2)\, \underline{x} \otimes z_{v_1} \wedge z_{v_2} \otimes \prod_{j=1}^{\lambda_1-\lambda_2} y_j \\ &= -\sum_{k=1}^{\lambda_1-\lambda_2} \underline{x} \otimes z_{v_2} \wedge y_k \otimes z_{v_1} \prod_{j\neq k} y_j + \sum_{k=1}^{\lambda_1-\lambda_2} \underline{x} \otimes z_{v_1} \wedge y_k \otimes z_{v_2} \prod_{j\neq k} y_j.\end{aligned}$$

Finish by substituting the right-hand side of this identity into the last term of (†) to get zero. □

Since

$$V^{\otimes 2} \cong \left\langle (\lambda_1 - \lambda_2)\, g^{\nu}_{v_2} \circ f^{\lambda}_{v_1} - (\lambda_1 - \lambda_2 + 1)\, f^{\delta}_{v_1} \circ g^{\lambda}_{v_2} + f^{\delta}_{v_2} \circ g^{\lambda}_{v_1} \mid v_1, v_2 \in \mathcal{B} \right\rangle,$$

the discussion prior to Lemma 5.4 implies

$$\ker(\Phi_{\lambda,\mu}) = \left\langle (\lambda_1 - \lambda_2)\, g^{\nu}_{v_2} f^{\lambda}_{v_1} - (\lambda_1 - \lambda_2 + 1)\, f^{\delta}_{v_1} g^{\lambda}_{v_2} + f^{\delta}_{v_2} g^{\lambda}_{v_1} \mid v_1, v_2 \in \mathcal{B} \right\rangle.$$

Remark 5.5. Consequently, because each path going in one direction around the square may be written as a linear combination of paths going the opposite way, there are no relations amongst paths traversing in the same direction.

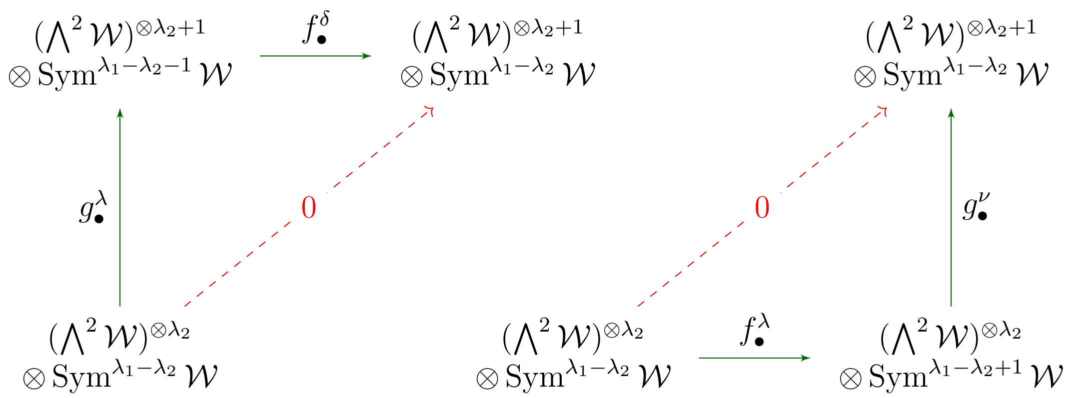

FIGURE 5.5: There are no relations between paths going the same direction around the square.

We now present the ideal $I \subseteq \ker(\Phi)$ as defined in (5.3). Recall the basis $\mathcal{B} = \{u_1, \ldots, u_n\}$ of V.

Proposition 5.6. *For each $(\lambda, \mu) \in P_2$, let ν, δ be the vertices that lie on paths λ and μ as defined in Sections 5.1.1-5.1.4. Define the sets $K_{\lambda,\mu}$ as follows:*

(i) *if $\mu = (\lambda_1 + 2, \lambda_2)$, $K_{\lambda,\mu} = \left\{ f^{\nu}_{u_j} f^{\lambda}_{u_i} - f^{\nu}_{u_i} f^{\lambda}_{u_j} \mid 1 \leq i, j \leq n \right\}$.*

(ii) *if $\mu = (\lambda_1, \lambda_2 + 2)$, $K_{\lambda,\mu} = \left\{ g^{\nu}_{u_j} g^{\lambda}_{u_i} - g^{\nu}_{u_i} g^{\lambda}_{u_j} \mid 1 \leq i, j \leq n \right\}$.*

(iii) *if $\lambda_1 = \lambda_2$ and $\mu = (\lambda_1 + 1, \lambda_1 + 1)$, $K_{\lambda,\mu} = \left\{ g^{\nu}_{u_j} f^{\lambda}_{u_i} + g^{\nu}_{u_i} f^{\lambda}_{u_j} \mid 1 \leq i, j \leq n \right\}$.*

(iv) *if $\lambda_1 > \lambda_2$ and $\mu = (\lambda_1 + 1, \lambda_2 + 1)$,*

$$K_{\lambda,\mu} = \left\{ (\lambda_1 - \lambda_2)\, g^{\nu}_{u_j} f^{\lambda}_{u_i} - (\lambda_1 - \lambda_2 + 1)\, f^{\delta}_{u_i} g^{\lambda}_{u_j} + f^{\delta}_{u_j} g^{\lambda}_{u_i} \mid 1 \leq i, j \leq n \right\}.$$

Then each $K_{\lambda,\mu}$ is a basis of $\ker(\Phi_{\lambda,\mu})$ and the ideal

$$I := \left(\bigcup_{(\lambda,\mu) \in P_2} K_{\lambda,\mu} \right) \subseteq \ker(\Phi)$$

contains all of the relations in Q' generated by paths of length two.

Proof. Sections 5.1.1-5.1.4. ∎

5.2 Relations between longer paths

Having described the ideal $I \subseteq \ker(\Phi)$ in Proposition 5.6, we now prove that $I = \ker(\Phi)$. Recall from equation (5.1) that we have $\ker(\Phi) = (\cup_{(\lambda,\mu)\in P} K_{\lambda,\mu})$. Since the ideal I is taken over P_2, a subset of P, we must show that $\ker(\Phi_{\lambda,\mu}) = e_\mu I e_\lambda$ for all pairs (λ, μ) in the complement of P_2, i.e. the set

$$P_l := \left\{(\lambda, \mu) \in {Q'_0}^2 \mid \lambda < \mu, |\mu| > |\lambda| + 2\right\}$$

where we write l simply to mean 'longer'.

We do this in two propositions. First we compute $\ker(\Phi_{\lambda,\mu})$ in the special cases that all paths $\lambda \to \mu$ are straight lines in Q', i.e. $(\lambda, \mu) \in P_l$ where μ is of the form either (μ_1, λ_2) or (λ_1, μ_2). Then we deal with the remaining cases where both $\lambda_1 < \mu_1$ and $\lambda_2 < \mu_2$.

Prior to the first of these propositions, recall from Example 4.5(ii) that the k-th symmetric power of V is given by

$$\operatorname{Sym}^k V := \frac{V^{\otimes k}}{\langle v_1 \otimes \cdots \otimes v_k - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \mid \sigma \in S_k, v_i \in \mathcal{B}\rangle}, \tag{5.6}$$

where S_k is the permutation group on $\{1, \ldots, k\}$. We write $v_1 \cdots v_k \in \operatorname{Sym}^k V$ for the equivalence class containing $v_1 \otimes \cdots \otimes v_k$. Following [Wey03, §1.1.1 p.3], there is a natural embedding given by

$$\begin{aligned} \Delta_k \colon \operatorname{Sym}^k V &\hookrightarrow V^{\otimes k} \\ v_1 \cdots v_k &\longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}. \end{aligned} \tag{5.7}$$

Notation 5.7. We will make use of Notation 5.3(ii) again: when writing down the subspaces $\ker(\Phi_{\lambda,\mu})$, we will abuse notation slightly by writing $f^\lambda_{u_\rho}$ and $g^\lambda_{u_\rho}$ for the arrows $a^{\lambda,1}_\rho$ and $a^{\lambda,2}_\rho$ respectively. Henceforth the symbols $f^\lambda_{u_\rho}, g^\lambda_{u_\rho}$ are juxtaposed to denote paths in $\Bbbk Q'$, but separated by $\circ$ for homomorphisms in A.

In addition, we will use some notation from Section 4.7; denote $m_1 = \mu_1 - \lambda_1$ and $m_2 = \mu_2 - \lambda_2$, and since we are using the f and g notation for arrows in Q'_1, without ambiguity we may drop the superscript on all arrows in a path except the first as in Notation 4.38(i).

Proposition 5.8. *Let* $(\lambda, \mu) \in P_l$ *and suppose* μ *is of the form either* (μ_1, λ_2) *or* (λ_1, μ_2). *Then*

$$\ker(\Phi_{\lambda,\mu}) = \begin{cases} \left\langle f_{v_{m_1}} \cdots f^{\lambda}_{v_1} - f_{v_{\sigma(m_1)}} \cdots f^{\lambda}_{v_{\sigma(1)}} \mid \sigma \in S_{m_1}, v_i \in \mathcal{B} \right\rangle & \text{if } \mu = (\mu_1, \lambda_2), \\ \left\langle g_{v_{m_2}} \cdots g^{\lambda}_{v_1} - g_{v_{\sigma(m_2)}} \cdots g^{\lambda}_{v_{\sigma(1)}} \mid \sigma \in S_{m_2}, v_i \in \mathcal{B} \right\rangle & \text{if } \mu = (\lambda_1, \mu_2). \end{cases} \tag{5.8}$$

In particular, $\ker(\Phi_{\lambda,\mu}) = e_\mu I e_\lambda$.

Proof. The fact that the subspaces (5.8) are contained in $\ker(\Phi_{\lambda,\mu})$ follows from straightforward induction arguments on the results of Sections 5.1.1-5.1.2. We claim that these relations span $\ker(\Phi_{\lambda,\mu})$ by dimension count. The domain of the surjective map $\Phi_{\lambda,\mu}$ is $e_\mu \Bbbk Q' e_\lambda$, which is isomorphic to either $V^{\otimes m_1}$ or $V^{\otimes m_2}$ when μ is equal to (μ_1, λ_2) or (λ_1, μ_2) respectively. Taking the quotient of these spaces by the appropriate subspace in (5.8) gives $\operatorname{Sym}^{m_1} V$ or $\operatorname{Sym}^{m_2} V$ respectively by (5.6). Since the codomain of $\Phi_{\lambda,\mu}$ is $\operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W})$, which is isomorphic to $\operatorname{Sym}^{m_1} V$ or $\operatorname{Sym}^{m_2} V$ respectively by Corollary 4.25, the claim follows from the first isomorphism theorem.

For the final statement, first suppose $\mu = (\mu_1, \lambda_2)$. The subspace $e_\mu I e_\lambda$ consists only of the relations amongst straight line paths $\lambda \to \mu$, and all of these are generated by those in Proposition 5.6(i), specifically those of the form $\{f_{v_2} f^{\gamma_i}_{v_1} - f_{v_1} f^{\gamma_i}_{v_2} \mid v_1, v_2 \in \mathcal{B}\}$ where $\gamma_i = \lambda + i e_1$ for $0 \leq i \leq m_1 - 2$. Hence, define $S^T_{m_1} \subseteq S_{m_1}$ to be the subset of adjacent transpositions, i.e. $\sigma \in S^T_{m_1}$ if for some $1 \leq k \leq m_1 - 1$ we have $\sigma(k) = k+1$, $\sigma(k+1) = k$, and $\sigma(j) = j$ for all $j \neq k, k+1$. Then

$$e_\mu I e_\lambda = \left\langle f_{v_{m_1}} \cdots f^{\lambda}_{v_1} - f_{v_{\sigma(m_1)}} \cdots f^{\lambda}_{v_{\sigma(1)}} \mid \sigma \in S^T_{m_1}, v_i \in \mathcal{B} \right\rangle.$$

Hence we have $e_\mu I e_\lambda \subseteq \ker(\Phi_{\lambda,\mu})$, but since S_{m_1} is generated by the elements of $S^T_{m_1}$ the reverse inclusion follows simply by performing a sequence of permutations in $S^T_{m_1}$. The proof is similar for $\mu = (\lambda_1, \mu_2)$. □

Proposition 5.9. *Let* $(\lambda, \mu) \in P_l$ *and suppose* $\lambda_1 < \mu_1$ *and* $\lambda_2 < \mu_2$. *Then* $\ker(\Phi_{\lambda,\mu}) = e_\mu I e_\lambda$.

Proof. First of all, as in Section 4.7, it is enough to consider the special case that $\lambda_2 = 0$ as each of the spaces $e_\mu \Bbbk Q' e_\lambda$, $e_\mu I e_\lambda$ and $\operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W})$ is unchanged if we replace (λ_1, λ_2) and (μ_1, μ_2) by $(\lambda_1 - \lambda_2, \lambda_2 - \lambda_2)$ and $(\mu_1 - \lambda_2, \mu_2 - \lambda_2)$ respectively. See, for example, Corollary 4.26(ii).

There is a commutative diagram

$$\begin{array}{ccc} e_\mu \Bbbk Q' e_\lambda & \xrightarrow{\quad\Phi_{\lambda,\mu}\quad} & \operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W}) \\ \pi \big\downarrow & \nearrow_{\Psi_{\lambda,\mu}} & \\ \dfrac{e_\mu \Bbbk Q' e_\lambda}{e_\mu I e_\lambda} & & \end{array}$$

where π is the quotient map. The goal is to show that $\Psi_{\lambda,\mu}$ is injective: then $\ker(\Psi_{\lambda,\mu}) = 0$ and so $\ker(\Phi_{\lambda,\mu}) = \pi^{-1}(0) = e_\mu I e_\lambda$ as required.

Consider a path (or more generally, a linear combination of paths) $p \in e_\mu \Bbbk Q' e_\lambda$ and let ν be the vertex (μ_1, λ_2). If at any point on the path(s) p there is a vertical arrow immediately before a horizontal arrow, it is possible to use relations from I, namely those of Proposition 5.6(iv), to rewrite those two arrows as a linear combination of arrows around the same square in Q' that instead go horizontally before vertically. We can repeat this process until p has been rewritten completely as linear combination of paths that all go strictly horizontally before vertically; in other words, there exists an element $p_2 \otimes_\Bbbk p_1 \in e_\mu \Bbbk Q' e_\nu \otimes_\Bbbk e_\nu \Bbbk Q' e_\lambda$ such that $[p] = [p_2 p_1] \in e_\mu \Bbbk Q' e_\lambda / e_\mu I e_\lambda$.

Now, in $\Bbbk Q'/I$ we have $[p_2 p_1] = [p_2][p_1]$ where

$$[p_2] \in \frac{e_\mu \Bbbk Q' e_\nu}{e_\mu I e_\nu} \cong \operatorname{Sym}^{m_2} V, \quad [p_1] \in \frac{e_\nu \Bbbk Q' e_\lambda}{e_\nu I e_\lambda} \cong \operatorname{Sym}^{m_1} V,$$

and we have used the isomorphisms from Proposition 5.8 with $m_1 = \mu_1 - \lambda_1$ and $m_2 = \mu_2$. Since every $[p] \in e_\mu \Bbbk Q' e_\lambda / e_\mu I e_\lambda$ can be written in the form $[p_2 p_1]$, there exists a surjective homomorphism

$$\xi_1 \colon \operatorname{Sym}^{m_2} V \otimes_\Bbbk \operatorname{Sym}^{m_1} V \twoheadrightarrow \frac{e_\mu \Bbbk Q' e_\lambda}{e_\mu I e_\lambda}$$

where $[p_2] \otimes_\Bbbk [p_1] \mapsto [p_2 p_1] = [p]$. We now split into two subcases according to the decomposition of $\operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W})$ into irreducibles.

(i): $\mu_2 \leq \lambda_1$. In this case $\operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^\mu \mathcal{W})$ is isomorphic to $\operatorname{Sym}^{m_2} V \otimes_\Bbbk \operatorname{Sym}^{m_1} V$; this follows from Lemma 4.36 and the first Pieri rule (Proposition 4.7(i)).

We have the diagram

$$
\begin{array}{ccc}
\operatorname{Sym}^{m_2} V \otimes_{\Bbbk} \operatorname{Sym}^{m_1} V & \xleftarrow{\;\cong\;} & \operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{\lambda}\mathcal{W}, \mathbb{S}^{\mu}\mathcal{W}) \\
\Big\downarrow \xi_1 & & \Big\downarrow \mathrm{id} \\
\dfrac{e_\mu \Bbbk Q' e_\lambda}{e_\mu I e_\lambda} & \xrightarrow{\;\Psi_{\lambda,\mu}\;} & \operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{\lambda}\mathcal{W}, \mathbb{S}^{\mu}\mathcal{W})
\end{array}
$$

and therefore ξ_1 and $\Psi_{\lambda,\mu}$ must be bijections. In particular, $\Psi_{\lambda,\mu}$ is injective as required.

Note that the relations between paths on the main diagonal of Q', i.e. those of Proposition 5.6(iii), are absent in case (i). Indeed, any vertex γ on a path $\lambda \to \mu$ satisfies $0 \le \gamma_2 \le \mu_2$ and $\lambda_1 \le \gamma_1 \le \mu_1$. Vertices on the diagonal also satisfy $\gamma_1 = \gamma_2$, and when $\mu_2 \le \lambda_1$ the only such possible vertex is (λ_1, μ_2). Thus, with at most one vertex on the diagonal on any path $\lambda \to \mu$, Proposition 5.6(iii) plays no role in $e_\mu \Bbbk Q' e_\lambda / e_\mu I e_\lambda$ in this case. The next case is different.

(ii): $\mu_2 > \lambda_1$. The irreducible summands of $\operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{\lambda}\mathcal{W}, \mathbb{S}^{\mu}\mathcal{W})$ form a proper subset of those in the irreducible decomposition of $\operatorname{Sym}^{m_2} V \otimes_{\Bbbk} \operatorname{Sym}^{m_1} V$; again this follows from Lemma 4.36 and the first Pieri rule (Proposition 4.7(i)). Indeed, as hinted above we must now also consider the possibility of relations between vertices along the diagonal. Let $d = \mu_2 - \lambda_1 > 0$. Then the vertices $(\lambda_1 + k, \lambda_1 + k)$ for all $0 \le k \le d$ may appear on paths $\lambda \to \mu$. Previously, we used the relations around squares to rewrite p as a linear combination of paths going strictly horizontally before vertically. While we may also do that here and the surjective map ξ_1 still applies, we can also use the relations around squares to rewrite p as a linear combination of paths that take the route

$$(\lambda_1, 0) \to (\lambda_1, \lambda_1) \to (\lambda_1{+}1, \lambda_1{+}1) \to \cdots \to (\lambda_1{+}d, \lambda_1{+}d) = (\mu_2, \mu_2) \to (\mu_1, \mu_2),$$

in other words, paths in p travel vertically from λ to the diagonal and then staircase along it as much as possible, exiting horizontally towards μ at height μ_2. Define the sequence $\nu_0 = \lambda, \nu_i = (\lambda_1 + i - 1, \lambda_1 + i - 1)$ for all $1 \le i \le d+1$, and $\nu_{d+2} = \mu$. Then there exists an element $q_2 \otimes_{\Bbbk} z_d \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} z_1 \otimes_{\Bbbk} q_1$ in

$$\bigotimes_{i=0}^{d+1} e_{\nu_{i+1}} \Bbbk Q' e_{\nu_i} \tag{5.9}$$

such that $[p] = [q_2 z_d \cdots z_1 q_1] \in e_\mu \Bbbk Q' e_\lambda / e_\mu I e_\lambda$.

Following the strategy above, we take the quotient of each subspace in (5.9) by the appropriate graded slice of I. For $i = 0$ we have straight vertical paths $(\lambda_1, 0) \to (\lambda_1, \lambda_1)$ and for $i = d+1$ we have straight horizontal paths $(\mu_2, \mu_2) \to (\mu_1, \mu_2)$. Hence, using Proposition 5.8 we have

$$[q_2] \in \frac{e_\mu \Bbbk Q' e_{\nu_{d+1}}}{e_\mu I e_{\nu_{d+1}}} \cong \operatorname{Sym}^{\mu_1 - \mu_2} V, \quad [q_1] \in \frac{e_{\nu_1} \Bbbk Q' e_\lambda}{e_{\nu_1} I e_\lambda} \cong \operatorname{Sym}^{\lambda_1} V.$$

Each of the remaining subspaces, $e_{\nu_{i+1}} \Bbbk Q' e_{\nu_i}$ for $1 \le i \le d$, is spanned by paths starting at a vertex on the diagonal and going horizontally then vertically to the next vertex on the diagonal. These are precisely the paths considered in Section 5.1.3 and therefore we have

$$[z_i] \in \frac{e_{\nu_{i+1}} \Bbbk Q' e_{\nu_i}}{e_{\nu_{i+1}} I e_{\nu_i}} \cong \bigwedge^2 V, \quad 1 \le i \le d.$$

Consider the quotient

$$D := \bigotimes_{i=0}^{d+1} \frac{e_{\nu_{i+1}} \Bbbk Q' e_{\nu_i}}{e_{\nu_{i+1}} I e_{\nu_i}}.$$

Then by the prior discussion, there is a surjective homomorphism

$$\xi_2 \colon D \cong \operatorname{Sym}^{\mu_1 - \mu_2} V \otimes_\Bbbk \left(\bigwedge^2 V \right)^{\otimes d} \otimes_\Bbbk \operatorname{Sym}^{\lambda_1} V \longrightarrow\!\!\!\!\!\rightarrow \frac{e_\mu \Bbbk Q' e_\lambda}{e_\mu I e_\lambda}$$

where $[q_2] \otimes_\Bbbk [z_d] \otimes_\Bbbk \cdots \otimes_\Bbbk [z_1] \otimes_\Bbbk [q_1] \mapsto [q_2 z_d \cdots z_1 q_1] = [p]$.

We must now find the irreducible decomposition of D, which we accomplish using the Pieri rules; see Proposition 4.7. We first decompose the central collection of terms $(\bigwedge^2 V)^{\otimes d}$. The second Pieri rule states that tensoring a Schur power $\mathbb{S}^\gamma V$ by $\bigwedge^2 V$ yields a direct sum taken over the ways of adding two new boxes to distinct rows of γ. Starting with $\mathbb{S}^{(1,1)} V = \bigwedge^2 V$ and applying this rule $d-1$ times, we have

$$\left(\bigwedge^2 V \right)^{\otimes d} \cong \mathbb{S}^{(d,d)} V \oplus X,$$

where X is a direct sum of Schur powers of V defined by Young diagrams with at least three rows.

Next we tensor $\mathbb{S}^{(d,d)} V \oplus X$ by the $\operatorname{Sym}^{\lambda_1} V$ term. By the first Pieri rule this has decomposition given over the ways of adding λ_1 boxes to each partition in the sum $\mathbb{S}^{(d,d)} V \oplus X$ with no two in the same column. By adding λ_1 boxes to the top row of (d,d) we have the term $\mathbb{S}^{(\lambda_1+d,d)}V = \mathbb{S}^{(\mu_2,\mu_2-\lambda_1)}V$, but again, every other term has at least three rows in this second decomposition. Amending X to X' (we don't care exactly what these terms with at least three rows are), we have

$$\left(\bigwedge^2 V\right)^{\otimes d} \otimes_{\Bbbk} \operatorname{Sym}^{\lambda_1} V \cong \mathbb{S}^{(\mu_2,\mu_2-\lambda_1)}V \oplus X'.$$

Lastly, we must tensor this decomposition by $\operatorname{Sym}^{\mu_1-\mu_2} V$. Using the first Pieri rule again and focusing only on the terms that will produce Young diagrams with at most two rows, we have $D \cong (\bigoplus_\gamma \mathbb{S}^\gamma V) \oplus X''$ where γ ranges over the partitions $(\max\{m_1,m_2\}, \min\{m_1,m_2\}), (\max\{m_1,m_2\}+1, \min\{m_1,m_2\}-1), \ldots, (\mu_1, \mu_2-\lambda_1)$. By Lemma 4.36 these partitions are precisely those that describe the irreducible decomposition of $\operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda\mathcal{W}, \mathbb{S}^\mu\mathcal{W})$. Therefore,

$$D \cong \operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda\mathcal{W}, \mathbb{S}^\mu\mathcal{W}) \oplus X''.$$

Hence, we have a diagram of surjective maps

$$\begin{array}{ccccc}
\operatorname{Sym}^{m_2} V \otimes_{\Bbbk} \operatorname{Sym}^{m_1} V & & & & \\
& \overset{\xi_1}{\searrow\!\!\!\searrow} & & & \\
& & \dfrac{e_\mu \Bbbk Q' e_\lambda}{e_\mu I e_\lambda} & \xrightarrow{\ \Psi_{\lambda,\mu}\ } & \operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda\mathcal{W}, \mathbb{S}^\mu\mathcal{W}) \\
& \overset{\xi_2}{\nearrow\!\!\!\nearrow} & & & \\
D \cong \operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda\mathcal{W}, \mathbb{S}^\mu\mathcal{W}) \oplus X'' & & & &
\end{array}$$

Since ξ_1 and ξ_2 are surjective, $e_\mu \Bbbk Q' e_\lambda / e_\mu I e_\lambda$ must be isomorphic to a subspace of the direct sum of the summands that appear in both the irreducible decompositions of $\operatorname{Sym}^{m_2} V \otimes_{\Bbbk} \operatorname{Sym}^{m_1} V$ and D. Since the decomposition of $\operatorname{Sym}^{m_2} V \otimes_{\Bbbk} \operatorname{Sym}^{m_1} V$ consists of only partitions with at most two rows, of which those comprising $\operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda\mathcal{W}, \mathbb{S}^\mu\mathcal{W})$ form a proper subset, we conclude that $e_\mu \Bbbk Q' e_\lambda / e_\mu I e_\lambda$ is isomorphic to a subspace of $\operatorname{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda\mathcal{W}, \mathbb{S}^\mu\mathcal{W})$. This forces the surjective map $\Psi_{\lambda,\mu}$ to be an isomorphism, and in particular is injective as required. □

We now conclude Chapters 4 and 5 with the full presentation of Kapranov's tilting algebra for $\operatorname{Gr}(n,2)$.

Theorem 5.10. *Let $Y = \mathrm{Gr}(n,2)$, let E be the tilting bundle (4.1) and let $A = \mathrm{End}_{\mathcal{O}_Y}(E)$. Let Q' be the quiver defined in Definition 4.31. Then the $\Bbbk$-algebra A is isomorphic to $\Bbbk Q'/I$, where*

$$I = \left(\bigcup_{(\lambda,\mu)\in P_2} K_{\lambda,\mu} \right)$$

and

(i) *if* $\mu = (\lambda_1+2, \lambda_2)$, $K_{\lambda,\mu} = \left\{ f_{u_j} f^{\lambda}_{u_i} - f_{u_i} f^{\lambda}_{u_j} \mid 1 \le i,j \le n \right\}$.

(ii) *if* $\mu = (\lambda_1, \lambda_2+2)$, $K_{\lambda,\mu} = \left\{ g_{u_j} g^{\lambda}_{u_i} - g_{u_i} g^{\lambda}_{u_j} \mid 1 \le i,j \le n \right\}$.

(iii) *if* $\lambda_1 = \lambda_2$ *and* $\mu = (\lambda_1+1, \lambda_1+1)$, $K_{\lambda,\mu} = \left\{ g_{u_j} f^{\lambda}_{u_i} + g_{u_i} f^{\lambda}_{u_j} \mid 1 \le i,j \le n \right\}$.

(iv) *if* $\lambda_1 > \lambda_2$ *and* $\mu = (\lambda_1+1, \lambda_2+1)$,

$$K_{\lambda,\mu} = \left\{ (\lambda_1 - \lambda_2)\, g_{u_j} f^{\lambda}_{u_i} - (\lambda_1 - \lambda_2 + 1)\, f_{u_i} g^{\lambda}_{u_j} + f_{u_j} g^{\lambda}_{u_i} \mid 1 \le i,j \le n \right\}.$$

Proof. In Chapter 4 we defined a $\Bbbk$-algebra homomorphism $\Phi\colon \Bbbk Q' \to A$ and proved it is surjective. After establishing that $\ker(\Phi) = (\bigcup_{(\lambda,\mu)\in P} K_{\lambda,\mu})$, we presented the ideal $I = (\bigcup_{(\lambda,\mu)\in P_2} K_{\lambda,\mu}) \subseteq \ker(\Phi)$ in Proposition 5.6. Propositions 5.8 and 5.9 then prove that $\langle K_{\lambda,\mu}\rangle = \ker(\Phi_{\lambda,\mu}) = e_\mu I e_\lambda \subseteq I$ for all $(\lambda,\mu) \in P \setminus P_2$. This completes the proof that $\ker(\Phi) = I$. □

Example 5.11. Let $Y = \mathrm{Gr}(5,2)$. The tilting quiver is given by Figure 5.6, and below we list the relations that span $I = \ker(\Phi)$. Following Notation 5.7, we only require a superscript for the first arrow in a path since the f and g-type notation determines the remaining arrows. For all $1 \le i,j \le 5$, we have the following.

- Horizontal paths: for $\lambda = (0,0), (1,0), (1,1)$ we have

$$f_{u_j} f^{\lambda}_{u_i} = f_{u_i} f^{\lambda}_{u_j}.$$

- Vertical paths: for $\lambda = (2,0), (3,0), (3,1)$ we have

$$g_{u_j} g^{\lambda}_{u_i} = g_{u_i} g^{\lambda}_{u_j}.$$

- Paths on the main diagonal: for $\lambda = (0,0), (1,1), (2,2)$ we have

$$g_{u_j} f^{\lambda}_{u_i} = -g_{u_i} f^{\lambda}_{u_j}.$$

- Lower-left and upper squares: for $\lambda = (1,0),(2,1)$ we have

$$g_{u_j} f^{\lambda}_{u_i} = 2 f_{u_i} g^{\lambda}_{u_j} - f_{u_j} g^{\lambda}_{u_i}.$$

- Lower-right square: for $\lambda = (2,0)$ we have

$$2 g_{u_j} f^{\lambda}_{u_i} = 3 f_{u_i} g^{\lambda}_{u_j} - f_{u_j} g^{\lambda}_{u_i}.$$

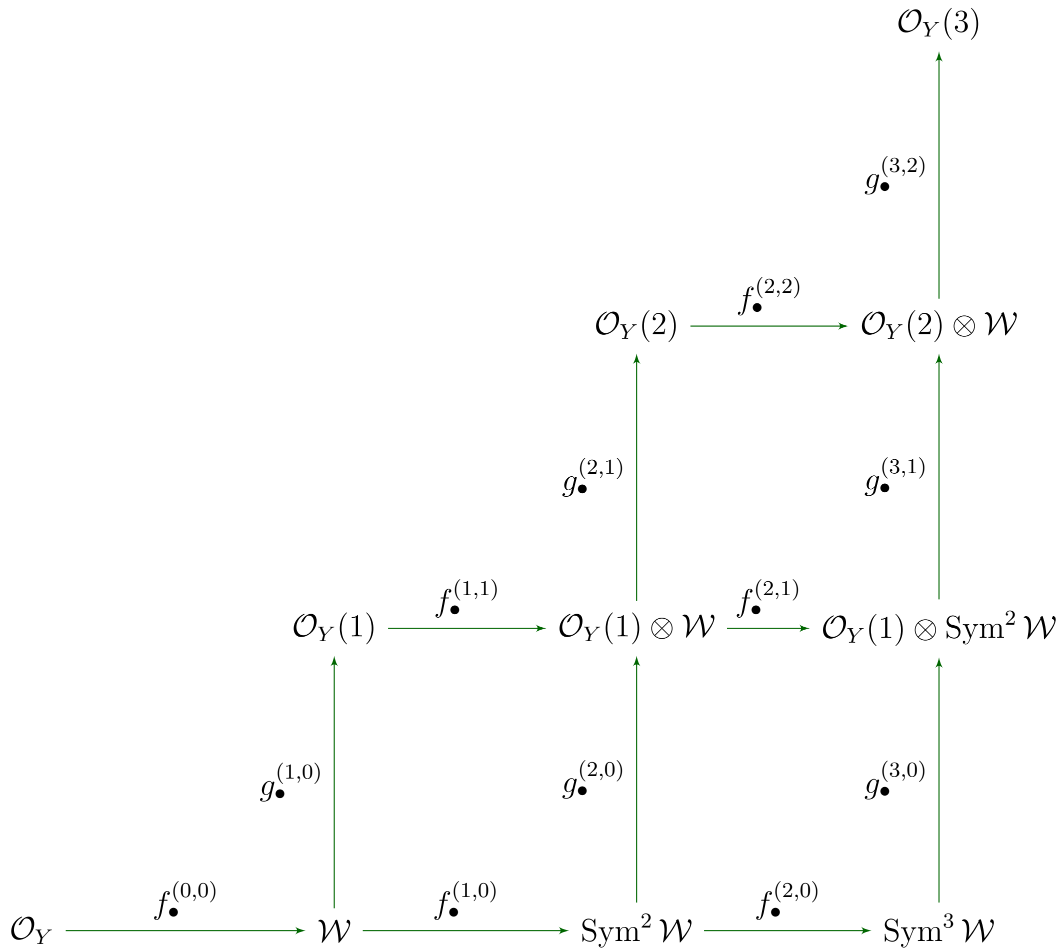

FIGURE 5.6: The tilting quiver for $Y = \mathrm{Gr}(5,2)$. Each arrow represents 5 arrows corresponding to the basis $\mathcal{B}$ of V.

Remark 5.12. Following Example 5.11, consider the full sub-quiver of Q' for $\mathrm{Gr}(5,2)$ defined by deleting the vertices $(3,0),\ldots,(3,3) \in Q'_0$ and any arrows with head or tail at those vertices. By also removing all arrows associated to $u_5 \in \mathcal{B}$, we recover the tilting quiver for $\mathrm{Gr}(4,2)$; see Figure 6.1. In particular, the relations defining $\ker(\Phi)$ for $\mathrm{Gr}(4,2)$ form a sublist of those in the above example; these were calculated by Buchweitz, Leuschke and Van den Bergh in [BLV15, Example 8.4].

5.3 Comparison with the work of Buchweitz, Leuschke and Van den Bergh

First of all, given the tilting quiver for $\mathrm{Gr}(n, 2)$ in Figure 4.2 it is easy to predict the tilting quiver for $\mathrm{Gr}(n, r)$: we have a vertex per indecomposable summand of the tilting bundle (one for each $\lambda \in \mathrm{Young}(n - r, r)$) and n arrows $\lambda \to \mu$ corresponding to a basis of V when $\lambda < \mu \in \mathrm{Young}(n - r, r)$ differ by one box. More precisely, we have the definition below.

Definition 5.13. For $Y = \mathrm{Gr}(n, r)$, define the tilting quiver Q' by

$$Q'_0 = \{\lambda \in \mathbb{Z}^r \mid n - r \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0\},$$

$$Q'_1 = \left\{ a_\rho^{\lambda,i} \;\middle|\; \begin{array}{l} 1 \leq \rho \leq n \\ i \in \{1 \ldots, r\},\ \lambda, \lambda + e_i \in Q'_0 \\ \mathrm{t}(a_\rho^{\lambda,i}) = \lambda,\ \mathrm{h}(a_\rho^{\lambda,i}) = \lambda + e_i \end{array} \right\}.$$

Similar to the $r = 2$ case, Q' is acyclic and $(0, \ldots, 0)$ is the unique source vertex.

One can reconstruct the quiver Q' in Definition 5.13 from the quiver in [BLV16, Theorem B] by removing any arrows labelled by the space 'G' (these all go in the opposite direction). Recall that we have $A = \mathrm{End}_{\mathcal{O}_Y}(E)$ where E is the tilting bundle (4.1) on $Y = \mathrm{Gr}(n, r)$.

Theorem 5.14 ([BLV16, Theorems B, 6.9])**.** *Let Q' be the quiver in Definition 5.13. Then there exists an ideal $J \subset \Bbbk Q'$ such that $\Bbbk Q'/J \cong A$.*

Example 5.15. Suppose $Y = \mathrm{Gr}(6, 3)$. The indecomposable summands of E are given by $\mathbb{S}^\lambda \mathcal{W}$ where $\lambda \in \mathrm{Young}(3, 3)$, and following Definition 5.13 the tilting quiver for Y is given by Figure 5.7.

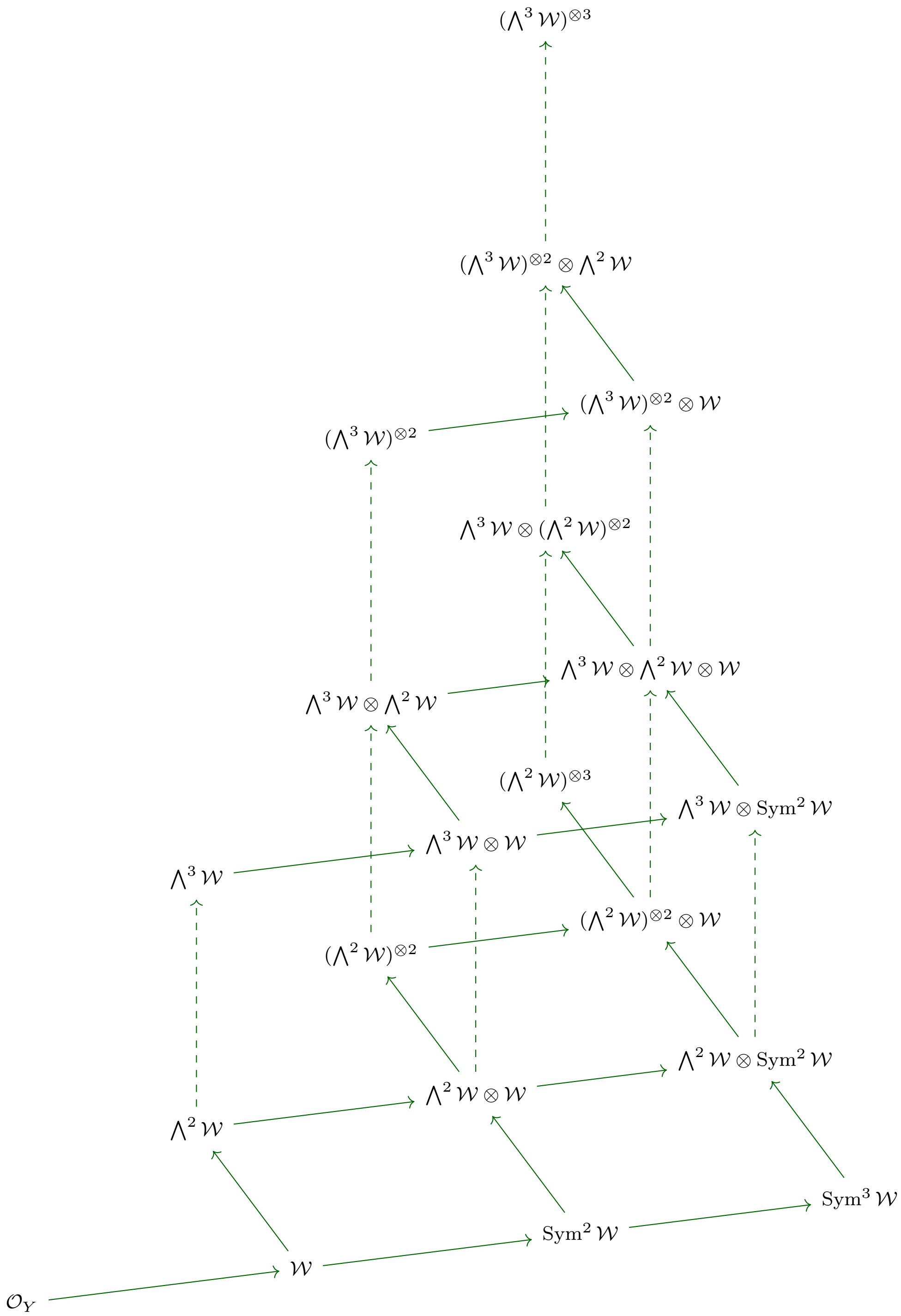

FIGURE 5.7: The tilting quiver Q' for $\mathrm{Gr}(6, 3)$ drawn in $\mathbb{Z}^3$. Each arrow in the figure represents 6 arrows in the quiver. The sub-bundle of exchange relations E_λ (see (4.2)) is implicit.

Our construction of the map Φ, the proof of Proposition 4.30 (surjectivity), and our description of the kernel in Theorem 5.10 gives us a deep understanding of the isomorphism $\Bbbk Q'/\ker(\Phi) \cong A$ in the $\mathrm{Gr}(n,2)$ case. In the general case for $\mathrm{Gr}(n,r)$ with $r > 2$ however, this success is limited by the surjectivity argument. Besides not knowing the form of the homomorphisms $\mathbb{S}^{\gamma}\mathcal{W} \to \mathbb{S}^{\gamma+e_i}\mathcal{W}$ for $i > 2$, the problem with generalising the proof of Proposition 4.30 to $r > 2$ is that we may no longer take advantage of Schur's lemma; indeed, the multiplicity of each irreducible summand of $\mathbb{S}^{\gamma}V \subset \mathbb{S}^{\mu/\lambda}V$, which is given by the Littlewood-Richardson number $c^{\mu}_{\lambda,\gamma}$, may be greater than one.

We now briefly describe the method of proof behind Theorem 5.14. Unlike our direct calculations of the spaces $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^{\lambda}\mathcal{W}, \mathbb{S}^{\mu}\mathcal{W})$ for all pairs $\lambda < \mu \in \mathrm{Young}(n-2,2)$ in Lemma 4.36, the indirect approach of Buchweitz, Leuschke and Van den Bergh instead computes the internal Ext groups of vertex simple modules.

Define a quiver with vertices $\lambda \in \mathrm{Young}(n-r,r)$ and let S_λ be the simple module associated to the vertex λ. Define the set of arrows from λ to μ by a basis for $\mathrm{Ext}^1_A(S_\lambda, S_\mu)^\vee$. The calculations in [BLV16, Section 5.5] yield

$$\mathrm{Ext}^1_A(S_\mu, S_\lambda) = \begin{cases} V & \text{if } \lambda < \mu \text{ with } |\mu| = |\lambda| + 1, \\ 0 & \text{otherwise.} \end{cases}$$

The tensor algebra then determines the resulting quiver which is equal to the tilting quiver Q' from Definition 5.13.

Since A admits a grading by $\mathbb{N}$, the A_∞ structure on $\mathrm{Ext}^\bullet_A(\oplus S_\lambda, \oplus S_\lambda)$ defines a map

$$\mathrm{Ext}^2_A(\oplus S_\lambda, \oplus S_\lambda)^\vee \longrightarrow \bigoplus_{k\geq 2} \left(\mathrm{Ext}^1_A(\oplus S_\lambda, \oplus S_\lambda)^\vee\right)^{\otimes k} \tag{5.10}$$

whose image is an ideal $J \subset \Bbbk Q'$. Theorem 2.13 of [Seg08] now implies that the quiver Q' with relations J satisfies $\Bbbk Q/J \cong A$; for further reference, see [BP08, Section 1]. Therefore in order to present the ideal J, $\mathrm{Ext}^2_A(S_\mu, S_\lambda)$ must be calculated for all $\lambda, \mu \in \mathrm{Young}(n-r,r)$. We have the following.

Proposition 5.16 ([BLV16, 5.5]). *Let $\lambda, \mu \in \mathrm{Young}(n-r, r)$ with $n - r > 1$. Then*

$$\mathrm{Ext}^2_A(S_\mu, S_\lambda) = \begin{cases} \mathrm{Sym}^2 V & \text{if } \mu = \lambda + \text{two boxes in a column,} \\ \bigwedge^2 V & \text{if } \mu = \lambda + \text{two boxes in a row,} \\ V \otimes V & \text{if } \mu = \lambda + \text{two disconnected boxes.} \end{cases} \tag{5.11}$$

Observe that these are the same spaces calculated in Sections 5.1.1-5.1.2, 5.1.3 and 5.1.4 respectively. With these calculations in mind, Buchweitz, Leuschke and Van den Bergh construct J as an ideal generated by a collection of kernels of certain linear maps; see [BLV16, Definition 5.5]. By demonstrating that the degree 2 graded slice of this ideal is equal to (5.11) as λ and μ vary, they conclude by observing that J generates all the necessary relations of $\Bbbk Q'$ by comparison with the dimension of $\mathrm{Ext}^2_A(\oplus S_\lambda, \oplus S_\lambda)^\vee$ in the proof of [BLV16, Theorem 6.9]. Indeed, they point out in [BLV16, Proposition A.10] that the relations are generated quadratically. While this approach has the clear advantage of being able to state the spaces of relations for the tilting quiver of $\mathrm{Gr}(n, r)$ for any $n > r \geq 1$, the drawback is that these relations are not given explicitly, though a recipe is provided for how these relations can be calculated. This is essential for Chapter 6, and we do this for $\mathrm{Gr}(n, 2)$ in Theorem 5.10.

In Proposition 4.28 we write down maps $\mathbb{S}^\lambda \mathcal{W} \to \mathbb{S}^{\lambda + e_i} \mathcal{W}$ for $i = 1, 2$ where $\mathcal{W}$ is the tautological bundle of $\mathrm{Gr}(n, 2)$; the fact we did this explicitly was key to describing $I = \ker(\Phi)$ in Theorem 5.10. An important point is that there is no canonical way to write down these maps: they form an example of what is known as a *Pieri system* (actually, they form part of what Buchweitz, Leuschke and Van den Bergh call a *compatible* Pieri system, since maps are required for the arrows going in the opposite direction that also satisfy various commuting diagrams). Given a (compatible) Pieri system and a collection of scalars which must be calculated as a result of the choice of system, [BLV16, Theorem 7.18] tells us that the generators for J are given by the kernels of certain linear maps. Unfortunately, even writing a compatible Pieri system is not at all trivial, so to exhaustively find relations for $\Bbbk Q'$ when $r > 2$ turns out to be an extensive combinatorial exercise. We discuss this further in Chapter 7.

Chapter 6

Reconstructing $\mathrm{Gr}(n, 2)$ from a tilting bundle

Given our presentation of $A \cong \Bbbk Q' / \ker(\Phi)$ from Theorem 5.10, we are now in a position to prove the following:

Theorem 6.1. *Let Y be the Grassmannian $\mathrm{Gr}(n, 2)$. Then the morphism $f_E : Y \to \mathcal{M}(E)$ from (2.7) is an isomorphism.*

To prove Theorem 6.1 we first prove a more technical result using induction in Sections 6.1 and 6.2; see Lemma 6.5. Then we complete the proof in Section 6.3. Due to the notation involved, rather than state Lemma 6.5 immediately it will be easier to prove the base case $\mathrm{Gr}(4, 2)$ as an example first, and only after that state the induction hypothesis in the following section.

6.1 Base case: $\mathrm{Gr}(4, 2)$

Begin by fixing a basis u_1, u_2, u_3, u_4 of V. By Theorem 5.10 we have the tilting quiver given in Figure 6.1 and for all $1 \leq i, j \leq 4$ the ideal of relations is generated by the following.

$$\begin{aligned}
&\bullet\ g^{(1,0)}_{u_i} f^{(0,0)}_{u_j} + g^{(1,0)}_{u_j} f^{(0,0)}_{u_i} \\
&\bullet\ f^{(1,0)}_{u_i} f^{(0,0)}_{u_j} - f^{(1,0)}_{u_j} f^{(0,0)}_{u_i} \\
&\bullet\ g^{(2,0)}_{u_i} f^{(1,0)}_{u_j} - 2 f^{(1,1)}_{u_j} g^{(1,0)}_{u_i} + f^{(1,1)}_{u_i} g^{(1,0)}_{u_j} \\
&\bullet\ g^{(2,1)}_{u_i} f^{(1,1)}_{u_j} + g^{(2,1)}_{u_j} f^{(1,1)}_{u_i} \\
&\bullet\ g^{(2,1)}_{u_i} g^{(2,0)}_{u_j} - g^{(2,1)}_{u_j} g^{(2,0)}_{u_i}
\end{aligned} \tag{6.1}$$

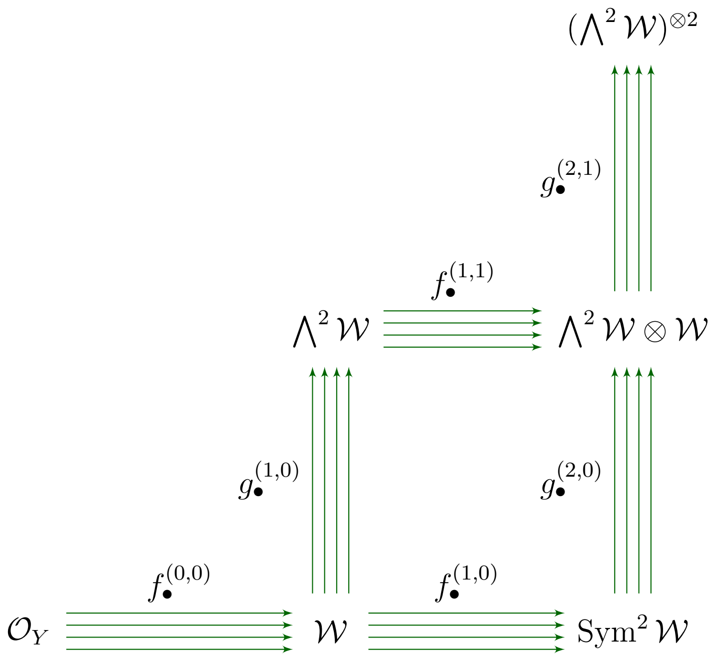

FIGURE 6.1: The tilting quiver for $\mathrm{Gr}(4, 2)$. Each $\bullet$ varies independently across $u_1, \dots, u_4$.

Once and for all fix a point $w \in \mathcal{M}(E) = \mathcal{M}(A, \mathbf{v}, \theta)$. Then w is a θ-stable representation of Q' with dimension vector $\mathbf{v}$ subject to the relations given by (6.1), and as such, we may decompose w into two distinct collections of matrices as follows.

Notation 6.2. (i) Firstly, as a matrix for each $a \in Q'_1$ as per (2.1). Denote the matrix of $f^\lambda_{u_i}$ by F^λ_i, and define the matrix of $g^\lambda_{u_i}$ by G^λ_i. Since $w \in \mathcal{M}(E)$, these matrices must satisfy the matrix relations corresponding to those in (6.1).

(ii) Secondly, by grouping together the matrices in (i) as per (2.3). Write W_λ for the matrix defined by concatenating, side by side, the columns of the matrices whose corresponding arrows have head at λ to form a single long matrix (formally, this new matrix is the *co-product*); see $W_{(1,0)}$ below, for example. Hereafter we will simply use the term 'concatenation' to describe this process. Since $w \in \mathcal{M}(A, \mathbf{v}, \theta)$, θ-stability implies that each W_λ must be full rank ([Cra11, Lemma 2.1]).

Remark 6.3. In order to know what the orders of the matrices F^λ_i and G^λ_i are we must calculate the dimension vector $\mathbf{v}$. This is given by the ranks of the vector bundles at each vertex. Using the formula [FH91, Theorem 6.3(1)], we have $\mathrm{rank}(\mathbb{S}^{(\lambda_1,\lambda_2)}\mathcal{W}) = \lambda_1 - \lambda_2 + 1$. This means that the bundles on the diagonal have rank 1, the bundles just below that have rank 2, and so on until reaching the bottom right corner $\mathrm{Sym}^{n-2}\mathcal{W}$, which has rank $n-1$. More precisely, all bundles along the diagonal line $\lambda_1 = \lambda_2 + k$ have rank $k+1$.

For the matrices corresponding to the arrows between $\mathcal{O}_Y$ and $\mathcal{W}$ we write

$$F_1^{(0,0)} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, F_2^{(0,0)} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, F_3^{(0,0)} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, F_4^{(0,0)} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}.$$

Then for $\lambda = (1,0)$ we have

$$W_{(1,0)} = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix},$$

which must be full rank; without loss of generality, re-index the basis of V if necessary so that $F_1^{(0,0)}, F_2^{(0,0)}$ (the first two columns of $W_{(1,0)}$) are linearly independent. Then we may use the group action to change basis at the vertex $(1,0)$ and write

$$W_{(1,0)} = \begin{pmatrix} 1 & 0 & x_1 & x_3 \\ 0 & 1 & x_2 & x_4 \end{pmatrix},$$

where $x_1, \ldots, x_4 \in \Bbbk$. Denote the entries of all the other F_i^λ, G_i^λ matrices using elements $y_1, \ldots, y_{72} \in \Bbbk$ as in Figure 6.2, where for fixed λ and $i < j$ the entries of F_i^λ and G_i^λ are indexed lower than the entries of F_j^λ and G_i^λ respectively.

Claim: *All of the matrices F_i^λ and G_i^λ comprising the point w can be chosen to take a distinguished form, modulo the group action, with entries in polynomial terms of only x_1, x_2, x_3, x_4 as in Figure 6.3.*

In other words, the entries of $W_{(1,0)}$ are enough to determine all of the data of the point w. The proof of this claim comprises of the remainder of this section.

STEP 1: *The maps $\mathcal{O}_Y \to \mathcal{W} \to \bigwedge^2 \mathcal{W}$.*

In this first part of the quiver we have the matrices

$$F_1^{(0,0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, F_2^{(0,0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, F_3^{(0,0)} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, F_4^{(0,0)} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \tag{6.2}$$

and

$$G_1^{(1,0)} = \begin{pmatrix} y_1 & y_2 \end{pmatrix}, G_2^{(1,0)} = \begin{pmatrix} y_3 & y_4 \end{pmatrix}, G_3^{(1,0)} = \begin{pmatrix} y_5 & y_6 \end{pmatrix}, G_4^{(1,0)} = \begin{pmatrix} y_7 & y_8 \end{pmatrix},$$

which are subject to the relations $G_i^{(1,0)} F_j^{(0,0)} + G_j^{(1,0)} F_i^{(0,0)} = 0$ for $1 \le i, j \le 4$.

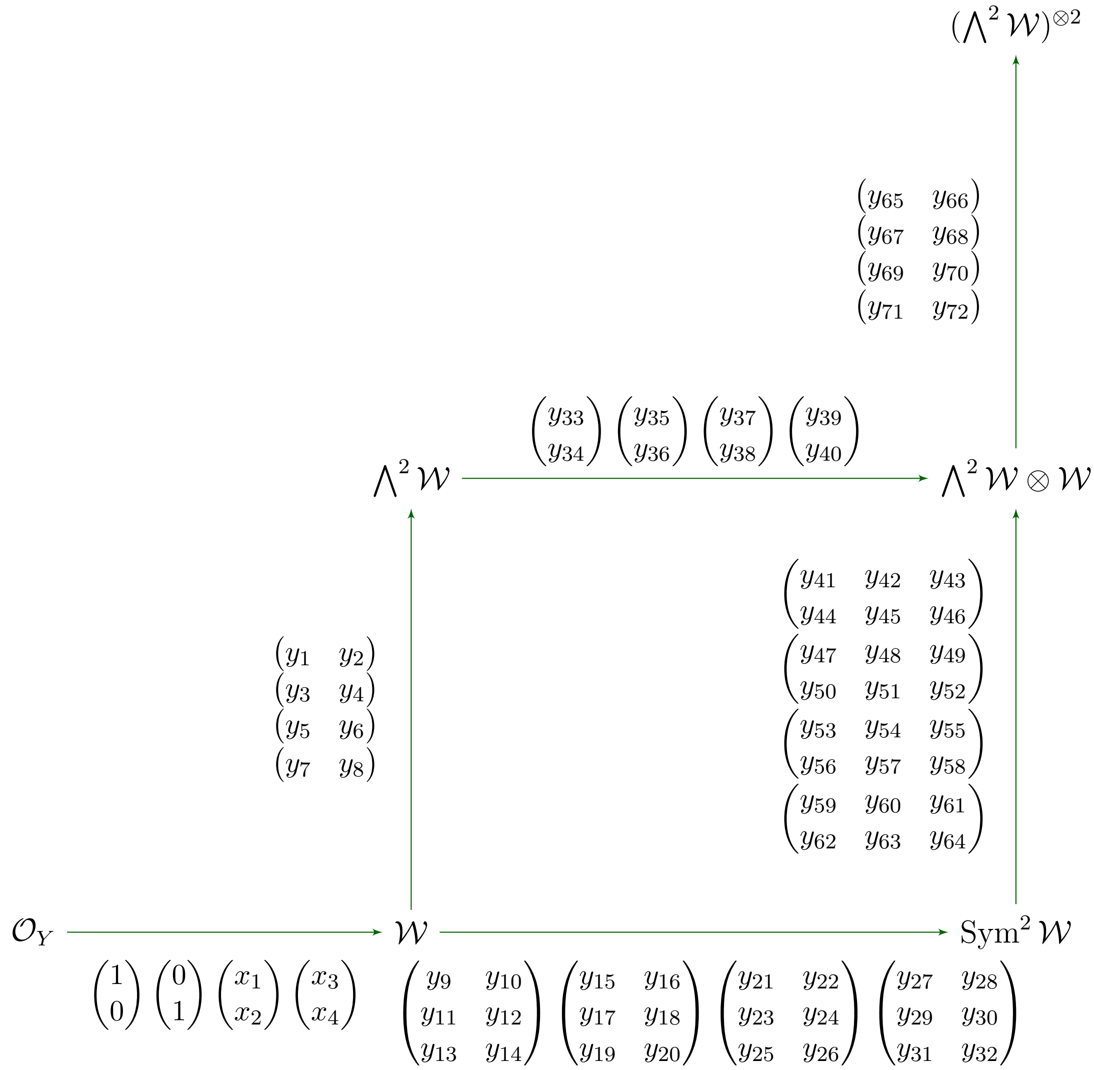

FIGURE 6.2: General form of $w \in \mathcal{M}(E)$ using decomposition (i), e.g. the matrix with entries y_5, y_6 is $G_3^{(1,0)}$.

By first considering when $i = 1 = j$ and $i = 2 = j$, we find that $y_1 = 0 = y_4$. When $i = 1, j = 2$ we have

$$\begin{pmatrix} 0 & y_2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} y_3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \implies y_3 = -y_2.$$

Next, setting $i = 1, j = 3$ yields

$$\begin{pmatrix} 0 & y_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_5 & y_6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \implies y_5 = -y_2 x_2,$$

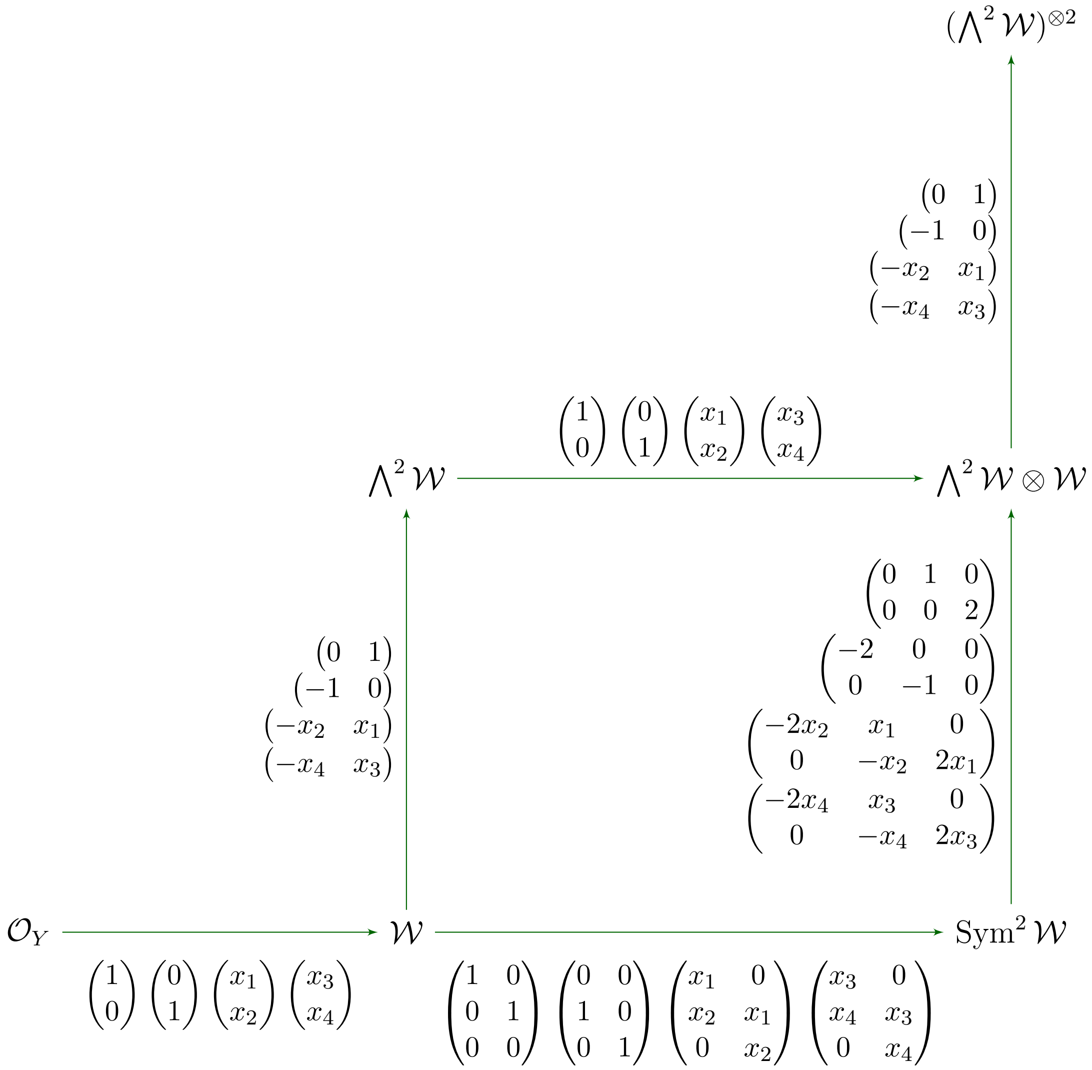

FIGURE 6.3: The unique solution, up to change of basis, of the system of relations with the full rank stability conditions.

and setting $i = 2, j = 3$ yields

$$\begin{pmatrix}-y_2 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} + \begin{pmatrix}y_5 & y_6\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 0 \implies y_6 = y_2 x_1.$$

Repeating the above two substitutions with $j = 4$ yields $y_7 = -y_2 x_4$ and $y_8 = y_2 x_3$. So far we have

$$W_{(1,1)} = \begin{pmatrix}0 & y_2 & -y_2 & 0 & -y_2x_2 & y_2x_1 & -y_2x_4 & y_2x_3\end{pmatrix}.$$

Since $W_{(1,1)}$ is full rank we must have $y_2 \neq 0$. We now use the GL(1) action at the vertex $\bigwedge^2 \mathcal{W}$ to multiply $W_{(1,1)}$ by y_2^{-1}, giving

$$G_1^{(1,0)} = \begin{pmatrix} 0 & 1 \end{pmatrix}, G_2^{(1,0)} = \begin{pmatrix} -1 & 0 \end{pmatrix}, G_3^{(1,0)} = \begin{pmatrix} -x_2 & x_1 \end{pmatrix}, G_4^{(1,0)} = \begin{pmatrix} -x_4 & x_3 \end{pmatrix} \tag{6.3}$$

as required.

Step 2: *The maps* $\mathcal{O}_Y \to \mathcal{W} \to \operatorname{Sym}^2 \mathcal{W}$.

Here we have the matrices

$$F_1^{(1,0)} = \begin{pmatrix} y_9 & y_{10} \\ y_{11} & y_{12} \\ y_{13} & y_{14} \end{pmatrix}, F_2^{(1,0)} = \begin{pmatrix} y_{15} & y_{16} \\ y_{17} & y_{18} \\ y_{19} & y_{20} \end{pmatrix}, F_3^{(1,0)} = \begin{pmatrix} y_{21} & y_{22} \\ y_{23} & y_{24} \\ y_{25} & y_{26} \end{pmatrix}, F_4^{(1,0)} = \begin{pmatrix} y_{27} & y_{28} \\ y_{29} & y_{30} \\ y_{31} & y_{32} \end{pmatrix}$$

subject to the relations $F_i^{(1,0)} F_j^{(0,0)} - F_j^{(1,0)} F_i^{(0,0)} = 0$. First set $i = 1, j = 2$. Then

$$\begin{pmatrix} y_9 & y_{10} \\ y_{11} & y_{12} \\ y_{13} & y_{14} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} y_{15} & y_{16} \\ y_{17} & y_{18} \\ y_{19} & y_{20} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{cases} y_{10} = y_{15} \\ y_{12} = y_{17} \\ y_{14} = y_{19} \end{cases},$$

so the second column of $F_1^{(1,0)}$ equals the first column of $F_2^{(1,0)}$. Next fix $j = 3$ and in turn substitute $i = 1$ then $i = 2$, yielding

$$\begin{pmatrix} y_9 & y_{10} \\ y_{11} & y_{12} \\ y_{13} & y_{14} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} y_{21} & y_{22} \\ y_{23} & y_{24} \\ y_{25} & y_{26} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{cases} y_{21} = y_9 x_1 + y_{10} x_2 \\ y_{23} = y_{11} x_1 + y_{12} x_2 \\ y_{25} = y_{13} x_1 + y_{14} x_2 \end{cases},$$

$$\begin{pmatrix} y_{10} & y_{16} \\ y_{12} & y_{18} \\ y_{14} & y_{20} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} y_{21} & y_{22} \\ y_{23} & y_{24} \\ y_{25} & y_{26} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{cases} y_{22} = y_{10} x_1 + y_{16} x_2 \\ y_{24} = y_{12} x_1 + y_{18} x_2 \\ y_{26} = y_{14} x_1 + y_{20} x_2 \end{cases}.$$

Similarly, fix $j = 4$ while substituting $i = 1$ and $i = 2$ as above to get

$$\begin{array}{ll} y_{27} = y_9 x_3 + y_{10} x_4 & y_{28} = y_{10} x_3 + y_{16} x_4 \\ y_{29} = y_{11} x_3 + y_{12} x_4 & y_{30} = y_{12} x_3 + y_{18} x_4 \\ y_{31} = y_{13} x_3 + y_{14} x_4 & y_{32} = y_{14} x_3 + y_{20} x_4 \end{array}.$$

Substituting all of the above into $W_{(2,0)}$, we have

$$W_{(2,0)} = \begin{pmatrix} y_9 & y_{10} & y_{10} & y_{16} & y_9x_1 + y_{10}x_2 & y_{10}x_1 + y_{16}x_2 & y_9x_3 + y_{10}x_4 & y_{10}x_3 + y_{16}x_4 \\ y_{11} & y_{12} & y_{12} & y_{18} & y_{11}x_1 + y_{12}x_2 & y_{12}x_1 + y_{18}x_2 & y_{11}x_3 + y_{12}x_4 & y_{12}x_3 + y_{18}x_4 \\ y_{13} & y_{14} & y_{14} & y_{20} & y_{13}x_1 + y_{14}x_2 & y_{14}x_1 + y_{20}x_2 & y_{13}x_3 + y_{14}x_4 & y_{14}x_3 + y_{20}x_4 \end{pmatrix}.$$

We now show that the minor of $W_{(2,0)}$ given by the first, second and fourth columns must be full rank. Suppose for a contradiction that it is not full rank. Then we may use the group action (specifically, $\mathrm{GL}(3)$ acting at the vertex $\mathrm{Sym}^2 \mathcal{W}$) to produce a row of zeros in this minor. Suppose this is the top row, i.e. y_9, y_{10}, y_{16} become zero in the new basis (the argument is similar for the other rows). The effect this has on the rest of $W_{(2,0)}$ is that now the entire top row is zero. This contradicts the condition that $W_{(2,0)}$ must be full rank, thus we conclude that the chosen minor must be full rank. Consequently, we may use the group action to turn this minor into the identity matrix. The matrix with respect to this new basis is

$$W_{(2,0)} = \begin{pmatrix} 1 & 0 & 0 & 0 & x_1 & 0 & x_3 & 0 \\ 0 & 1 & 1 & 0 & x_2 & x_1 & x_4 & x_3 \\ 0 & 0 & 0 & 1 & 0 & x_2 & 0 & x_4 \end{pmatrix},$$

thereby yielding the matrices $F_1^{(1,0)}, \ldots, F_4^{(1,0)}$ in Figure 6.3 as required.

STEP 3: *The central square, including the maps* $\mathcal{W} \to \bigwedge^2 \mathcal{W} \to \bigwedge^2 \otimes \mathcal{W}$ *and* $\mathcal{W} \to \mathrm{Sym}^2 \mathcal{W} \to \bigwedge^2 \otimes \mathcal{W}$.

Figure 6.4 summarises the progress of the first two steps. Around the square we have the relations

$$G_i^{(2,0)} F_j^{(1,0)} = 2F_j^{(1,1)} G_i^{(1,0)} - F_i^{(1,1)} G_j^{(1,0)}, \ 1 \leq i, j \leq 4. \tag{6.4}$$

When $i = j$ this simplifies to $G_i^{(2,0)} F_i^{(1,0)} = F_i^{(1,1)} G_i^{(1,0)}$. Additionally, the matrix $W_{(2,1)}$, which is formed by concatenating the eight matrices $F_1^{(1,1)}, \ldots, F_4^{(1,1)}$, $G_1^{(2,0)}, \ldots, G_4^{(2,0)}$, must be full rank.

STEP 3A: *Write* $y_{41}, \ldots, y_{52}$ *(the entries of* $G_1^{(2,0)}, G_2^{(2,0)}$*) in terms of* $y_{33}, \ldots, y_{36}$ *(the entries of* $F_1^{(1,1)}, F_2^{(1,1)}$*).*

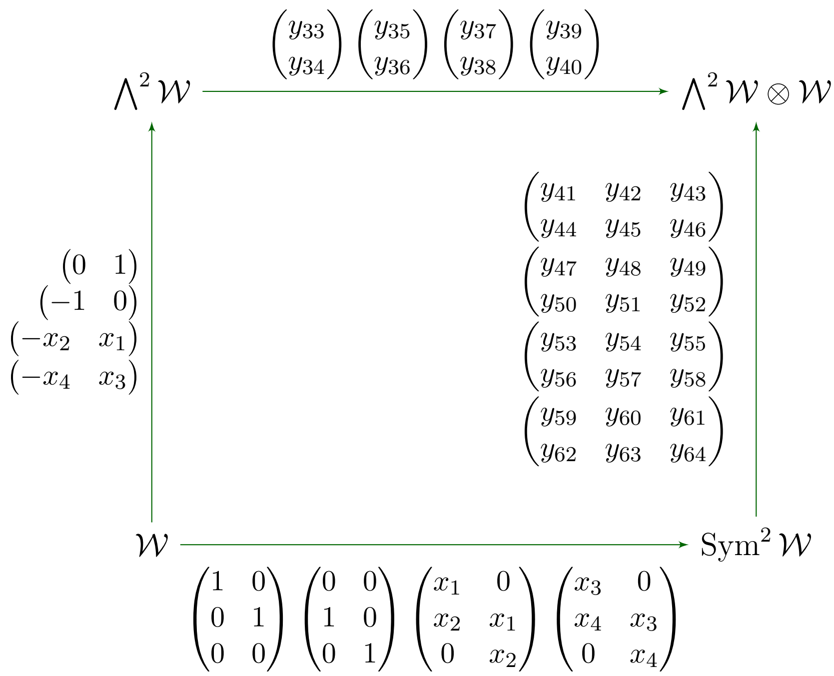

FIGURE 6.4: Progress after Steps 1 and 2.

First, consider (6.4) when $i, j = 1$ and $i, j = 2$ in turn. We have

$$\begin{pmatrix} y_{41} & y_{42} & y_{43} \\ y_{44} & y_{45} & y_{46} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} y_{33} \\ y_{34} \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \implies \begin{pmatrix} y_{41} & y_{42} \\ y_{44} & y_{45} \end{pmatrix} = \begin{pmatrix} 0 & y_{33} \\ 0 & y_{34} \end{pmatrix},$$

$$\begin{pmatrix} y_{47} & y_{48} & y_{49} \\ y_{50} & y_{51} & y_{52} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y_{35} \\ y_{36} \end{pmatrix} \begin{pmatrix} -1 & 0 \end{pmatrix} \implies \begin{pmatrix} y_{48} & y_{49} \\ y_{51} & y_{52} \end{pmatrix} = \begin{pmatrix} -y_{35} & 0 \\ -y_{36} & 0 \end{pmatrix}.$$

Next, for $i = 1$ and $j = 2$ we have $G_1^{(2,0)} F_2^{(1,0)} = 2F_2^{(1,1)} G_1^{(1,0)} - F_1^{(1,1)} G_2^{(1,0)}$, giving

$$\begin{aligned} \begin{pmatrix} y_{41} & y_{42} & y_{43} \\ y_{44} & y_{45} & y_{46} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} &= 2 \begin{pmatrix} y_{35} \\ y_{36} \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} - \begin{pmatrix} y_{33} \\ y_{34} \end{pmatrix} \begin{pmatrix} -1 & 0 \end{pmatrix} \\ \implies \begin{pmatrix} y_{42} & y_{43} \\ y_{45} & y_{46} \end{pmatrix} &= \begin{pmatrix} y_{33} & 2y_{35} \\ y_{34} & 2y_{36} \end{pmatrix}, \end{aligned}$$

and when $i = 2$ and $j = 1$, we have $G_2^{(2,0)} F_1^{(1,0)} = 2F_1^{(1,1)} G_2^{(1,0)} - F_2^{(1,1)} G_1^{(1,0)}$ so that

$$\begin{pmatrix} y_{47} & y_{48} & y_{49} \\ y_{50} & y_{51} & y_{52} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} y_{33} \\ y_{34} \end{pmatrix} \begin{pmatrix} -1 & 0 \end{pmatrix} - \begin{pmatrix} y_{35} \\ y_{36} \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}$$
$$\implies \begin{pmatrix} y_{47} & y_{48} \\ y_{50} & y_{51} \end{pmatrix} = \begin{pmatrix} -2y_{33} & -y_{35} \\ -2y_{34} & -y_{36} \end{pmatrix}.$$

Combining all of the above, we are able to write the entries of $G_1^{(2,0)}, G_2^{(2,0)}$ in terms of those in $F_1^{(1,1)}, F_2^{(1,1)}$ as follows:

$$G_1^{(2,0)} = \begin{pmatrix} 0 & y_{33} & 2y_{35} \\ 0 & y_{34} & 2y_{36} \end{pmatrix}, \; G_2^{(2,0)} = \begin{pmatrix} -2y_{33} & -y_{35} & 0 \\ -2y_{34} & -y_{36} & 0 \end{pmatrix}.$$

STEP 3B: *Write* y_{37}, y_{38} *(the entries of* $F_3^{(1,1)}$*) in terms of* $y_{33}, \ldots, y_{36}, x_1, x_2$*.* Taking $i = 3$ and $j = 1$ we have

$$\begin{pmatrix} 0 & y_{33} & 2y_{35} \\ 0 & y_{34} & 2y_{36} \end{pmatrix} \begin{pmatrix} x_1 & 0 \\ x_2 & x_1 \\ 0 & x_2 \end{pmatrix} = 2 \begin{pmatrix} y_{37} \\ y_{38} \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} - \begin{pmatrix} y_{33} \\ y_{34} \end{pmatrix} \begin{pmatrix} -x_2 & x_1 \end{pmatrix}$$
$$\implies \begin{pmatrix} y_{33}x_2 & y_{33}x_1 + 2y_{35}x_2 \\ y_{34}x_2 & y_{34}x_1 + 2y_{36}x_2 \end{pmatrix} = \begin{pmatrix} y_{33}x_2 & 2y_{37} - y_{33}x_1 \\ y_{34}x_2 & 2y_{38} - y_{34}x_1 \end{pmatrix}$$
$$\implies \begin{cases} y_{37} = y_{33}x_1 + y_{35}x_2 \\ y_{38} = y_{34}x_1 + y_{36}x_2 \end{cases}$$

and so

$$F_3^{(1,1)} = \begin{pmatrix} y_{33}x_1 + y_{35}x_2 \\ y_{34}x_1 + y_{36}x_2 \end{pmatrix}.$$

STEP 3C: *Write* $y_{53}, \ldots, y_{58}$ *(the entries of* $G_3^{(2,0)}$*) in terms of* $y_{33}, \ldots, y_{36}, x_1, x_2$*.* Taking $i = 1$ and $j = 3$ we have

$$\begin{pmatrix} y_{53} & y_{54} & y_{55} \\ y_{56} & y_{57} & y_{58} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} y_{33} \\ y_{34} \end{pmatrix} \begin{pmatrix} -x_2 & x_1 \end{pmatrix} - \begin{pmatrix} y_{33}x_1 + y_{35}x_2 \\ y_{34}x_1 + y_{36}x_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}$$
$$\implies \begin{pmatrix} y_{53} & y_{54} \\ y_{56} & y_{57} \end{pmatrix} = \begin{pmatrix} -2y_{33}x_2 & y_{33}x_1 - y_{35}x_2 \\ -2y_{34}x_2 & y_{34}x_1 - y_{36}x_2 \end{pmatrix}.$$

Now take $i = 2$ and $j = 3$ to get

$$\begin{pmatrix} y_{53} & y_{54} & y_{55} \\ y_{56} & y_{57} & y_{58} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 \begin{pmatrix} y_{35} \\ y_{36} \end{pmatrix} \begin{pmatrix} -x_2 & x_1 \end{pmatrix} - \begin{pmatrix} y_{33}x_1 + y_{35}x_2 \\ y_{34}x_1 + y_{36}x_2 \end{pmatrix} \begin{pmatrix} -1 & 0 \end{pmatrix}$$

$$\implies \begin{pmatrix} y_{54} & y_{55} \\ y_{57} & y_{58} \end{pmatrix} = \begin{pmatrix} y_{33}x_1 - y_{35}x_2 & 2y_{35}x_1 \\ y_{34}x_1 - y_{36}x_2 & 2y_{36}x_1 \end{pmatrix}.$$

Combining the above and summarising steps 3B and 3C, we have

$$F_3^{(1,1)} = \begin{pmatrix} y_{33}x_1 + y_{35}x_2 \\ y_{34}x_1 + y_{36}x_2 \end{pmatrix}, \ G_3^{(2,0)} = \begin{pmatrix} -2y_{33}x_2 & y_{33}x_1 - y_{35}x_2 & 2y_{35}x_1 \\ -2y_{34}x_2 & y_{34}x_1 - y_{36}x_2 & 2y_{36}x_1 \end{pmatrix}.$$

STEP 3D: *Write* $y_{39}, y_{40}, y_{59}, \ldots, y_{64}$ *(the entries of* $F_4^{(1,1)}$ *and* $G_4^{(2,0)}$*) in terms of* $y_{33}, \ldots, y_{36}, x_3, x_4$.

This step is identical to STEPS 3B and 3C, only whenever i or j equals 3, we instead substitute 4. This gives

$$F_4^{(1,1)} = \begin{pmatrix} y_{33}x_3 + y_{35}x_4 \\ y_{34}x_3 + y_{36}x_4 \end{pmatrix}, \ G_4^{(2,0)} = \begin{pmatrix} -2y_{33}x_4 & y_{33}x_3 - y_{35}x_4 & 2y_{35}x_3 \\ -2y_{34}x_4 & y_{34}x_3 - y_{36}x_4 & 2y_{36}x_3 \end{pmatrix}.$$

STEP 3E: We complete STEP 3 by repeating the same argument used to conclude STEP 2. The matrix $W_{(2,1)}$, formed by concatenating $F_1^{(1,1)}, \ldots, F_4^{(1,1)}$, $G_1^{(2,0)}, \ldots, G_4^{(2,0)}$, is a 2×16 matrix where, due to STEPS 3A-3D, every term on the top row is a multiple of either y_{33} or y_{35} and every term on the bottom row is a multiple of either y_{34} or y_{36}. The argument at the end of STEP 2 now applies: the minor formed by the first two columns of $W_{(2,1)}$ must be full rank, otherwise it is possible to use the group action in such a way that an entire row of $W_{(2,1)}$ becomes zero, which contradicts the condition that $W_{(2,1)}$ must be full rank. Therefore, the group action allows us to change basis such that $(F_1^{(1,1)} \ F_2^{(1,1)})$ becomes the identity matrix. This forces $y_{33}, y_{36} \mapsto 1$ and $y_{34}, y_{35} \mapsto 0$, and the resulting change to the rest of $W_{(2,1)}$ yields the matrices $F_1^{(1,1)}, \ldots, F_4^{(1,1)}, G_1^{(2,0)}, \ldots, G_4^{(2,0)}$ in Figure 6.3 as required.

STEP 4: *The maps* $\bigwedge^2 \mathcal{W} \to \bigwedge^2 \otimes \mathcal{W} \to (\bigwedge^2 \mathcal{W})^{\otimes 2}$.

The final step is identical to STEP 1 because the matrices for the maps $\bigwedge^2 \mathcal{W} \to \bigwedge^2 \otimes \mathcal{W}$ coincide with those for $\mathcal{O}_Y \to \mathcal{W}$, and the relations involved are identical. This implies $G_i^{(2,1)} = G_i^{(1,0)}$ for $1 \le i \le 4$ as required (see Figure 6.3) and ensures that $W_{(2,2)}$ is full rank. It is routine to check that the relations

$G_i^{(2,1)}G_j^{(2,0)} - G_j^{(2,1)}G_i^{(2,0)} = 0$ hold. This completes the proof of the claim. $\square$

Remark 6.4. It should be noted that there is an alternative method of producing the distinguished matrices in Figure 6.3. Indeed, the forms of those matrices are not arbitrary and in fact correspond to the choice of maps defined in Proposition 4.28.

As observed in (4.8), for $1 \leq i \leq 4$ the basis u_i of V gives us a basis of sections z_{u_i} of $H^0(Y, \mathcal{W})$ which hereafter we simply denote z_i. Then for any point $y \in Y$ there exists an open set $U_{i,j} \subset Y$, $1 \leq i < j \leq 4$, such that $z_i(y), z_j(y)$ forms a basis of the fibre $\mathcal{W}_y$. However, we will simply reorder the basis elements if necessary as in (6.2) and assume that $y \in U_{1,2}$, thus we write $b_1 := z_1(y)$ and $b_2 := z_2(y)$ for the basis of $\mathcal{W}_y$. We then write $z_3(y) = x_1b_1 + x_2b_2$ and $z_4(y) = x_3b_1 + x_4b_2$. Now, the basis b_1, b_2 of $\mathcal{W}_y$ induces a basis on each fibre $(\mathbb{S}^\lambda\mathcal{W})_y = \mathbb{S}^\lambda\mathcal{W}_y$ for all $\lambda \in \mathrm{Young}(2,2)$ in the canonical way. For example, $\bigwedge^2 \mathcal{W}_y$ has basis $b_2 \wedge b_1$ and $\mathrm{Sym}^2 \mathcal{W}_y$ has basis b_1b_1, b_1b_2, b_2b_2.

Consider the maps $f_{u_i}^{(1,0)} \colon \mathcal{W} \to \mathrm{Sym}^2 \mathcal{W}$. Using Proposition 4.28 we can evaluate these maps on the fibres of the bundles at y. Then for all $1 \leq i \leq 4$ and $1 \leq j \leq 2$ we have

$$(f_{u_i}^{(1,0)})_y(b_j) = b_j z_i(y).$$

For $i = 1$ we have $(f_{u_1}^{(1,0)})_y(b_1) = b_1b_1$ and $(f_{u_1}^{(1,0)})_y(b_2) = b_1b_2$. In the ordered basis b_1b_1, b_1b_2, b_2b_2 of $\mathrm{Sym}^2 V$ mentioned above, this produces the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which is precisely the matrix $F_1^{(1,0)}$ as calculated above. Similarly, for $i = 3$ we have $(f_{u_3}^{(1,0)})_y(b_1) = x_1b_1b_1 + x_2b_1b_2$ and $(f_{u_3}^{(1,0)})_y(b_2) = x_1b_1b_2 + x_2b_2b_2$, yielding

$$\begin{pmatrix} x_1 & 0 \\ x_2 & x_1 \\ 0 & x_2 \end{pmatrix}$$

which is equal to $F_3^{(1,0)}$. All of the remaining matrices in Figure 6.3 may be calculated in the same way; we prove this in general in Remark 6.7. This demonstrates that the system of distinguished matrices we have constructed is precisely the θ-stable A-module of dimension vector $\mathbf{v}$ parametrised by y, and therefore we have described the image of the morphism f_E at the point y.

6.2 The induction step

Now suppose $n > 4$ and $Y = \mathrm{Gr}(n, 2)$. As before, begin by fixing a basis $\mathcal{B} = \{u_1, \dots, u_n\}$ of V and for all pairs $\lambda, \lambda + e_j \in \mathrm{Young}(n-2, 2)$ we obtain a basis of $\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{S}^\lambda \mathcal{W}, \mathbb{S}^{\lambda+e_j}\mathcal{W})$, namely $f^\lambda_{u_i}$ for $1 \le i \le n$ if $j = 1$, or $g^\lambda_{u_i}$ for $1 \le i \le n$ if $j = 2$; see Proposition 4.28. These provide the arrows in the tilting quiver. Now fix a point $w \in \mathcal{M}(E)$ and recall the two decompositions of w given in Notation 6.2: we have a matrix per arrow, denoted F^λ_i or G^λ_i as appropriate, and also the concatenation of these matrices whose corresponding arrows have head at the same vertex, denoted W_λ.

First consider $W_{(1,0)}$, the $2 \times n$ matrix with columns $F^{(0,0)}_1, \dots, F^{(0,0)}_n$ that by assumption must be full rank. Without loss of generality, assume that the first two columns are linearly independent (if they're not, simply re-index the basis). Using the group action we may change basis to make these columns into the 2×2 identity matrix and rename the remaining entries to give a general form of $W_{(1,0)}$ as follows:

$$W_{(1,0)} = \begin{pmatrix} 1 & 0 & x_1 & x_3 & \cdots & x_{2n-7} & x_{2n-5} \\ 0 & 1 & x_2 & x_4 & \cdots & x_{2n-6} & x_{2n-4} \end{pmatrix}.$$

Lemma 6.5. *All of the matrices F^λ_i and G^λ_i comprising the point $w \in \mathcal{M}(E)$ can be chosen to take a distinguished form, modulo the group action, with entries in polynomial terms of only $x_1, \dots, x_{2n-4}$ as described below.*

We prove this lemma by induction on $Y = \mathrm{Gr}(n, 2)$, where the base case $n = 4$ was completed in the previous section.

Induction hypothesis: Suppose that the result holds for $n-1$; then Lemma 6.5 holds for all matrices corresponding to arrows in the tilting quiver of $\mathrm{Gr}(n-1, 2)$, i.e. the sub-quiver S of Q' defined by

$$\begin{aligned} S_0 &:= \left\{ \lambda \in \mathbb{Z}^2 \mid n-3 \ge \lambda_1 \ge \lambda_2 \ge 0 \right\}, \\ S_1 &:= \left\{ a^{\lambda,i}_\rho \;\middle|\; \begin{array}{l} 1 \le \rho \le n-1 \\ i \in \{1,2\},\ \lambda, \lambda + e_i \in B_0 \\ \mathrm{t}(a^{\lambda,i}_\rho) = \lambda,\ \mathrm{h}(a^{\lambda,i}_\rho) = \lambda + e_i \end{array} \right\}. \end{aligned} \tag{6.5}$$

Remark 6.6. Observe that S is not a full sub-quiver of Q' as the arrows corresponding to $u_n \in \mathcal{B}$ are missing.

Before stating what the hypothesised forms of these matrices are, we make an important observation and establish some notation. Recall from Remark 6.3 that all vector bundles on the diagonal line $\lambda_1 = \lambda_2 + k$ in S have rank $k+1$. Consider two vertices $\lambda < \mu$ with $|\mu| = |\lambda| + 2$ and write $k_1 := \operatorname{rank}(\mathbb{S}^\lambda \mathcal{W})$ and $k_2 := \operatorname{rank}(\mathbb{S}^\mu \mathcal{W})$. Now consider another pair of vertices $\lambda' < \mu'$ with $|\mu'| = |\lambda'| + 2$, $\operatorname{rank}(\mathbb{S}^{\lambda'} \mathcal{W}) = k_1$ and $\operatorname{rank}(\mathbb{S}^{\mu'} \mathcal{W}) = k_2$. Then paths $\lambda \to \mu$ are in bijective correspondence with paths $\lambda' \to \mu'$ and the sizes of the matrices corresponding to arrows in these paths are the same. Moreover, by Theorem 5.10, the relations between these matrices correspond. We therefore suppose as part of the induction hypothesis that any two matrices corresponding to the same basis vector u_i and that have the same order are identical. For example, in Figure 6.3 compare the matrices for $\mathcal{O}_Y \to \mathcal{W}$ with $\bigwedge^2 \mathcal{W} \to \bigwedge^2 \mathcal{W} \otimes \mathcal{W}$ (2×1 matrices), and the matrices for $\mathcal{W} \to \bigwedge^2 \mathcal{W}$ with $\bigwedge^2 \mathcal{W} \otimes \mathcal{W} \to (\bigwedge^2 \mathcal{W})^{\otimes 2}$ (1×2 matrices). Hence, the following notation is well-defined for all $1 \leq i \leq n-1$:

$$\begin{aligned}
&\text{For all } 1 \leq k \leq n-3,\ F_i^{(k)} := F_i^\lambda \text{ for any } \lambda \in S_0 \text{ with } \operatorname{rank}(\mathbb{S}^\lambda \mathcal{W}) = k,\\
&\text{For all } 2 \leq k \leq n-2,\ G_i^{(k)} := G_i^\lambda \text{ for any } \lambda \in S_0 \text{ with } \operatorname{rank}(\mathbb{S}^\lambda \mathcal{W}) = k.
\end{aligned} \tag{6.6}$$

Hence every matrix corresponding to an arrow in $S_1 \subset Q_1'$ is of the form $F_i^{(k)}$ or $G_i^{(k)}$; observe that these are always $(k+1) \times k$ or $(k-1) \times k$ matrices respectively.

We now describe the matrices $F_i^{(k)}$ and $G_i^{(k)}$. For $1 \leq k \leq n-3$, $F_1^{(k)}$ is the $k \times k$ identity matrix with an extra row of zeroes at the bottom, while $F_2^{(k)}$ is the $k \times k$ identity matrix with an extra row of zeroes at the top.

$$F_1^{(k)} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad F_2^{(k)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

For $F_3^{(k)}$ we have

$$F_3^{(k)} = \begin{pmatrix} x_1 & 0 & 0 & \cdots & \cdots & 0 \\ x_2 & x_1 & 0 & \cdots & \cdots & 0 \\ 0 & x_2 & x_1 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & x_2 & x_1 \\ 0 & 0 & \cdots & \cdots & 0 & x_2 \end{pmatrix}$$

For $4 \leq i \leq n-1$, $F_i^{(k)}$ takes the same form as $F_3^{(k)}$ but with x_1, x_2 replaced by the entries of the i^{th} column of $W_{(1,0)}$. Observe that for $k = 1$ and $1 \leq i \leq 4$

we recover the matrices F_i^λ for $\lambda = (0,0)$ and $\lambda = (1,1)$; see (6.2).

Next, for $2 \le k \le n-2$, we have

$$G_1^{(k)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k-1 \end{pmatrix}, \quad G_2^{(k)} = \begin{pmatrix} -(k-1) & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -2 & 0 & 0 \\ 0 & \cdots & 0 & -1 & 0 \end{pmatrix},$$

and

$$G_3^{(k)} = \begin{pmatrix} -(k-1)x_2 & x_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & -(k-2)x_2 & 2x_1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & -2x_2 & (k-2)x_1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & -x_2 & (k-1)x_1 \end{pmatrix}.$$

In the same way as above, $G_i^{(k)}$ for $4 \le i \le n-1$ takes the same form as $G_3^{(k)}$ but with x_1, x_2 replaced by the entries of the i^{th} column of $W_{(1,0)}$. As above, for $k = 1$ and $1 \le i \le 4$ we recover the matrices G_i^λ for $\lambda = (1,0)$, see (6.3), and $\lambda = (2,1)$.

It is routine to check that all the hypothesised matrices satisfy the relations in Theorem 5.10 and when appropriately concatenated, provide full rank W_λ for each $\lambda \in S_0$ so long as $W_{(1,0)}$ is full rank.

Proof of Lemma 6.5: Recall the sub-quiver S of Q' from (6.5). It remains to show that all matrices corresponding to arrows in $Q'_1 \setminus S_1$ satisfy the conditions in Lemma 6.5, and in particular, that they also follow the pattern of matrices described in the induction hypothesis.

We first deal with arrows in Q'_1 corresponding to the basis vector u_n that have head at a vertex in S_0. For these matrices we make the observation that for any given $\lambda \in S_0$, the work done to find F_3^λ, G_3^λ is identical to the work required to find F_i^λ, G_i^λ for any $i > 3$. Indeed, see Step 3D in the base case where we noted that the working required to find F_4^λ and G_4^λ was identical to finding F_3^λ and G_3^λ for certain λ. We therefore extend our above definition of the matrices $F_i^{(k)}$ and $G_i^{(k)}$ to include $i = n$, and it is routine to check that these matrices satisfy all the required conditions of Lemma 6.5.

Next we deal with all of the remaining arrows that do not have head or tail at the bottom right corner vertex $(n-2, 0)$. All of the vector bundles situated at the head or tail of these arrows have rank less than or equal to $n-2$. Therefore, by the observation in the induction hypothesis (comparing relations between paths $\lambda \to \mu$ and $\lambda' \to \mu'$ between bundles of the same ranks), we may again simply extend the definitions of $F_i^{(k)}$ and $G_i^{(k)}$ in (6.6) to all $\lambda \in Q_0'$ rather than only the proper subset S_0.

It remains to check the arrows with head or tail at the vertex $(n-2,0)$, i.e. the arrows between $\mathrm{Sym}^{n-3}\mathcal{W}$, $\mathrm{Sym}^{n-2}\mathcal{W}$ and $\bigwedge^2\mathcal{W}\otimes\mathrm{Sym}^{n-3}\mathcal{W}$ in the lower right corner of the tilting quiver as shown in Figure 6.5. This forms the rest of Section 6.2. For consistency we will denote the matrices corresponding to these arrows by $F_i^{(n-2)} := F_i^\lambda$ for $\lambda = (n-3,0)$ and $G_i^{(n-1)} := G_i^\lambda$ for $\lambda = (n-2,0)$.

To show that Lemma 6.5 holds for $F_i^{(n-2)}, G_i^{(n-2)}, 1 \le i \le n$, we will take inspiration from Step 2 and Step 3 of the base case.

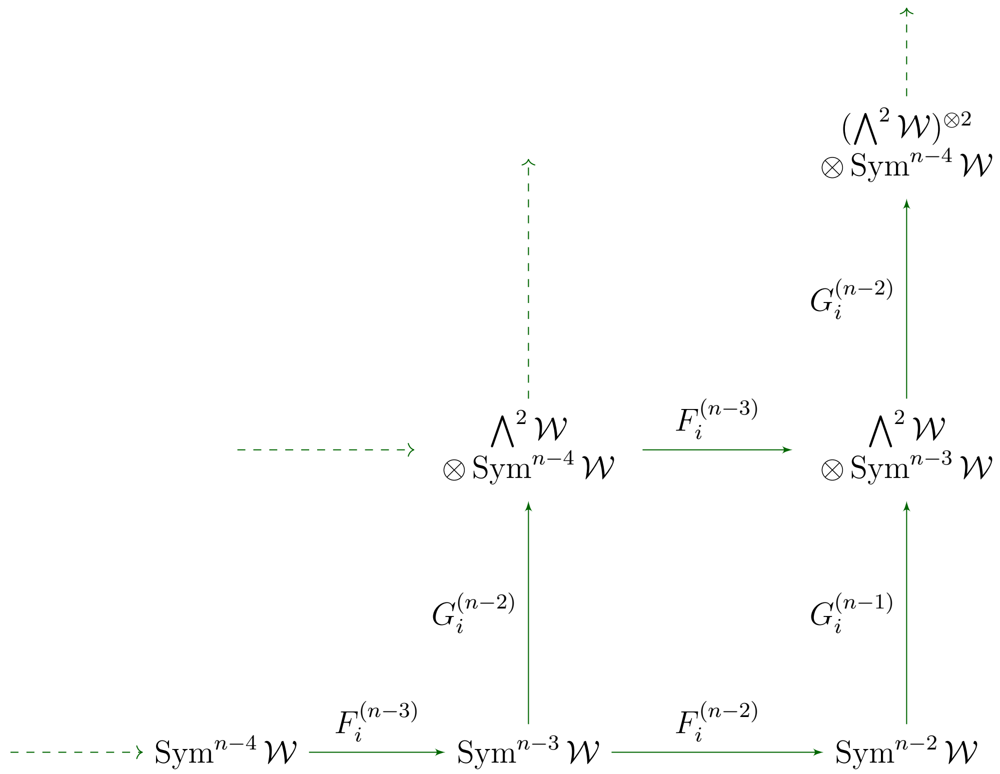

Figure 6.5: The lower right corner of the tilting quiver for $\mathrm{Gr}(n,2)$.

Step 1: *Show that Lemma 6.5 holds for $F_i^{(n-2)}$, $1 \le i \le n$.*
To find the $F_i^{(n-2)}$ we make use of the relations

$$F_i^{(n-2)}F_j^{(n-3)} = F_j^{(n-2)}F_i^{(n-3)},\ 1 \le i,j \le n.$$

For $1 \le u \le n-1$ and $1 \le v \le n-2$, denote the (u,v)-th entry of $F_i^{(n-2)}$ for

$i = 1, 2, 3$ as follows:

$$(F_1^{(n-2)})_{u,v} := a_{u,v},\ (F_2^{(n-2)})_{u,v} := b_{u,v},\ (F_3^{(n-2)})_{u,v} := c_{u,v}.$$

First we study $i = 1$ and $j = 2$. Since the $F_1^{(n-3)}$ and $F_2^{(n-3)}$ are just $(n-3) \times (n-3)$ identity matrices augmented by a row of zeroes at the bottom and top respectively, the relation implies that $\mathrm{col}_k(F_2^{(n-2)}) = \mathrm{col}_{k+1}(F_1^{(n-2)})$ for all $1 \leq k \leq n-3$, i.e.

$$b_{u,v} = a_{u,v+1},\ 1 \leq v \leq n-3, \tag{6.7}$$

thus $F_2^{(n-2)}$ is entirely determined by $F_1^{(n-2)}$ apart from its final column.

Next we set $i = 3$ and $j = 1$. The relation is $F_3^{(n-2)} F_1^{(n-3)} = F_1^{(n-2)} F_3^{(n-3)}$, and we analyse each side separately. The left hand side is more straightforward as multiplying by $F_1^{(n-3)}$ simply turns the last column of $F_3^{(n-2)}$ into zeroes while leaving the rest unaltered, i.e.

$$(F_3^{(n-2)} F_1^{(n-3)})_{u,v} = \begin{cases} 0 & \text{if } v = n-2, \\ c_{u,v} & \text{otherwise.} \end{cases}$$

For the right hand side recall that $F_3^{(n-3)}$ is the $(n-2) \times (n-3)$ matrix

$$F_3^{(n-3)} = \begin{pmatrix} x_1 & 0 & 0 & \cdots & \cdots & 0 \\ x_2 & x_1 & 0 & \cdots & \cdots & 0 \\ 0 & x_2 & x_1 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & x_2 & x_1 \\ 0 & 0 & \cdots & \cdots & 0 & x_2 \end{pmatrix}.$$

Left-multiplying by $F_1^{(n-2)}$ yields

$$(F_1^{(n-2)} F_3^{(n-3)})_{u,v} = \begin{cases} 0 & \text{if } v = n-2, \\ a_{u,v} x_1 + a_{u,v+1} x_2 & \text{otherwise,} \end{cases}$$

and comparing with the above we have

$$c_{u,v} = a_{u,v} x_1 + a_{u,v+1} x_2,\ 1 \leq u \leq n-1,\ 1 \leq v \leq n-3.$$

By repeating the above with $i = 3$ and $j = 2$ we mostly get information about

$F_3^{(n-2)}$ that we already have since $F_2^{(n-3)}$ is also an identity matrix with an extra row of zeroes (this time at the top rather than the bottom), and $F_2^{(n-2)}$ is largely determined by $F_1^{(n-2)}$ by (6.7). Importantly however, we do pick up the final column of $F_3^{(n-2)}$ in this calculation which is given by

$$c_{u,n-2} = a_{u,n-2}x_1 + b_{u,n-2}x_2.$$

Combining this with the above, we can now write $F_3^{(n-2)}$ in terms of only the entries of $F_1^{(n-2)}, F_2^{(n-2)}$ and x_1, x_2 as follows:

$$(F_3^{(n-2)})_{u,v} = c_{u,v} = \begin{cases} a_{u,v}x_1 + a_{u,v+1}x_2 & \text{if } 1 \leq v \leq n-3, \\ a_{u,n-2}x_1 + b_{u,n-2}x_2 & \text{if } v = n-2. \end{cases} \tag{6.8}$$

We find a similar set of equations for $F_i^{(n-2)}$, $4 \leq i \leq n$; simply replace x_1, x_2 in (6.8) with the i-th column of $W_{(1,0)}$.

To finish Step 1 we make the same observation as at the end of Step 2 of the base case. Consider the matrix $W_{(n-2,0)}$, formed by concatenating the $F_i^{(n-2)}$, and suppose for contradiction that the $(n-1) \times (n-1)$ minor formed by taking $F_1^{(n-2)}$ and the final column of $F_2^{(n-2)}$ is not full rank. Then it is possible to use the group action (specifically, $\mathrm{GL}(n-1)$ acting at the vertex $\mathrm{Sym}^{n-2}\mathcal{W}$) to produce a row of zeros in this minor; suppose for example that this is the top row (the argument for the other rows is similar). The effect this has on the rest of $W_{(n-2,0)}$ is that the entire top row becomes zero. This contradicts the stability condition that $W_{(n-2,0)}$ must be full rank, thus we conclude that the chosen minor must be full rank. As a result, we may use the group action to change the chosen minor into the identity matrix. The resulting change to $W_{(n-2,0)}$ is that the $F_i^{(n-2)}$ take precisely the required forms as in the induction hypothesis, and so Lemma 6.5 holds for these matrices. Moreover, $F_1^{(n-2)}$ and $F_2^{(n-2)}$ are $(n-2) \times (n-2)$ identity matrices augmented by a row of zeroes at the bottom and top respectively, and for $i > 3$, $F_i^{(n-2)}$ takes the required forms similar to $F_i^{(n-3)}$ in the induction hypothesis. This completes Step 1.

Step 2: *Show that Lemma 6.5 holds for* $G_i^{(n-1)}$, $1 \leq i \leq n$.

This step is slightly simpler than Step 3 of the base case since it remains only to prove that the $G_i^{(n-1)}$ take the required forms. By Theorem 5.10(iv), across the lower right corner square we have the relations

$$(n-3)G_i^{(n-1)}F_j^{(n-2)} = (n-2)F_j^{(n-3)}G_i^{(n-2)} - F_i^{(n-3)}G_j^{(n-2)}, \; 1 \leq i, j \leq n. \tag{6.9}$$

We will refresh notation from Step 1 and for $1 \leq u \leq n-2$, $1 \leq v \leq n-1$, denote the (u,v)-th entry of $G_i^{(n-1)}$ for $i = 1,2,3$ as follows:

$$(G_1^{(n-1)})_{u,v} := a_{u,v},\ (G_2^{(n-1)})_{u,v} := b_{u,v},\ (G_3^{(n-1)})_{u,v} := c_{u,v}.$$

We first consider the cases when $i, j \in \{1,2\}$. Note that when $i = j$, (6.9) simplifies to $G_i^{(n-1)} F_i^{(n-2)} = F_i^{(n-3)} G_i^{(n-2)}$. Recall that $F_i^{(n-3)}, F_i^{(n-2)}$ for $i = 1,2$ are identity matrices augmented by a row of zeroes at the bottom and top respectively, and that

$$G_1^{(n-2)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n-3 \end{pmatrix}, \qquad G_2^{(n-2)} = \begin{pmatrix} -(n-3) & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -2 & 0 & 0 \\ 0 & \cdots & 0 & -1 & 0 \end{pmatrix}.$$

When $i = 1 = j$ we have $G_1^{(n-1)} F_1^{(n-2)} = F_1^{(n-3)} G_1^{(n-2)}$. The left hand side is equal to $G_1^{(n-1)}$ with the final column removed, and the right hand side is equal to $G_1^{(n-2)}$ augmented by a row of zeroes at the bottom. Comparing both sides entry-wise yields

$$G_1^{(n-1)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & a_{1,n-1} \\ 0 & 0 & 2 & \cdots & 0 & a_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-3 & a_{n-3,n-1} \\ 0 & 0 & 0 & \cdots & 0 & a_{n-2,n-1} \end{pmatrix}.$$

Repeating for $i = 2 = j$ yields

$$G_2^{(n-1)} = \begin{pmatrix} b_{1,1} & 0 & \cdots & 0 & 0 & 0 \\ b_{2,1} & -(n-3) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ b_{n-3,1} & 0 & \cdots & -2 & 0 & 0 \\ b_{n-2,1} & 0 & \cdots & 0 & -1 & 0 \end{pmatrix}.$$

To find the remaining entries of $G_1^{(n-1)}$ and $G_2^{(n-1)}$ we use (6.9) with $i = 1$ and $j = 2$. This reads

$$(n-3) G_1^{(n-1)} F_2^{(n-2)} = (n-2) F_2^{(n-3)} G_1^{(n-2)} - F_1^{(n-3)} G_2^{(n-2)},$$

which becomes

$$(n-3)\begin{pmatrix} 1 & 0 & \cdots & 0 & a_{1,n-1} \\ 0 & 2 & \cdots & 0 & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & n-3 & a_{n-3,n-1} \\ 0 & 0 & \cdots & 0 & a_{n-2,n-1} \end{pmatrix} =$$

$$(n-2)\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n-3 \end{pmatrix} - \begin{pmatrix} -(n-3) & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -2 & 0 & 0 \\ 0 & \cdots & 0 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix},$$

and so we get $a_{u,n-1} = 0$ for $1 \leq u \leq n-3$ and $a_{n-2,n-1} = n-2$. We now repeat the above with $i = 2$ and $j = 1$ to get a similar equation and ultimately discover that the first column of $G_2^{(n-1)}$ satisfies $b_{u,1} = 0$ for $2 \leq u \leq n-2$ and $b_{1,1} = -(n-2)$. We thus have the required forms for $G_1^{(n-1)}$ and $G_2^{(n-1)}$ as shown below:

$$G_1^{(n-1)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n-2 \end{pmatrix}, \qquad G_2^{(n-1)} = \begin{pmatrix} -(n-2) & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -2 & 0 & 0 \\ 0 & \cdots & 0 & -1 & 0 \end{pmatrix}.$$

It remains to investigate $G_3^{(n-1)}$ (as usual the process of finding $G_i^{(n-1)}$ for $4 \leq i \leq n$ will be identical). With $i = 3$ and $j = 1$, equation (6.9) becomes

$$(n-3)G_3^{(n-1)}F_1^{(n-2)} = (n-2)F_1^{(n-3)}G_3^{(n-2)} - F_3^{(n-3)}G_1^{(n-2)}$$

and so we have

$$(n-3)\begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & \cdots & c_{1,n-2} \\ c_{2,1} & c_{2,2} & \cdots & \cdots & c_{2,n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_{n-2,1} & \cdots & \cdots & \cdots & c_{n-2,n-2} \end{pmatrix} =$$

$$(n-2)\begin{pmatrix} -(n-3)x_2 & x_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & -(n-4)x_2 & 2x_1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & -2x_2 & (n-4)x_1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & -x_2 & (n-3)x_1 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \end{pmatrix}$$

$$-\begin{pmatrix} 0 & x_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & x_2 & 2x_1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 2x_2 & 3x_1 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & (n-4)x_2 & (n-3)x_1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & (n-3)x_2 \end{pmatrix}.$$

This gives us most of $G_3^{(n-1)}$. For $1 \le u, v \le n-2$, we have

$$c_{u,v} = \begin{cases} -(n-2-u+1)x_2 & \text{if } u = v, \\ ux_1 & \text{if } 1 \le u \le n-3, v = u+1, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we must find the last column $c_{u,n-1}$. We repeat the above with $i = 3$ and $j = 2$ and, similar to previous steps, pick up largely the same set of equations but with the final column included. In particular,

$$c_{u,n-1} = \begin{cases} (n-2)x_1 & \text{if } u = n-2, \\ 0 & \text{otherwise.} \end{cases}$$

In conclusion, we have

$$G_3^{(n-1)} = \begin{pmatrix} -(n-2)x_2 & x_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & -(n-3)x_2 & 2x_1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & -2x_2 & (n-3)x_1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & -x_2 & (n-2)x_1 \end{pmatrix},$$

which is precisely the form required by the induction hypothesis. As mentioned above, the proof is identical to show that $G_i^{(n-1)}$ for $4 \le i \le n$ takes the same form as $G_3^{(n-1)}$ except that x_1, x_2 are replaced by the entries of the i-th column of $W_{(1,0)}$. Hence, we have shown that each $G_i^{(n-1)}$ is of the form defined in the induction hypothesis. This completes STEP 2 and the proof of Lemma 6.5. □

Remark 6.7. We now prove the claim in Remark 6.4 that the matrices F_i^k, G_i^k may be deduced from the choices of maps in Proposition 4.28. Recall that for $\lambda \in \mathrm{Young}(n-2,2)$ we have $\mathbb{S}^\lambda \mathcal{W} = (\bigwedge^2 \mathcal{W})^{\otimes \lambda_2} \otimes \mathrm{Sym}^{\lambda_1 - \lambda_2} \mathcal{W}$, and following the notation of Remark 6.4 for each $y \in Y$ we have a basis of the fibre $\mathbb{S}^\lambda \mathcal{W}_y$ given by

$$p_j := (b_2 \wedge b_1)^{\otimes \lambda_2} \otimes b_1^{\lambda_1 - \lambda_2 - j} b_2^j \ , \ 0 \le j \le \lambda_1 - \lambda_2.$$

It suffices to prove the result for only $i = 3$ where $z_{u_3}(y) = x_1 b_1 + x_2 b_2$; for other values of i we simply substitute the appropriate column of $W_{(1,0)}$.

First fix λ such that $\lambda + e_1 \in \mathrm{Young}(n-2,2)$. We will calculate F_3^k where $k = \lambda_1 - \lambda_2 + 1$. The induced basis of $\mathbb{S}^{\lambda + e_1} \mathcal{W}_y$ is given by

$$q_j := (b_2 \wedge b_1)^{\otimes \lambda_2} \otimes b_1^{\lambda_1 - \lambda_2 + 1 - j} b_2^j \ , \ 0 \le j \le \lambda_1 - \lambda_2 + 1$$

Then for all $0 \le j \le \lambda_1 - \lambda_2$, we have

$$\begin{aligned} (f_{u_3}^\lambda)_y(p_j) &= x_1 \, (b_2 \wedge b_1)^{\otimes \lambda_2} \otimes b_1^{\lambda_1 - \lambda_2 - j + 1} b_2^j + x_2 \, (b_2 \wedge b_1)^{\otimes \lambda_2} \otimes b_1^{\lambda_1 - \lambda_2 - j} b_2^{j+1} \\ &= x_1 q_j + x_2 q_{j+1} \end{aligned}$$

which yields the matrix F_3^k as required.

Now fix λ such that $\lambda + e_2 \in \mathrm{Young}(n-2,2)$. We will calculate G_3^k where $k = \lambda_1 - \lambda_2 + 1$. The induced basis of $\mathbb{S}^{\lambda + e_2} \mathcal{W}_y$ is given by

$$s_j := (b_2 \wedge b_1)^{\otimes \lambda_2 + 1} \otimes b_1^{\lambda_1 - \lambda_2 - 1 - j} b_2^j \ , \ 0 \le j \le \lambda_1 - \lambda_2 - 1$$

Then for all $0 \leq j \leq \lambda_1 - \lambda_2$, we have

$$\begin{aligned}
(g_{u_3}^{\lambda})_y(p_j) &= jx_1(b_2 \wedge b_1)^{\otimes \lambda_2+1} \otimes b_1^{\lambda_1-\lambda_2-j} b_2^{j-1} \\
&\quad - (\lambda_1 - \lambda_2 - j)x_2(b_2 \wedge b_1)^{\otimes \lambda_2+1} \otimes b_1^{\lambda_1-\lambda_2-j-1} b_2^{j} \\
&= \begin{cases} -x_2(k-1)s_0 & \text{if } j = 0 \\ jx_1 s_{j-1} - (k-1-j)x_2 s_j & \text{if } 1 \leq j \leq \lambda_1 - \lambda_2 - 1 \\ x_1(k-1)s_{\lambda_1-\lambda_2-1} & \text{if } j = \lambda_1 - \lambda_2 \end{cases}
\end{aligned}$$

which yields the matrix G_3^k as required.

6.3 Proof of Theorem 6.1

Given the closed immersion $f_E : Y \to \mathcal{M}(E)$, we now construct a morphism $f' \colon \mathcal{M}(E) \to Y$ satisfying $f' \circ f_E = \mathrm{id}_Y$ and $f_E \circ f' = \mathrm{id}_{\mathcal{M}(E)}$, thereby proving that f_E is an isomorphism.

As a fine moduli space, the multigraded linear series $\mathcal{M}(E) = \mathcal{M}(A, \mathbf{v}, \theta)$ carries a tautological bundle $\mathcal{V} := \bigoplus_{i \in Q'} \mathcal{V}_i$ where each $\mathcal{V}_i$ is globally generated by [CIK18, Corollary 2.4] and satisfies $\mathrm{rank}(\mathcal{V}_i) = \mathbf{v}_i$. Since $\mathcal{M}(E)$ is also the space of isomorphism classes of representations of the tilting quiver Q' with dimension vector $\mathbf{v}$ subject to the relations in Theorem 5.10, it is therefore a subvariety of the quiver flag variety X formed using the same quiver and dimension vector but with no relations. Write $\mathcal{V}'_{(1,0)}$ for the tautological bundle on X at the vertex $(1,0)$. By Proposition 2.4(iii), the n arrows $(0,0) \to (1,0)$ in Q' imply that $\dim(H^0(X, \mathcal{V}'_{(1,0)})) = n$. Since $\mathcal{V}_{(1,0)}$ is the restriction of $\mathcal{V}'_{(1,0)}$ to $\mathcal{M}(E)$ and there are no relations amongst paths $(0,0) \to (1,0)$, we therefore also have $\dim(H^0(\mathcal{M}(E), \mathcal{V}_{(1,0)})) = n$.

Now consider the sub-bundle $E' := \mathcal{V}_{(0,0)} \oplus \mathcal{V}_{(1,0)}$ of E, where $\mathcal{V}_{(0,0)} = \mathcal{O}_{\mathcal{M}(E)}$ and $\mathcal{V}_{(1,0)}$ is globally generated. Then by [CIK18, Theorem 2.6] there is a morphism

$$f_{E'} \colon \mathcal{M}(E) \longrightarrow \mathcal{M}(E'),$$

where $\mathcal{M}(E')$ is the multigraded linear series of E'. The bundles $\mathcal{V}_{(0,0)}$ and $\mathcal{V}_{(1,0)}$ have ranks 1 and 2 respectively and we have $\dim(H^0(\mathcal{M}(E), \mathcal{V}_{(1,0)})) = n$ from above. Hence, the quiver for $\mathcal{M}(E')$ has only two vertices with n arrows between them and no relations. Following Example 2.5(ii), $\mathcal{M}(E')$ is therefore isomorphic to the Grassmannian $\mathrm{Gr}(n, 2) = Y$, and so in fact we have constructed a morphism

$$f_{E'} \colon \mathcal{M}(E) \longrightarrow Y.$$

Using the content of Sections 6.1 and 6.2, it remains to show that $f_{E'} \circ f_E = \mathrm{id}_Y$ and $f_E \circ f_{E'} = \mathrm{id}_{\mathcal{M}(E)}$.

Now, as observed in Remarks 6.4 and 6.7, f_E takes a point $y \in Y$ to the θ-stable A-module of dimension vector $\mathbf{v}$ parametrised by y, i.e. to the fibre of the bundle E over Y, and this image $f_E(y)$ is described by the distinguished matrices defined in the induction hypothesis for Lemma 6.5. See Figure 6.3 when $n = 4$, for example. Since y provides the data for the matrices corresponding to the arrows $(0, 0) \to (1, 0)$ and $f_{E'}$ is simply the projection back onto these arrows, we have $f_{E'} \circ f_E = \mathrm{id}_Y$.

Now suppose w is an arbitrary point of $\mathcal{M}(E)$. Then Lemma 6.5 implies that w is equivalent modulo the group action to a distinguished point $w' \in \mathcal{M}(E)$, where every entry of each matrix comprising w' is a polynomial in the entries of the matrix $W_{(1,0)}$. As a matrix, the point $y' = f_{E'}(w') \in Y$ is equal to $W_{(1,0)}$, and so applying f_E to y' simply reconstructs w' in the same way as described above. Thus $f_E(f_{E'}([w])) = f_E(f_{E'}([w'])) = [w'] = [w]$, and therefore $f_E \circ f_{E'} = \mathrm{id}_{\mathcal{M}(E)}$ as required. This completes the proof of Theorem 6.1. $\square$

Chapter 7

Future directions

As a result of Theorem 6.1 it is natural to conjecture the following.

Conjecture 7.1. *For any* $1 \leq r < n$ *let* $Y = \mathrm{Gr}(n, r)$. *Then the morphism* $f_E : Y \to \mathcal{M}(E)$ *is an isomorphism.*

To prove this using the methods in this thesis requires two main steps: firstly, using Theorem 5.14 we must write down the ideal of relations for $\Bbbk Q'$ explicitly, and secondly, take a similar approach to the proof in Chapter 6 to get the result.

7.1 Describing the ideal of relations for the tilting quiver of $\mathrm{Gr}(n, r)$

While the strategy has been roughly laid out, actually writing down generators in general for the ideal of relations J from Theorem 5.14 poses a far greater combinatorial challenge than the $r = 2$ case. As alluded to in Section 5.3, we must first write down a compatible Pieri system for the tilting quiver. In other words, we must define a system of well-defined maps $\mathbb{S}^{\lambda}\mathcal{W} \to \mathbb{S}^{\lambda+e_i}\mathcal{W}$ for all $\lambda \in \mathrm{Young}(n - r, r)$ and $1 \leq i \leq r$, where $\mathcal{W}$ is the rank r tautological quotient bundle on $\mathrm{Gr}(n, r)$. Proposition 4.28 covers the $i = 1, 2$ cases. For $i \geq 3$, more complicated exchange relations on the Young diagram λ (see Definition 4.2) need to be considered. One example is $\mathbb{S}^{(2,2,0)}\mathcal{W} \to \mathbb{S}^{(2,2,1)}\mathcal{W}$ where $v \in V$ and we have

$$\begin{aligned} x_1 \wedge x_2 \otimes y_1 \wedge y_2 \mapsto\ & 2x_1 \wedge x_2 \wedge z_v \otimes y_1 \wedge y_2 \\ & + 2y_1 \wedge y_2 \wedge z_v \otimes x_1 \wedge x_2 \\ & + x_1 \wedge y_1 \wedge z_v \otimes x_2 \wedge y_2 \\ & + y_1 \wedge x_2 \wedge z_v \otimes x_1 \wedge y_2 \\ & + x_1 \wedge y_2 \wedge z_v \otimes y_1 \wedge x_2 \\ & + y_2 \wedge x_2 \wedge z_v \otimes y_1 \wedge x_1. \end{aligned}$$

To complete this task in general, Buchweitz, Leuschke and Van den Bergh mention that Olver was the first to write down such a Pieri system in the preprint [Olv82], though it is unclear to what degree this is accomplished. Since then there has been more potentially helpful work that has considered these maps (or similar ones); see [ABW82], [MO92], [SW11], [Sam09]. The last reference describes a package called *PieriMaps* written for the Macaulay2 software by the author of the paper.

If a Pieri system can be written down in as simple a way as possible, then by observing the spaces calculated in Proposition 5.16 it is now a matter of composing these maps as appropriate to find explicit generators of J. While it is likely that in the cases where these relations span either $\mathrm{Sym}^2 V$ or $\bigwedge^2 V$ we have the usual bases of these spaces as generators, the hard part is calculating the relations around squares in the tilting quiver, i.e. cases when the relations span $V \otimes V$ and non-trivial linear combinations appear; see for example Section 5.1.4.

7.2 Reconstructing quiver flag varieties from a tilting bundle

Despite the description of $A = \mathrm{End}_{\mathcal{O}_Y}(E)$ for $Y = \mathrm{Gr}(n, r)$ in [BLV16], the problems detailed above mean that we do not currently have explicit generators for the ideal of relations $J \subset \Bbbk Q'$, and therefore little can be said about a potential proof of Theorem 6.1 in the general case. We do however suspect that such a proof, while combinatorially unpleasant, would be quite similar to the $\mathrm{Gr}(n, 2)$ case.

The methods used in this thesis are very hands-on. An alternative approach altogether would be to recall Remark 2.12: the results of Bergman-Proudfoot imply that f_E identifies Y with a connected component of $\mathcal{M}(E)$, because Y is smooth, E is a tilting bundle, and our stability condition θ is great; see [BP08, Theorem 2.4]. A proof of Conjecture 7.1 would therefore follow from showing that for $Y = \mathrm{Gr}(n, r)$, the moduli space $\mathcal{M}(E)$ is connected. In fact, a successful implication of this approach may even lead to generalising the result further to all quiver flag varieties:

Conjecture 7.2. *Let Y be any quiver flag variety and E the tilting bundle from Theorem 2.9. Then the morphism $f_E : Y \to \mathcal{M}(E)$ is an isomorphism.*

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