Young Tableaux

Harry Rainbird (supervised by Alastair Craw)

April 4, 2019

Abstract

In this project we show how the combinatorics of Young tableaux can be applied to a wide variety of mathematical objects. These applications include constructing the irreducible representations of the symmetric group and all irreducible finite-dimensional holomorphic representations of the general linear group for finite-dimensional complex vector spaces. Moreover, the underlying use of Young tableaux in these different areas allows us to relate these constructions in powerful ways. This includes identifying representations of the symmetric group with a ring of polynomials and we also obtain a similar identification for representations of the general linear group.

In addition to applications in representation theory, some of the techniques derived in constructing irreducible representations can be used to express flag varieties as subvarieties in projective space. In particular, we explore how the quadratic relations, which arise naturally in tableaux combinatorics, can be used to construct flag varieties.

Contents

0 Introduction **3**

1 Young Tableau **4**
1.1 Nomenclature 4
1.2 Tableaux Ordering 4

2 Schur Polynomials **5**
2.1 The Tableaux Ring 6
2.2 Schur Polynomials 7
2.3 Ring of Symmetric Functions 10

3 Representations of the Symmetric Group **11**
3.1 Specht Module Construction 11
3.2 Irreducible Representations of the Symmetric Group 13
3.3 M^λ and S^λ Structure 16
3.4 Ring of Representations 17
3.5 Dual Construction of Specht Modules 19
3.6 Alternative Construction of Specht Modules 22

4 Representations of the General Linear Group **23**
4.1 Schur Module Construction 23
4.2 Schur Module Generators 25
4.3 Irreducible Representations of $\mathrm{GL}(E)$ 27
4.4 Specht and Schur Module Relationship 30
4.5 Characters of Representations of $\mathrm{GL}(E)$ 32
4.6 Representation Ring of $\mathrm{GL}(E)$ 33

5 The Grassmannian **34**
5.1 The Ideal of Quadratic Relations 34
5.2 Projective Embedding of the Grassmannian 36
5.3 The Grassmannian as a Subvariety of Projective Space 38

0 Introduction

The aim of this project is to explore how the combinatorics of Young tableaux can be applied to polynomial rings, representation theory and algebraic geometry. We begin with Chapter 1 where we introduce the basic definitions and notation used for Young tableaux. We also discuss the orderings on tableaux that will be required in other sections.

Section 2 discusses how to define the product of two tableaux and as a result introduces the *tableau ring*. We then define a homomorphism from the tableau ring to the polynomial ring and, in particular, explain how Schur polynomials can be constructed using tableau. The structure of the tableaux ring can then be used to deduce properties about symmetric polynomials. For example, we use the Pieri formulas and the Littlewood-Richardson rule to deduce analogue results for symmetric polynomials. The rest of Section 2 defines the ring of symmetric functions and proves that it has basis given by Schur polynomials. The exploration of symmetric functions allows us to easily prove properties of representations of the symmetric group and the general linear group in Section 3 and Section 4.

Section 3 begins by constructing representations of the symmetric group using Young tableaux. We then define Specht modules and prove that they define all irreducible representations of the symmetric group up to isomorphism. After this, we construct an isomorphism between the ring of symmetric functions, discussed in Section 2, and the ring of representations of the symmetric group. Thereafter, lots of the results of Section 2 can be transferred easily to their analogues in the ring of representations of the symmetric group. The rest of Section 3 is devoted to building two alternative constructions of Specht modules. The first of which allows us to view Specht modules as a quotient space and second of which will add to our intuition of Specht modules. The relations that define the quotient space above are called the quadratic relations and are of fundamental importance in the combinatorics of tableaux.

Section 4 uses tableaux to construct irreducible representations of the general linear group for a finite-dimensional complex vector space, called Schur modules. We then discuss the structure of Schur modules, weight space decompositions of representations and prove that Schur modules define all irreducible finite-dimensional holomorphic representations of the general linear group up to isomorphism. We then look into the relationship between Schur modules and Specht modules and construct an exact functor between the category of S_n-modules and the category of $\mathrm{GL}(E)$-modules, where E is a finite-dimensional complex vector space. Section 4 ends by exploring and combining the relationships between the ring of symmetric polynomials, the ring of representations of symmetric functions, the representation ring of the general linear group and characters of representations of the general linear group.

Section 5 starts by using the relationships given at the end of Section 4 to generalise the use of the quadratic relations to the symmetric algebra. We then define the Grassmannian and consider its embedding, called the Plücker embedding, into projective space. The rest of the section expresses the Plücker embedding as a subvariety of projective space using the quadratic relations and goes on to discuss that we have similar results for flag varieties.

The majority of this project has been adapted from the textbook Young Tableaux [3, Fulton]. Content that has not been adapted from Young Tableaux has been cited at the location of its appearance in the text and sections that are entirely original are marked with the symbol $\diamond$.

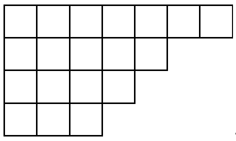
1 Young Tableau

1.1 Nomenclature

We begin by introducing some essential nomenclature.

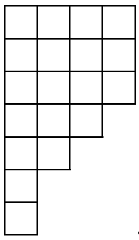
Definition 1.1.1 (Young Diagram)**.** A Young diagram is a collection of boxes arranged so that their rows are left-aligned and there is a (weakly) decreasing number of boxes in each row.

Counting the number of boxes in each row of a Young diagram corresponds to a partition of the total number of boxes n. For example, with $n = 19$ the partition $(7, 5, 4, 3)$ corresponds to the Young diagram



.

If λ is a partition of n we write $\lambda \vdash n$ and $|\lambda| = n$. We can also define the *conjugate* of a partition, denoted by $\tilde{\lambda}$, where $\tilde{\lambda}$ is obtained by flipping a diagram over its main diagonal. For example, the conjugate of the above Young diagram is



.

Now the purpose of Young diagrams it to put stuff in the boxes. As such we define a *filling* of a Young diagram to be any way of putting a positive integer in each box. Then when the entries are distinct we refer to a filling as a *numbering.*

Definition 1.1.2 (Young tableau)**.** A Young tableau is a filling that is

(1) weakly increasing across each row;

(2) strictly increasing down each column.

We say that λ is the *shape* of the tableau for some partition λ. A *standard tableau* is a Young tableau in which the entries are $1, \ldots, n$, each occurring once.

◇ **Example 1.1.3.** On the shape $\lambda = (3, 3, 2, 1)$ consider

$$\begin{array}{|c|c|c|}\hline 6&7&6\\\hline 3&4&2\\\hline 3&5\\\cline{1-2} 1\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|}\hline 9&8&7\\\hline 6&5&4\\\hline 3&2\\\cline{1-2} 1\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&2\\\hline 2&3&4\\\hline 4&5\\\cline{1-2} 5\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&7\\\hline 3&6&8\\\hline 4&9\\\cline{1-2} 5\\\cline{1-1}\end{array}\;.$$

These are a filling, numbering, tableau and standard tableau (respectively).

1.2 Tableaux Ordering

We can define orderings on tableaux in a number of different ways. This is because an ordering can be defined using the shape, the numbering or both the shape and the numbering.

Definition 1.2.1 (Dominance ordering)**.** Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_\ell)$ be partitions of n. Then we say that λ dominates μ, denoted by $\mu \trianglelefteq \lambda$, if

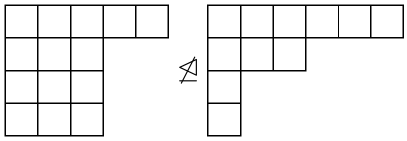
$$\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \tag{1.1}$$

for all i. Similarly, we say that λ strictly dominates μ, denoted by $\mu \triangleleft \lambda$, if

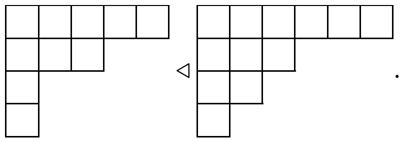
$$\mu_1 + \cdots + \mu_i < \lambda_1 + \cdots + \lambda_i \tag{1.2}$$

for all i.

◇ **Example 1.2.2.** We have that

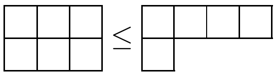


since equation (1.1) holds for the first and second row, but not the third and fourth row. However, changing the number of boxes in the third and fourth rows of the two diagrams yields

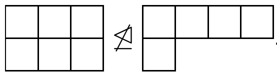


Definition 1.2.3 (Lexicographic ordering)**.** Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_\ell)$ be partitions of n. The lexicographic ordering is defined so that if the first i for which $\mu_i \neq \lambda_i$, if any, has $\mu_i < \lambda_i$, then we write $\mu \leq \lambda$.

Remark 1.2.4. Note that $\mu \trianglelefteq \lambda$ implies $\mu \leq \lambda$. However, the implication cannot be reversed. For example,



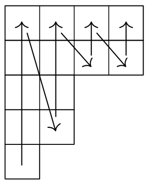
and



The lexicographic ordering can be used to define an ordering on numberings of n boxes with distinct entries in $\{1, \ldots, n\}$. We say that $T' < T$ if either

(1) the shape of T is larger in the lexicographic ordering; or

(2) T and T' have the same shape and the largest entry that is in a different box in the two numberings occurs earlier in T than in T', whereby we list the entries from the bottom to top in each column starting in the left column and moving to the right.



Example 1.2.5. We can use the above to order all the standard tableaux on $\lambda = (3, 2)$

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} > \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array} > \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array} > \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array} > \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}.$$

2 Schur Polynomials

This section uses the combinatorics of tableaux to construct and prove properties of an important collection of polynomials called *Schur polynomials*. We then go on to define the *ring of symmetric functions*. These constructions will be useful in Section 3 for proving assertions made about irreducible representations of the symmetric group and in Section 4.5 for finding the characters of irreducible representations of the general linear group.

2.1 The Tableaux Ring

We begin by imparting some structure on the set of tableaux with entries in $[m] := \{1, \ldots, m\}$. There are a few ways to define the product of tableaux, which (rather interestingly [3, pp.17-23]) can be shown to be equivalent. For our purposes, we just give one of these definitions.

We start by defining how to *row-insert* a single positive integer x into a tableau T. The procedure is as follows: if x is greater than (or equal to) one of the integers in the first row of T then simply add x to the end of the first row. Else, replace the left-most entry in the first row of T that is strictly greater than x, call it y, with x. Now use the same process to insert y into the second row. Repeat until the bumped entry can be put on the end of a row, or until we run out of rows, in which case the bumped entry starts a new row at the bottom.

Example 2.1.1. We row-insert the integer 1 into the tableau given below with the following iterations:

$$\begin{array}{|c|c|c|c|}\hline 1&2&2&3\\\hline 2&3&5&5\\\hline 4&4&6\\\cline{1-3} 5&6\\\cline{1-2}\end{array} \longleftarrow 1 \qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&3&5&5\\\hline 4&4&6\\\cline{1-3} 5&6\\\cline{1-2}\end{array} \longleftarrow 2 \qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&2&5&5\\\hline 4&4&6\\\cline{1-3} 5&6\\\cline{1-2}\end{array} \longleftarrow 3 \qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&2&5&5\\\hline 3&4&6\\\cline{1-3} 5&6\\\cline{1-2}\end{array} \longleftarrow 4$$

Thus the resulting tableau is given by

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&2&5&5\\\hline 3&4&6\\\cline{1-3} 4&6\\\cline{1-2} 5\\\cline{1-1}\end{array}\,.$$

By construction, row-inserting an integer into a tableau will always result in another tableau. From here it is simple to define the *product tableaux*. Given two tableaux T and U we define $T \cdot U$ to be the result of progressively row-inserting the entries of U into T, starting with the left-most entry in the bottom row of U and continuing left to right, bottom to top (such that the bottom row is emptied first) until U is empty. As each row-insertion results in a tableau, it follows that the final product $T \cdot U$ is also a tableau.

◇ **Example 2.1.2.** (1) It follows from Example 2.1.1 that

$$\begin{array}{|c|c|c|c|}\hline 1&2&2&3\\\hline 2&3&5&5\\\hline 4&4&6\\\cline{1-3} 5&6\\\cline{1-2}\end{array} \cdot \begin{array}{|c|}\hline 1\\\hline\end{array} = \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&2&5&5\\\hline 3&4&6\\\cline{1-3} 4&6\\\cline{1-2} 5\\\cline{1-1}\end{array}\,.$$

(2) The examples below demonstrate that the products of tableaux with the same shapes, but different numberings, don't always result in the same shaped tableau

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} \cdot \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array} = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&2\\\cline{1-2} 3&5\\\cline{1-2} 4\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 4&4\\\cline{1-2}\end{array} \cdot \begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array} = \begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&2\\\hline 2&3\\\cline{1-2} 4&4\\\cline{1-2}\end{array}\,.$$

(3) Combined with (2), the example below demonstrates that the product is not commutative

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array} \cdot \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&3\\\hline 2&2&4&5\\\cline{1-4}\end{array}\,.$$

This turns the set of tableaux (on all shapes) into a monoid where the identity is given by the empty tableau. The proof that this is associative is omitted.[1] Now restrict the monoid of tableaux to entries in $[m]$ and define $R_{[m]}$ to be the free $\mathbb{Z}$-module with basis the tableaux with entries in $[m]$ where multiplication is given by the tableaux product as above. We call $R_{[m]}$ the *tableau ring*.

[1]See [3, p.23]

2.2 Schur Polynomials

We can now use the tableaux ring to define Schur polynomials and make deductions about their behaviour. Observe that there is a natural homomorphism from the tableau ring $R_{[m]}$ into the polynomial ring $\mathbb{Z}[x_1, \ldots, x_m]$. For a tableau T consider the map

$$T \mapsto x^T \tag{2.1}$$

where x^T is the monomial in $\mathbb{Z}[x_1, \ldots, x_m]$ which is the product of variables $x_i^{T(i)}$ with $T(i)$ being the number of times i occurs in T. For example,

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 6 \\ \hline 3 & 4 & 4 & 7 \\ \hline 4 & 5 & 5 \\ \cline{1-3}\end{array} \mapsto x_1^2 x_3^2 x_4^3 x_5^2 x_6 x_7.$$

This map respects addition by construction and respects multiplication since the number of entries i in the product tableaux $T \cdot U$ will be will $T(i) + U(i)$. Now define $S_\lambda = S_\lambda[m]$ to be the sum of tableaux T of shape λ in the tableau ring $R_{[m]}$, where $\lambda \vdash n$. Define the image of S_λ under the homomorphism above to be the *Schur polynomial* $s_\lambda(x_1, \ldots, x_m)$.

◇ **Examples 2.2.1.** (1) Let $m = 2$ and consider the shape $\lambda = (4)$. Then we have that $S_{(4)}$ is given by

$$S_{(4)} = \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\ \hline\end{array} + \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline\end{array} + \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\ \hline\end{array} + \begin{array}{|c|c|c|c|}\hline 1&2&2&2\\ \hline\end{array} + \begin{array}{|c|c|c|c|}\hline 2&2&2&2\\ \hline\end{array}.$$

Hence the Schur polynomial is given by $s_{(4)} = x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_2^4$. This is the *complete symmetric polynomial*[2] of degree $p = 4$ in two variables; where, in general, the complete symmetric polynomial of degree p in m variables is defined to be $h_p(x_1, \ldots, x_m) := s_{(p)}(x_1, \ldots, x_m)$.

(2) Let $m = 4$ and consider the shape $\lambda = (1^3)$. Then we have that $S_{(1^3)}$ is given by

$$S_{(1^3)} = \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array} + \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 4\\ \hline\end{array} + \begin{array}{|c|}\hline 1\\ \hline 3\\ \hline 4\\ \hline\end{array} + \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline 4\\ \hline\end{array}.$$

Hence the Schur polynomial is given by $s_{(1^3)} = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$. This is the *elementary polynomial* of degree $p = 3$ in four variables; where, in general, the elementary polynomial of degree p in m variables is defined to be $e_p(x_1, \ldots, x_m) := s_{(1^{(p)})}(x_1, \ldots, x_m)$.

The combinatorics of the tableaux ring can now be applied to prove properties of Schur polynomials. Indeed, using the algorithm for computing the product of tableaux in Section 2.1 it can be shown that

$$S_\lambda \cdot S_{(p)} = \sum_\mu S_\mu \tag{2.2}$$

where the sum is over all shapes μ that are obtained by adding p boxes to λ, with no two in the same column; and

$$S_\lambda \cdot S_{(1^p)} = \sum_\mu S_\mu \tag{2.3}$$

where the sum is over all shapes μ that are obtained by adding p boxes to λ, with no two in the same row. Together (2.2) and (2.3) are called the *Pieri formulas*.

Applying our homomorphism (2.1) to the Pieri formulas immediately yields

$$s_\lambda(x_1, \ldots, x_m) \cdot h_p(x_1, \ldots, x_m) = \sum_\mu s_\mu(x_1, \ldots, x_m)$$
$$s_\lambda(x_1, \ldots, x_m) \cdot e_p(x_1, \ldots, x_m) = \sum_\mu s_\mu(x_1, \ldots, x_m)$$

[2] We will see in Corollary 2.2.6 that these polynomials are indeed symmetric.

where the sums are as described for equations (2.2) and (2.3) respectively. Now define the *content* $\mu = (\mu_1, \ldots, \mu_\ell)$ of a tableau T so that $\mu_i = T(i)$ is the number of times the entry i appears in T. Also define the *Kostka number* $K_{\lambda\mu}$ to be the number of tableaux of shape λ with content μ. As a consequence of (2.2) we have

$$S_{(\mu_1)} \cdot S_{(\mu_2)} \cdot \ldots \cdot S_{(\mu_\ell)} = \sum_\lambda K_{\lambda\mu} S_\lambda. \tag{2.4}$$

◇ **Example 2.2.2.** (1) Let $m = 2$ and $\mu = (1, 2)$ and consider

$$S_{(1)} = \ytableaushort{1} + \ytableaushort{2} \qquad S_{(2)} = \ytableaushort{11} + \ytableaushort{12} + \ytableaushort{22}$$

Now there are only two tableaux of content $\mu = (1, 2)$, namely $\ytableaushort{122}$ and $\ytableaushort{12,2}$. Hence, by equation (2.4), we have

$$S_{(1)} \cdot S_{(2)} = S_{(3)} + S_{(2,1)}.$$

Indeed,

$$\begin{aligned} S_{(1)} \cdot S_{(2)} &= \left(\ytableaushort{1} \cdot \ytableaushort{11} + \ytableaushort{1} \cdot \ytableaushort{12} + \ytableaushort{1} \cdot \ytableaushort{22} \right) + \left(\ytableaushort{2} \cdot \ytableaushort{11} + \ytableaushort{2} \cdot \ytableaushort{12} + \ytableaushort{2} \cdot \ytableaushort{22} \right) \\ &= \ytableaushort{111} + \ytableaushort{112} + \ytableaushort{122} + \ytableaushort{11,2} + \ytableaushort{12,2} + \ytableaushort{222}. \\ &= S_{(3)} + S_{(2,1)}. \end{aligned}$$

(2) Now let $m = 3$ and $\mu = (1, 2, 1)$ so that $\ytableaushort{1223}$, $\ytableaushort{123,2}$, $\ytableaushort{122,3}$, $\ytableaushort{12,23}$ are the tableaux of content μ. Hence, by equation (2.4), we have

$$S_{(1)} \cdot S_{(2)} \cdot S_{(1)} = S_{(4)} + 2S_{(3,1)} + S_{(2,2)}.$$

As before, (2.4) gives us an analogous formula for complete symmetric polynomials

$$h_{(\mu_1)} \cdot h_{(\mu_2)} \cdot \ldots \cdot h_{(\mu_\ell)} = \sum_\lambda K_{\lambda\mu} s_\lambda \tag{2.5}$$

and similarly for elementary polynomials

$$e_{(\mu_1)} \cdot e_{(\mu_2)} \cdot \ldots \cdot e_{(\mu_\ell)} = \sum_\lambda K_{\tilde{\lambda}\mu} s_\lambda \tag{2.6}$$

where $\tilde{\lambda}$ is the conjugate of λ.

Now fix three partitions λ, μ, and ν. Let V be a tableau of shape ν. Define $c^\nu_{\lambda\mu}$, called a *Littlewood-Richardson number*, to be the number of ways V can be written as a product $T \cdot U$, where T is a tableau of shape λ and U is a tableau of shape μ. It can be shown that $c^\nu_{\lambda\mu}$ is independent of the choice of V.

Theorem 2.2.3 (Littlewood-Richardson Rule). *For $S_\lambda, S_\mu \in R_{[m]}$ we have*

$$S_\lambda \cdot S_\mu = \sum_\nu c^\nu_{\lambda\mu} S_\nu. \tag{2.7}$$

Theorem 2.2.3 states exactly that each tableau V of shape ν can be written as a product of tableaux of shape λ and a tableau of shape μ in $c^\nu_{\lambda\mu}$ ways. Analogously to above, we obtain an equivalent formula for Schur polynomials: $s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$.

◇ **Example 2.2.4.** Let $n = 4$, $m = 2$ and consider $s_{(3,1)} = x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3$ and $s_{(2,2)} = x_1^2x_2^2$. By the Littlewood-Richardson rule (Theorem 2.2.3) we have

$$s_{(3,1)} \cdot s_{(2,2)} = s_{(5,3)}.$$

This is because there are exactly three tableaux (with entries in $\{1, 2\}$) that can be written as a product of tableaux of shape $(3, 1)$ and $(2, 2)$, specifically

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\ \hline 2&2&2\\ \cline{1-3}\end{array}\qquad\begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&2\\ \hline 2&2&2\\ \cline{1-3}\end{array}\qquad\begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&2\\ \hline 2&2&2\\ \cline{1-3}\end{array}$$

which all have shape $(5, 3)$. Indeed, we can verify this result

$$s_{(3,1)} \cdot s_{(2,2)} = (x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3) \cdot (x_1^2x_2^2) = x_1^5x_2^3 + x_1^4x_2^4 + x_1^3x_2^5 = s_{(5,3)}.$$

We now show that the Schur polynomials are symmetric polynomials and therefore belong to the ring of symmetric functions defined in the next section (2.3). To prove this fact we will need the following lemma about Kostka numbers:

Lemma 2.2.5. *For partitions λ, μ of n we have that $K_{\lambda\mu} \neq 0$ if and only if $\mu \trianglelefteq \lambda$.*

◇ *Proof:* Suppose that $\mu \ntrianglelefteq \lambda$. Now, to show that there are no partitions of shape λ and content μ, it is sufficient to consider the tableau whereby we put all the entries in a list

$$(\overbrace{1, \dots, 1}^{\mu_1 \text{ times}}, \overbrace{2, \dots, 2}^{\mu_2 \text{ times}}, \dots, \overbrace{\ell, \dots, \ell}^{\mu_\ell \text{ times}}) \tag{2.8}$$

and place them into the tableau of shape λ from left to right, top to bottom. For example, if $\mu = (4, 5, 2, 3)$ and $\lambda = (6, 4, 3, 1)$ we would have

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&2&2\\ \hline 2&2&2&3\\ \cline{1-4} 3&4&4\\ \cline{1-3} 4\\ \cline{1-1}\end{array}\quad.$$

This is sufficient because if this fails to be a tableau, then no tableau with shape λ and content μ exists. Now if $\mu_1 \nleq \lambda_1$ then the entries 1 in our tableau of shape λ must spill over to the second row which contradicts the strictly increasing property of tableaux. Hence we may assume there exists an $i > 1$ such that

$$\begin{aligned} \mu_1 + \dots + \mu_{i-1} + a &= \lambda_1 + \dots + \lambda_{i-1} \\ \mu_1 + \dots + \mu_{i-1} + \mu_i &= \lambda_1 + \dots + \lambda_{i-1} + \lambda_i + b \end{aligned}$$

for $a \in \{0, 1, 2, \dots\}$ and $b \in \{1, 2, \dots\}$. Combining these expressions yields $\mu_i = \lambda_i + a + b$. Now consider entering the μ_i entries i into the diagram as above. We fill the a empty slots on the $(i-1)^{\text{th}}$ row, followed by the λ_i empty slots on the i^{th} row and then put the remaining $b \geq 1$ entries on the $(i+1)^{\text{th}}$ row. This contradicts our strictly increasing property of tableaux. Hence, $K_{\lambda\mu} = 0$.

Conversely, if $\mu \trianglelefteq \lambda$ then we can easily construct a tableau of shape λ and content μ using the filling process described above. □

Corollary 2.2.6. *The Schur polynomials s_λ are symmetric.*

Proof: We claim that (2.5) and (2.6) can be solved to express the Schur polynomials s_λ in terms of symmetric polynomials; then they themselves will be symmetric polynomials. With this aim, order all partitions of n lexicographically. Then, since $\mu \trianglelefteq \lambda$ implies that $\mu \leq \lambda$, by Lemma 2.2.5 we have that the matrix $K_{\lambda\mu}$ is a lower triangular matrix with entries 1 on the diagonal. This proves the claim. □

Remark 2.2.7. We can also use Lemma 2.2.5 to simplify some of the formulae derived in Section 2.2. In particular, (2.5) and (2.6) become

$$h_{(\mu_1)} \cdot h_{(\mu_2)} \cdot \dots \cdot h_{(\mu_\ell)} = s_\mu + \sum_{\lambda \rhd \mu} K_{\lambda\mu} s_\lambda \tag{2.9}$$

$$e_{(\mu_1)} \cdot e_{(\mu_2)} \cdot \dots \cdot e_{(\mu_\ell)} = s_{\tilde{\mu}} + \sum_{\tilde{\lambda} \rhd \mu} K_{\tilde{\lambda}\mu} s_\lambda \tag{2.10}$$

where the content $\mu = (\mu_1, \dots, \mu_\ell)$ is viewed as a partition.

2.3 Ring of Symmetric Functions

In this section, we construct a $\mathbb{Z}$-module that will be essential for Section 3.4, where we will define a ring isomorphism that relates representations of the symmetric group to Schur polynomials.

Define a *symmetric function of degree* n to be a collection of symmetric polynomials $p(x_1, \dots, x_m)$ of degree n, one for each m, such that

$$p(x_1, \dots, x_\ell, 0, \dots, 0) = p(x_1, \dots, x_\ell) \tag{2.11}$$

for all $\ell \leq m$. Let Λ_n denote the $\mathbb{Z}$-module of all such functions with integer coefficients. Then set

$$\Lambda := \bigoplus_{n=0}^{\infty} \Lambda_n$$

which is called the *ring of symmetric functions*.

◇ **Example 2.3.1.** (1) Consider the 2^{nd} complete symmetric polynomials defined in Examples 2.2.1 (2)

$$\begin{aligned} h_2(x_1) &= x_1^2 \\ h_2(x_1, x_2) &= x_1^2 + x_1x_2 + x_2^2 \\ h_2(x_1, x_2, x_3) &= x_1^2 + x_1x_2 + x_2^2 + x_1x_3 + x_2x_3 + x_3^2 \end{aligned}$$

which satisfy the equations (2.11) and so are elements of Λ_2.

(2) Let $\lambda = (3, 1, 1)$ be a partition of $n = 5$. Then the *monomial symmetric polynomial* of λ is defined to be

$$m_\lambda(x_1, x_2, x_3) = x_1^3x_2x_3 + x_1x_2^3x_3 + x_1x_2x_3^3.$$

We analogously define monomial symmetric polynomials for any partition λ. Note that they satisfy the equations (2.11) by setting m_λ to be zero if k, such that $\lambda = (\lambda_1, \dots, \lambda_k)$, is larger than the number of variables m.

Following from the example above, write $x = (x_1, \dots, x_m)$ and let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition. Then we can define symmetric polynomials

$$\begin{aligned} h_\lambda(x) &:= h_{\lambda_1}(x) \cdot \dots \cdot h_{\lambda_k}(x) \\ e_\lambda(x) &:= e_{\lambda_1}(x) \cdot \dots \cdot e_{\lambda_k}(x) \end{aligned}$$

where $h_p(x)$ and $e_p(x)$ are the p^{th} complete and elementary symmetric polynomials in the variables $x_1, \dots, x_m$. Recall that the Schur polynomials $s_\lambda(x_1, \dots, x_m)$ are symmetric (Corollary 2.2.6) and, using Example 2.3.1 above, we note that the monomial symmetric polynomials $m_\lambda(x_1, \dots, x_m)$ are also symmetric polynomials. This leads us to the following proposition.

Proposition 2.3.2. *The sets $\{s_\lambda\}$ and $\{h_\lambda\}$, as λ varies over all partitions of n, are $\mathbb{Z}$-bases for Λ_n.*

Proof: To prove this we first show a similar result for the monomial symmetric polynomials, defined in Examples 2.3.1 (2), for then the result will follow more easily. Define M to be the set $\{m_\lambda\}$ for all partitions λ of n with at most m rows. We show that M is a basis for the symmetric polynomials in m variables $x_1, \ldots, x_m$.

To show that M is spanning consider an arbitrary symmetric polynomial. Let $x^\lambda = x_1^{\lambda_1} \cdots x_m^{\lambda_m}$ be its maximal monomial in said polynomial whereby the partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ are ordered lexicographically. Let $r \in \mathbb{Z}$ be the coefficient of x^λ and note that, by symmetry, our symmetric polynomial contains a copy of the monomial symmetric polynomial rm_λ. Thus subtract rm_λ to obtain a polynomial which has maximal monomial that is smaller with respect to this ordering. We can repeat this process until we obtain 0 and thus we can express any symmetric polynomial as a linear combination of the m_λ.

To show linear independence of M suppose that $\sum r_\lambda m_\lambda = 0$ with the sum over partitions λ of n with at most m rows. Let λ be the maximal partition such that $r_\lambda \neq 0$. However, this then means that x^λ appears in the sum with non-zero coefficient. In other words, no such λ exists and M is linearly independent.

Therefore monomial symmetric polynomials m_λ, as λ varies over all partitions of n, are a basis for all symmetric functions of degree n with any number of variables m and hence for Λ_n.

We can now show that the set of Schur polynomials s_λ, as λ varies over partitions of n, is basis for Λ_n. For this it suffices to show that the Schur polynomials span Λ_n, since the cardinality of the set of Schur polynomials is the same as the cardinality of the set of monomial symmetric polynomials. Now it can be shown that the Schur polynomials satisfy [9, p.57]

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu \tag{2.12}$$

where $K_{\lambda\mu}$ are the Kostka numbers seen in Section 2.2. Thus, by Lemma 2.2.5, we can solve (2.12) to express the monomial symmetric polynomials in terms of Schur polynomials. This means that the set $\{s_\lambda\}$ spans Λ_n.

It remains to show that the set $\{h_\lambda\}$ forms a basis for Λ_n. However, the proof of this is extremely similar to the previous paragraph except we replace equation (2.12) with equation (2.5) and use that the knowledge that the s_λ are a basis. □

Corollary 2.3.3. *The h_λ form a $\mathbb{Z}$-basis for Λ as λ varies over all partitions.*

Remark 2.3.4. Since the Schur polynomials s_λ form a basis for Λ_n, we can define a symmetric inner product $(\cdot, \cdot)$ on Λ_n by requiring that the Schur polynomials s_λ form an orthonormal basis. That is,

$$(s_\lambda, s_\lambda) = 1 \qquad \text{and} \qquad (s_\lambda, s_\mu) = 0$$

for $\mu \neq \lambda$. We will need this in Section 3.4 to prove that the isomorphism that relates Λ to irreducible representations of S_n is indeed an isomorphism.

3 Representations of the Symmetric Group

We now use the tools we have created using tableaux to derive all the irreducible representations of the symmetric group, called *Specht modules*. Later in this section we'll relate these representations to Schur polynomials and give some alternative constructions of Specht modules. In this section all numberings T of a diagram with n boxes will have distinct entries in $\{1, \ldots, n\}$. This means that any tableau is also a standard tableau.

3.1 Specht Module Construction

We begin by constructing Specht modules. Notice that the symmetric group S_n defines a natural action on a numbering T of a Young diagram with n boxes. For $\sigma \in S_n$ we have that $\sigma \cdot T$ is the numbering that puts $\sigma(i)$ in the box in which T puts i.

For each numbering T this action generates a natural subgroup of S_n called the *row group* of T denoted by $R(T)$. It is defined as

$$R(T) := \{\sigma \in S_n \mid \sigma \text{ preserves the rows of } T\}.$$

That is, for $\sigma \in R(T)$, we have that T and $\sigma \cdot T$ have the same entries in their rows. If $\lambda = (\lambda_1, \dots, \lambda_k)$ is the shape of T then $R(T)$ is the product of symmetric groups

$$S_{\lambda_1} \times \cdots \times S_{\lambda_k}.$$

Analogously, we can define the *column group* of T denoted by $C(T)$. The construction of the row group allows us to define an equivalence class of numberings on a Young diagram.

Definition 3.1.1 (Tabloid)**.** A tabloid is an equivalence class of numberings on a Young diagram such that two numberings T and T' are equivalent if and only if there exists $p \in R(T)$ such that

$$T' = p \cdot T.$$

We denote the equivalence classes with curly brackets and so write $\{T'\} = \{T\}$.

We display tabloids in a similar fashion to tableaux, except we drop the vertical bars. For example,

$$\begin{array}{ccc} \hline 1 & 4 & 7 \\ \hline 3 & 6 & \\ \cline{1-2} 2 & 5 & \\ \cline{1-2} \end{array} = \begin{array}{ccc} \hline 4 & 7 & 1 \\ \hline 6 & 3 & \\ \cline{1-2} 2 & 5 & \\ \cline{1-2} \end{array}.$$

The action of S_n on tableaux also translates nicely to tabloids whereby for $\sigma \in S_n$ we have

$$\sigma \cdot \{T\} = \{\sigma \cdot T\}.$$

This is well-defined because S_n permutes the numbers, not the boxes. Tabloids are the main ingredient in our first construction of the irreducible representations of S_n. Define M^λ to be the complex vector space with basis of tabloids $\{T\}$ of shape λ, where $\lambda \vdash n$. The action of S_n on tabloids thus makes M^λ into $\mathbb{C}[S_n]$-module.

◇ **Examples 3.1.2.** (1) If $\lambda = (5)$ then $M^{(5)}$ is the 1-dimensional $\mathbb{C}$-vector space with basis vector $\begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline \end{array}$ and for any $\sigma \in S_5$ we have

$$\sigma \cdot \left(\begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline \end{array} \right) = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline \end{array}.$$

Thus we have that $M^{(5)}$ is isomorphic to the trivial representation.

(2) If $\lambda = (1, 1, 1)$ then $M^{(1,1,1)}$ is the 6-dimensional $\mathbb{C}$-vector space with basis vectors

$$\begin{array}{c} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}, \quad \begin{array}{c} \hline 1 \\ \hline 3 \\ \hline 2 \\ \hline \end{array}, \quad \begin{array}{c} \hline 2 \\ \hline 1 \\ \hline 3 \\ \hline \end{array}, \quad \begin{array}{c} \hline 2 \\ \hline 3 \\ \hline 1 \\ \hline \end{array}, \quad \begin{array}{c} \hline 3 \\ \hline 1 \\ \hline 2 \\ \hline \end{array}, \quad \begin{array}{c} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}.$$

It is clear that we can identify each of these basis elements with an element in S_n and so $M^{(1,1,1)}$ is isomorphic to the regular representation.

(3) If $\lambda = (2, 1)$ then $M^{(2,1)}$ is a 3-dimensional vector space over $\mathbb{C}$ with basis

$$\{T_1\} = \begin{array}{cc} \hline 2 & 3 \\ \hline 1 & \\ \cline{1-1} \end{array}, \quad \{T_2\} = \begin{array}{cc} \hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array}, \quad \{T_3\} = \begin{array}{cc} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}.$$

The representations M^λ of S_n are, in general, not irreducible. Thus we now define elements of the group algebra $\mathbb{C}[S_n]$ that will cut out the irreducible components of an M^λ.

Given a numbering T of a diagram with n boxes, we define *Young symmetrizers*

$$a_T := \sum_{p \in R(T)} p \qquad b_T := \sum_{q \in C(T)} \operatorname{sgn}(q) q \qquad c_T := b_T \cdot a_T.$$

These have the following useful properties. Firstly, for $p \in R(T)$ and $q \in C(T)$, we have

$$p \cdot a_T = a_T \cdot p = a_T, \qquad q \cdot b_T = b_T \cdot q = \operatorname{sgn}(q) b_T. \tag{3.1}$$

We also have

$$a_T \cdot a_T = |R(T)| \cdot a_T, \qquad b_T \cdot b_T = |C(T)| \cdot b_T. \tag{3.2}$$

We can now define Specht modules:

Definition 3.1.3 (Specht Module)**.** Let λ be a partition of n. For each numbering T of shape λ define

$$v_T := b_T \cdot \{T\} = \sum_{q \in C(T)} \operatorname{sgn}(q) \{q \cdot T\}. \tag{3.3}$$

Then we define the *Specht module* S^λ to be the subspace of M^λ spanned by all the elements v_T, as T varies over all numberings of λ.

◇ **Remark 3.1.4.** Notice that a tabloid $\{T\}$ ignores the row position in a numbering. Also notice that the Young symmetrizer b_T, if we're given the resulting sum $b_T \cdot T$, makes it impossible to distinguish the original column position of the entries of T.[3] Therefore it seems that $v_T = b_T \cdot \{T\}$ ignores the row and column position of an entry. So the question is: what are we left with? The answer to this question relates to what are called the quadratic relations (which we will see many times in later sections).

◇ **Example 3.1.5.** Let $\lambda = (2, 1)$ so that M^λ has basis $\{T_1\}, \{T_2\}, \{T_3\}$ as in Examples 3.1.2 (3). We therefore obtain

$$b_{T_1} = 1 - (12) \qquad b_{T_2} = 1 - (12) \qquad b_{T_3} = 1 - (13)$$

and thus S^λ is spanned by the elements

$$v_{T_1} = \begin{array}{cc}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array} - \begin{array}{cc}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} \qquad v_{T_2} = \begin{array}{cc}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} - \begin{array}{cc}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array} \qquad v_{T_3} = \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} - \begin{array}{cc}\hline 3 & 2 \\ \hline 1 & \\ \cline{1-1}\end{array}.$$

We will see later (Example 3.3.5) that $\dim(S^{(2,1)}) = 2$. Therefore, in this case, we have $S^\lambda \neq M^\lambda$.

3.2 Irreducible Representations of the Symmetric Group

This section amounts to proving the following:

Theorem 3.2.1. *For each partition λ of n the Specht module S^λ is an irreducible representation of S_n. Furthermore, every irreducible representation of S_n is isomorphic to exactly one S^λ.*

Before embarking on the proof of Theorem 3.2.1 we require some extra tableaux theory.

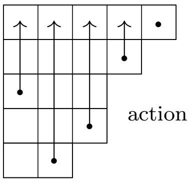
Lemma 3.2.2. *Let T and T' be numberings of the shapes λ and λ'. Assume that λ does not strictly dominate λ'. Then either*

(i) There are two distinct integers that occur in the same row of T' and the same column of T; or

(ii) $\lambda = \lambda'$ and there is some $p \in R(T')$ and $q \in C(T)$ such that $p \cdot T' = q \cdot T$.

[3] A column permutation is also assigned an orientation that depends on T. This orientation can be ignored for now but is something we'll see in more detail later.

Proof: Assume that (i) is false. We construct $q \in C(T)$ as given in (ii). Now, since (i) is false, all the entries in the first row of T' are in distinct columns in T. Therefore there exists a $q_1 \in C(T)$ that moves all these entries to the first row of T.



action of q_1

Note that this also implies that $\lambda'_1 \leq \lambda_1$. Similarly, all the entries in the second row of T' are also in distinct columns in T and thus in distinct columns in $q_1 \cdot T$. Thus we can find $q_2 \in C(T) = C(q_1 \cdot T)$ that moves these entries to the second row of $q_1 \cdot T$ and preserves the first row of $q_1 \cdot T$. Again, note that this implies that $\lambda'_2 \leq \lambda_2$.

Iterate this process for all rows 1 to k in λ and set $q = q_1 \cdots q_k \in C(T)$. Then if an entry i is in the j^{th} row of T' then i is also in the j^{th} row of $q \cdot T$. Furthermore, we have $\lambda'_1 + \cdots + \lambda'_j \leq \lambda_1 + \cdots + \lambda_j$, for $j = 1, \ldots, k$, so that $\lambda' \trianglelefteq \lambda$. However, by assumption λ does not strictly dominate λ' which therefore gives $\lambda = \lambda'$. Hence, $\{T'\} = \{q \cdot T\}$ which means there exists $p \in R(T)$ such that $p \cdot T' = q \cdot T$. $\square$

Corollary 3.2.3. *Using the ordering given at the end of Section 1.2, we have that if T and T' are standard tableaux with $T' > T$, then there is a pair of integers in the same row of T' and the same column of T.*

Proof: Since $T' > T$, we can apply Lemma 3.2.2. Indeed, $T' > T$ ensures that the shape of T cannot strictly dominate the shape of T' (see Remark 1.2.4). In search of a contradiction assume that we're in the case (ii) of Lemma 3.2.2 so that there exists $p \in R(T')$ and $q \in C(T)$ such that $p \cdot T' = q \cdot T$.

Now note that, since T and T' are standard tableau, $p \in R(T')$ and $q \in C(T)$, we have that

$$T' \leq p \cdot T' \qquad \text{and} \qquad q \cdot T \leq T$$

since the largest entry of T that is moved by p moves to the left and the largest entry of T that is moved by q moves upwards. (Note that above $\geq$ and $\leq$ do not represent the lexicographic ordering but instead represent the ordering given at the end of Section 1.2).

Therefore we can conclude that $T' \leq p \cdot T' = q \cdot T \leq T$. This contradicts $T' > T$ and so we must be in case (i) of Lemma 3.2.2. $\square$

Lemma 3.2.4. *Let T and T' be numberings of shape λ and λ', and assume that λ does not strictly dominate λ'. If there is a pair of integers in the same row of T' and the same column of T, then $b_T \cdot \{T'\} = 0$. Else, $\lambda = \lambda'$ and we have $b_T \cdot \{T'\} = \pm v_T$.*

Proof: This is an extension of the result in Lemma 3.2.2. Firstly, suppose there is a pair of integers in the same row of T' and the same column of T. Let $t \in R(T')$ be the transposition that permutes them. Then

$$\begin{aligned} b_T \cdot \left\{T'\right\} &= (b_T \cdot t) \cdot \left\{T'\right\} && (t \in R(T')) \\ &= \operatorname{sgn}(t) b_T \cdot \left\{T'\right\} && (3.1) \\ &= -b_T \cdot \left\{T'\right\}. && (t \text{ a transposition}) \end{aligned}$$

Thus we must have $b_T \cdot \{T'\} = 0$. Now if there is no such pair we are in case (ii) of Lemma 3.2.2 and so there is a $p \in R(T')$ and $q \in C(T)$ such that $p \cdot T' = q \cdot T$. Thus

$$\begin{aligned} b_T \cdot \left\{T'\right\} &= b_T \cdot \left\{p \cdot T'\right\} && (p \in R(T')) \\ &= b_T \cdot \{q \cdot T\} \\ &= (b_T \cdot q) \cdot \{T\} \\ &= \operatorname{sgn}(q) b_T \cdot \{T\} && (3.1) \\ &= \pm v_T. \end{aligned}$$

This concludes the proof. $\square$

Analogously to the relationship between Corollary 3.2.3 and Lemma 3.2.2 we also have

Corollary 3.2.5. *If T and T' are standard tableaux with $T' > T$, then $b_T \cdot \{T'\} = 0$.*

We now have all the necessary tools to show that the Specht modules S^λ do indeed give rise to all the irreducible representations of S_n.

Proof of Theorem 3.2.1: We first show that for each partition λ of n the Specht module S^λ is a submodule of M^λ. For this it suffices to show that $\sigma \cdot v_T = v_{\sigma \cdot T}$ for all T and $\sigma \in S_n$. Indeed,

$$\begin{aligned}
\sigma \cdot v_T &= \sigma \cdot \left(\sum_{q \in C(T)} \operatorname{sgn}(q)\{q \cdot T\} \right) \\
&= \sum_{q \in C(T)} \operatorname{sgn}(q)\{\sigma \cdot q \cdot T\} \\
&= \sum_{q \in C(T)} \operatorname{sgn}(q)\{(\sigma \cdot q \cdot \sigma^{-1}) \cdot \sigma \cdot T\} \\
&= \sum_{p \in C(\sigma \cdot T)} \operatorname{sgn}(p)\{\sigma \cdot T\} \\
&= v_{\sigma \cdot T},
\end{aligned}$$

where we have used the fact that $\sigma \cdot C(T) \cdot \sigma^{-1} = C(\sigma \cdot T)$ and that $\operatorname{sgn}(\sigma \cdot q \cdot \sigma^{-1}) = \operatorname{sgn}(q)$. This also shows that $S^\lambda = \mathbb{C}[S_n] \cdot v_T$. Indeed, if T' is a different numbering of λ and $\sigma \cdot T = T'$ then we have $\sigma \cdot v_T = v_{\sigma \cdot T} = v_{T'} \in \mathbb{C}[S_n] \cdot v_T$.

Secondly, we show that no two Specht modules are isomorphic. Let λ and μ be partitions of n such that $\mu > \lambda$[4]. Choose a numbering T of λ. Then, applying the latter case of Lemma 3.2.4, we have

$$b_T \cdot S^\lambda = \mathbb{C} \cdot v_T \neq 0. \tag{3.4}$$

Now applying the former case of Lemma 3.2.4 we have

$$b_T \cdot S^\mu = 0.$$

We conclude that $S^\lambda \ncong S^\mu$.

To show that Specht modules are irreducible suppose that $S^\lambda = V \oplus W$ for submodules V and W of M^λ. Then by (3.4) above we have

$$\mathbb{C} \cdot v_T = b_T \cdot S^\lambda = b_T \cdot V \oplus b_T \cdot W \subseteq V \oplus W.$$

Therefore, without loss of generality, V contains v_T. However, since V is a submodule it is closed under the action of the group algebra $\mathbb{C}[S_n]$ which means

$$S^\lambda = \mathbb{C}[S_n] \cdot v_T \subseteq V.$$

Thus $S^\lambda = V$ and so S^λ is irreducible.

Finally, we conclude that every irreducible representation is isomorphic to a Specht module. Indeed, there is an irreducible representation of S^λ (up to isomorphism) for each conjugacy class in S_n. However, the conjugacy classes in S^n consist of permutations of the same cycle type and each cycle type clearly corresponds to a partition of n. Therefore, as λ varies over all partitions of n, the S^λ must define all irreducible representations of S_n. ☐

[4] μ is strictly larger than λ in lexicographic ordering, which implies that λ does not strictly dominate μ. Hence, Lemma 3.2.4 applies.

3.3 M^λ and S^λ Structure

We now analyse the structure of the representations M^λ using the Specht modules S^λ. Some of this analysis will be used in Section 3.4 to draw parallels between the representations of the symmetric group and the symmetric polynomials from Section 2.3. We will also give a basis for S^λ.

Lemma 3.3.1. *Let $\phi : M^\lambda \to M^{\lambda'}$ be a homomorphism of representations of S_n. If S^λ is not in the kernel of ϕ, then $\lambda' \trianglelefteq \lambda$.*

Proof: Let T be a numbering of λ where v_T is not in the kernel of ϕ. Then

$$b_T \cdot \phi(\{T\}) = \phi(v_T) \neq 0.$$

Thus there is a numbering T' of shape λ' such that $b_T \cdot \{T'\} \neq 0$. Therefore, if λ does not strictly dominate λ', then by Lemma 3.2.4 we must have $\lambda' = \lambda$. Hence, either $\lambda' \triangleleft \lambda$ or $\lambda = \lambda'$. □

Theorem 3.3.2. *There are non-negative integers $k_{\nu\lambda}$, for $\nu \triangleright \lambda$, such that*

$$M^\lambda \cong S^\lambda \oplus \bigoplus_{\nu \triangleright \lambda} (S^\nu)^{\oplus k_{\nu\lambda}}.$$

Proof: For each ν, let $k_{\nu\lambda}$ be the number of times S^ν occurs in the decomposition of M^λ. Now

$$b_T \cdot M^\lambda = b_T \cdot S^\lambda$$

and so $k_{\lambda\lambda} = 1$. Now suppose S^ν appears in M^λ and consider the projection from M^ν to S^ν followed by the embedding of S^ν in M^λ

$$M^\nu \to S^\nu \to M^\lambda$$

which defines a homomorphism from M^ν to M^λ such that S^ν is not in the kernel. Thus Lemma 3.3.1 applies to give $\lambda \trianglelefteq \nu$. □

Proposition 3.3.3. *The elements v_T, as T varies over tableaux of shape λ, form a basis for S^λ.*

Proof: Firstly, to show linear independence, suppose that $\sum r_T v_T = 0$, where the sum is over all tableaux T of shape λ. Consider the ordering defined at the end of Section 1.2. We aim to show that the maximal T in the above sum has non-zero coefficient. Now each v_T is a linear combination of $\{T\}$ and $\{q \cdot T\}$ for $q \in C(T)$. Furthermore, $\{T\}$ appears with coefficient 1 in v_T. Recall, from the proof of Corollary 3.2.3, that for all $q \in C(T)$ we have $q \cdot T < T$ (when T is a tableau). Hence, the maximal T in $\sum r_T v_T = 0$ occurs only in v_T as $\{T\}$ with coefficient 1.

To show that the elements v_T are spanning we make use of the *Robinson correspondence* [3, p.52]. From this we can deduce

$$n! = \sum_{\lambda \vdash n} \left(f^\lambda\right)^2$$

where f^λ is the number of standard tableaux of shape λ. Therefore, using that the sum of the squares of the dimension of the irreducible representations is equal to the order of the group, we have

$$n! = \sum_{\lambda \vdash n} \left(f^\lambda\right)^2 \leq \sum_{\lambda \vdash n} \left(\dim\left(S^\lambda\right)\right)^2 = n!.$$

Hence, we must have $\dim\left(S^\lambda\right) = f^\lambda$ for all λ. This means that the v_T must be a basis for S^λ. □

Remark 3.3.4. We will see another proof of Theorem 3.3.3 in Section 3.5 using a dual construction of S^λ.

◇ **Example 3.3.5.** Recall Example 3.1.5 in which we showed that S^λ, for $\lambda = (2, 1)$, is spanned by $v_{T_1}, v_{T_2}, v_{T_3}$. By Proposition 3.3.3, we have that S^λ has basis v_{T_2}, v_{T_3}. This is because

$$T_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} \qquad T_3 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$$

are all the tableaux of shape λ with distinct entries in $\{1, 2, 3\}$. Therefore, v_{T_1} can be expressed as a linear combination of v_{T_2} and v_{T_3}. Indeed, $v_{T_1} = -v_{T_2}$. This means that $S^{(2,1)}$ is the unique 2-dimensional representation of S_3 (up to isomorphism).

3.4 Ring of Representations

The aim of this section is to introduce a relationship between the ring of symmetric functions Λ (Section 2.3) and the irreducible representations of the symmetric group. This will allow us to transfer some of the properties of Λ to representations of the symmetric group.

Firstly define R_n to be the free abelian group on the isomorphism classes of irreducible representations of S_n. That is, if $V \cong \bigoplus (S^\lambda)^{\oplus a_\lambda}$ is a representation of S_n, then its class $[V]$ is given by

$$[V] = \sum a_\lambda [S^\lambda].$$

Now define $R := \bigoplus_{n=0}^{\infty} R_n$ where $R_0 = \mathbb{Z}$. We will see later that this defines a ring, called the *ring of representations* of S_n, and that is isomorphic to the ring of symmetric functions. However, we first need to construct a multiplication operation on R.

Definition 3.4.1 (Induced Representation)**.** Let G be a group and let V be a $\mathbb{C}[H]$-module for some subgroup H of G. Then we define the induced representation of V from H to G as

$$\mathrm{Ind}_H^G(V) := \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V.$$

Examples 3.4.2. (1) $\diamond$ Consider S_n and the alternating group A_n for $n \geq 2$. Let $\mathbf{1}$ denote the trivial representation of A_n. Then the induced representation of $\mathbf{1}$ from A_n to S_n is

$$\mathrm{Ind}_{A_n}^{S_n}(\mathbf{1}) = \mathbb{C}[S_n] \otimes_{\mathbb{C}[A_n]} \mathbb{C}$$

which is isomorphic to the sign module. Indeed, let $\tau \in S_n$ and $\sigma \in \mathrm{Ind}_{A_n}^{S_n}(\mathbf{1})$. The cosets of A_n in S_n are A_n and $(12)A_n$, which partition S_n into even and odd permutations. Thus the action of τ on σ in $\mathrm{Ind}_{A_n}^{S_n}(\mathbf{1})$ is given by

$$\tau \cdot \sigma = \begin{cases} (12) \cdot \sigma & \tau \text{ odd;} \\ \sigma & \tau \text{ even.} \end{cases}$$

Hence the action of τ is uniquely determined by its sign.

(2) Fix a numbering T of λ so that M^λ has basis $\sigma \cdot \{T\}$ as σ varies over class representatives of $S_n/R(T)$. Hence, again letting $\mathbf{1}$ denote the trivial representation, we have

$$M^\lambda \cong \mathrm{Ind}_{R(T)}^{S_n}(\mathbf{1}) = \mathbb{C}[S_n] \otimes_{\mathbb{C}[R(T)]} \mathbb{C}.$$

We now define multiplication $R_n \times R_m \to R_{n+m}$ on R, for representations V and W of S_n and S_m (respectively), as

$$[V] \circ [W] := \left[\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} V \otimes W\right] = \mathbb{C}[S_{n+m}] \otimes_{\mathbb{C}[S_n \times S_m]} (V \otimes W) \tag{3.5}$$

where $V \otimes W$ is viewed as a representation of $S_n \times S_m$ with

$$(\sigma \times \tau) \cdot (v \otimes w) = \sigma \cdot v \otimes \tau \cdot w$$

for $\sigma \in S_n$, $\tau \in S_m$, $v \in V$ and $w \in W$. Furthermore, we view $S_n \times S_m$ as a subgroup of S_{n+m} by letting S_n act on the first n integers and S_m act on the remaining m integers. This multiplication makes R into a commutative (graded) ring with unit. Furthermore, we can equip R_n with a symmetric inner product $(\cdot,\cdot)$ by requiring that the irreducible representations $[S^\lambda]$ form an orthonormal basis. In particular, if $V \cong \oplus(S^\lambda)^{\oplus a_\lambda}$ and $W \cong \oplus(S^\lambda)^{\oplus b_\lambda}$, we have

$$([V],[W]) = \sum a_\lambda b_\lambda.$$

Now since the polynomials h_λ form a basis for the ring of symmetric functions (Corollary 2.3.3), we can define an additive homomorphism $\phi : \Lambda \to R$ such that

$$\phi(h^\lambda) = [M^\lambda].$$

We can now state and prove the main result of this section:

Theorem 3.4.3. *The map $\phi : \Lambda \to R$ is an (isometric) isomorphism such that $\phi(s^\lambda) = [S^\lambda]$.*

Proof: As we saw in Examples 3.1.2 (1), we have that ϕ takes h_n to the trivial representation $M^{(n)}$. Then, since Λ is a polynomial ring in the variables h_n, to show that ϕ is ring homomorphism we need only show that

$$M^{(\lambda_1)} \circ M^{(\lambda_2)} \circ \cdots \circ M^{(\lambda_k)} = M^\lambda \tag{3.6}$$

for $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition. Since then, if (3.6) holds, we have

$$\phi(h_{\lambda_1} \cdots h_{\lambda_k}) = \phi(h_\lambda) = [M^\lambda] = [M^{(\lambda_1)}] \circ \cdots \circ [M^{(\lambda_k)}] = \phi(h_{\lambda_1}) \cdots \phi(h_{\lambda_k})$$

so that ϕ respects multiplication. To prove that (3.6) holds we use the description of M^λ given in Example 3.4.2 (2). By induction, it suffices to show that

$$M^{(\lambda_1,\ldots,\lambda_k)} \circ M^{\lambda_{k+1}} = M^{(\lambda_1,\ldots,\lambda_{k+1})}$$

where $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n and $\lambda_k \geq \lambda_{k+1}$. Hence, fix a numbering T of λ whereby the numbers 1 to n are entered into T in order from left to right, top to bottom (so that the top row has entries $1 < \cdots < \lambda_1$). Similarly, fix a numbering U of (λ_{k+1}) which has entries $n+1, \ldots, n+k$ placed in order from left to right. By definition we have

$$\begin{aligned} M^{(\lambda_1,\ldots,\lambda_k)} \circ M^{\lambda_{k+1}} &= \mathbb{C}[S_{n+\lambda_{k+1}}] \otimes_{\mathbb{C}[S_n \times S_{\lambda_{k+1}}]} (M^\lambda \otimes M^{(\lambda_{k+1})}) \\ &= \mathbb{C}[S_{n+\lambda_{k+1}}] \otimes_{\mathbb{C}[S_n \times S_{\lambda_{k+1}}]} \left((\mathbb{C}[S_n] \otimes_{\mathbb{C}[R(T)]} \mathbb{C}) \otimes (\mathbb{C}[S_{\lambda_{k+1}}] \otimes_{\mathbb{C}[R(U)]} \mathbb{C})\right). \end{aligned}$$

Then, since $R(T) = S_{\lambda_1} \times \cdots \times S_{\lambda_k}$, we have

$$\mathbb{C}[R(T)] = \mathbb{C}[S_{\lambda_1}] \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}[S_{\lambda_k}]$$

and we also have $R(U) = S_{\lambda_{k+1}}$. Hence,

$$\begin{aligned} M^{(\lambda_1,\ldots,\lambda_k)} \circ M^{\lambda_{k+1}} &= \mathbb{C}[S_{n+\lambda_{k+1}}] \otimes_{\mathbb{C}[S_n] \otimes_{\mathbb{C}} \mathbb{C}[S_{\lambda_{k+1}}]} \left((\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_{\lambda_1}] \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}[S_{\lambda_k}]} \mathbb{C}) \otimes (\mathbb{C}[S_{\lambda_{k+1}}] \otimes_{\mathbb{C}[S_{\lambda_{k+1}}]} \mathbb{C})\right) \\ &= \mathbb{C}[S_{n+\lambda_{k+1}}] \otimes_{\mathbb{C}[S_n] \otimes_{\mathbb{C}} \mathbb{C}[S_{\lambda_{k+1}}]} \left((\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_{\lambda_1}] \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}[S_{\lambda_k}]} \mathbb{C}) \otimes \mathbb{C}\right) \\ &= \mathbb{C}[S_{n+\lambda_{k+1}}] \otimes_{\mathbb{C}[S_{\lambda_1}] \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}[S_{\lambda_{k+1}}]} \mathbb{C} \\ &= M^{(\lambda_1,\ldots,\lambda_{k+1})}. \end{aligned}$$

Thus, (3.6) holds and so ϕ is a ring homomorphism.

We now show that the $[M^\lambda]$ form a basis for R. Then it will follow that ϕ is an isomorphism of $\mathbb{Z}$-algebras. To prove this it suffices to show that the $[M^\lambda]$ are linearly independent because the set of $[S^\lambda]$ is a basis for R and there is an M^λ for every S^λ. Suppose that $\sum_\lambda r_\lambda [M^\lambda] = 0$. Then using a similar argument to the one used in the proof of Lemma 2.2.5, order the partitions lexicographically and let $[M^{\lambda'}]$ be the maximal term in the sum such that $r_{\lambda'} \neq 0$. Thus we can use Theorem 3.3.2 to express $M^{\lambda'}$ as a sum of S^λ. Therefore, since $\lambda \trianglelefteq \mu$ implies $\lambda \leq \mu$, we have that $S^{\lambda'}$ only occurs once in $\sum_\lambda r_\lambda [M^\lambda] = 0$ (specifically in the decomposition of $M^{\lambda'}$) and thus occurs with coefficient $r_{\lambda'} \neq 0$. This shows that the $[M^\lambda]$ are linearly independent.

Finally, we show that $\phi(s_\lambda) = [S^\lambda]$ by making use of the fact that ϕ is an isometry[5]. In Theorem 3.3.2 we saw that $[M^\lambda] = [S^\lambda] + \sum_{\nu \triangleright \lambda} k_{\nu\lambda} [S^\nu]$ and (2.9) gives us that $h_\lambda = s_\lambda + \sum_{\nu \triangleright \lambda} K_{\nu\lambda} s_\nu$. Then, as $\phi(h_\lambda) = [M^\lambda]$, we must have

$$\phi(s_\lambda) + \sum_{\nu \triangleright \lambda} K_{\nu\lambda} \phi(s_\nu) = [S^\lambda] + \sum_{\nu \triangleright \lambda} k_{\nu\lambda} [S^\nu]. \tag{3.7}$$

Using this we aim to show that $[S^\lambda]$ appears in the decomposition of $\phi(s_\lambda)$ with coefficient 1. Assume that this is true for now. Therefore that we can write

$$\phi(s_\lambda) = [S^\lambda] + \sum m_{\nu\lambda} [S^\nu]$$

[5]The proof of this is omitted, see [3, Page 92]

for some integers $m_{\nu\lambda}$. Therefore, recalling Remark 2.3.4, we can use that ϕ is an isometry to obtain

$$1 = (s_\lambda, s_\lambda) = (\phi(s_\lambda), \phi(s_\lambda)) = 1 + \sum (m_{\nu\lambda})^2$$

and so the $m_{\nu\lambda}$ must vanish. Hence, $\phi(s_\lambda) = [S^\lambda]$.

Now we prove that $[S^\lambda]$ does indeed appear in $\phi(s_\lambda)$ with coefficient 1. Suppose, for contradiction, that it does not. Then by (3.7) we have that $[S^\lambda]$ must appear with non-zero coefficient in $\phi(s_{\nu_1})$ for some $\nu_1 \rhd \lambda$. We can then apply (3.7) again to $\phi(s_{\nu_1})$ to obtain ν_2 whereby $[S^\lambda]$ appears with non-zero coefficient in $\phi(s_{\nu_2})$ and $\nu_2 \rhd \nu_1$. We can thus iterate this process to find that $[S^\lambda]$ appears with non-zero coefficient in $\phi(s_\mu)$, where $\mu = (n)$ for $|\lambda| = n$. However,

$$\phi(s_\mu) = \phi(s_\mu) + \overbrace{\sum_{\nu \rhd \mu} K_{\nu\mu}\phi(s_\nu)}^{=0} = [S^\mu] + \sum_{\nu \rhd \mu} k_{\nu\mu}[S^\nu] = [S^\mu]$$

where we have used that no partition dominates $\mu = (n)$. Hence, we have the desired contradiction and this concludes the proof. ☐

This powerful result allows us to transfer the properties of symmetric polynomials from Section 2.2 directly to properties of irreducible representations of S_n. Firstly, (2.9) yields:

Corollary 3.4.4 (Young's rule)**.**

$$M^\lambda \cong S^\lambda \oplus \bigoplus_{\mu \rhd \lambda} (S^\mu)^{\oplus K_{\mu\lambda}}$$

where $K_{\mu\lambda}$ is the Kostka number.

Also, the Littlewood-Richardson rule (2.7) gives an equivalent rule for Specht modules

Corollary 3.4.5 (Littlewood-Richardson rule - for Specht modules)**.**

$$S^\lambda \circ S^\mu \cong \bigoplus_{\nu} (S^\nu)^{\oplus c^\nu_{\lambda\mu}}.$$

3.5 Dual Construction of Specht Modules

We can also construct Specht modules using column tabloids in place of row tabloids. This will allow us to express Specht modules as quotients of a larger space. This will allow us to see the similarities between Specht modules and later constructions in Sections 4 and 5.

Definition 3.5.1 (Column Tabloid)**.** A column tabloid is an equivalence class of numberings (with entries in $\{1, \dots, n\}$) on a Young diagram such that two numberings T and T' are equivalent if and only if there exists $q \in C(T)$ such that

$$T' = q \cdot T.$$

The equivalent numberings T' and T have the same or opposite orientation according to whether q has positive or negative sign. We denote the equivalence classes with square brackets and so write $[T'] = [T]$ or $[T'] = -[T]$, depending on the sign of q.

We display column tabloids exactly as expected. For example[6]

$$\begin{array}{|c|c|c|} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & & \end{array} = - \begin{array}{|c|c|c|} 4 & 3 & 2 \\ 1 & 6 & 5 \\ 7 & & \end{array} = \begin{array}{|c|c|c|} 4 & 6 & 2 \\ 1 & 3 & 5 \\ 7 & & \end{array}.$$

[6]Please admire this diagram (there is no LaTeX package for column tabloids!)

Let λ be a partition of n and define $\tilde{M}^\lambda$ to be the complex vector space with basis of column tabloids $[T]$ of shape λ, but with the corresponding basis element defined only up to sign, depending on orientation. The action of S_n on tabloids thus makes $\tilde{M}^\lambda$ into $\mathbb{C}[S_n]$-module, where $\sigma \cdot [T] := [\sigma \cdot T]$.

We can equivalently define $\tilde{M}^\lambda$ to be the quotient of the complex vector space with basis $[T]$, for each numbering T of λ, by the subspace generated by all $[T] - \operatorname{sgn}(q)[T]$ for all $q \in C(T)$.

Analogously to Specht modules, we define $\tilde{S}^\lambda$ to be the submodule spanned by all elements

$$\tilde{v}_T = a_T \cdot [T] = \sum_{p \in R(T)} [p \cdot T].$$

All of the results of Section 3.2 have analogues in this dual setting.

Theorem 3.5.2. *In this dual setting we have:*

(1) If there is a pair of integers in the same row of T' and the same column of T, then $a_{T'} \cdot [T] = 0$. Else, T and T' have the same shape and we have $a_{T'} \cdot [T] = \pm \tilde{v}_{T'}$.

(2) Each $\tilde{S}^\lambda$ is an irreducible representation and every irreducible representation of S_n is isomorphic to exactly on $\tilde{S}^\lambda$.

(3) Let $\phi : \tilde{M}^{\lambda'} \to \tilde{M}^\lambda$ be a homomorphism of representations of S_n. If $\tilde{S}^{\lambda'}$ is not in the kernel of ϕ, then $\lambda. \trianglelefteq \lambda'$.

(4) The elements $\tilde{v}_T$, as T varies over the tableaux of shape λ, form a basis for $\tilde{S}^\lambda$.

(5) We have that

$$\tilde{M}^\lambda \cong \tilde{S}^\lambda \oplus \bigoplus_{\tilde{\nu} \rhd \tilde{\lambda}} (\tilde{S}^\nu)^{\oplus K_{\tilde{\nu}\tilde{\lambda}}}$$

where $K_{\tilde{\nu}\tilde{\lambda}}$ is the Kostka number.

The proofs are omitted since they are almost identical to the proofs of the analogue results Lemma 3.2.4, Theorem 3.2.1, Lemma 3.3.1, Proposition 3.3.3 and Theorem 3.3.2 respectively.

This dual construction now allows us to present S^λ as a quotient of $\tilde{M}^\lambda$. Such realisations are an important step towards relating representations of the general linear group in Chapter 4 to the Specht modules S^λ (specifically Proposition 4.4.2).

Define $\alpha : \tilde{M}^\lambda \to S^\lambda$; $[T] \mapsto v_T$ (with v_T as defined in (3.3)). Then α is well-defined since if $[T] = [T']$ then there is a $\sigma \in C(T)$ with $\operatorname{sgn}(\sigma) = 1$ such that $T' = \sigma \cdot T$. Then, using results obtained in Section 3.1, we have

$$v_{\sigma \cdot T} = \sigma \cdot v_T = \operatorname{sgn}(\sigma) \cdot v_T = v_T. \tag{3.8}$$

In addition, we have that α is a homomorphism of S_n-modules since, for a general $\sigma \in S_n$, the first equality in (3.8) shows that α commutes with the group action. Evidently, α is a surjection and so to express S^λ as a quotient of $\tilde{M}^\lambda$ it suffices to find the kernel of α.

To this end let $\mu = \tilde{\lambda}$ be the conjugate of λ, set $\ell = \lambda_1$ to be the length of μ. Then for any $1 \leq j \leq \ell - 1$, and $1 \leq k \leq \mu_{j+1}$, and any numbering T of λ define

$$\pi_{j,k}(T) = \sum [S] \in \tilde{M}^\lambda,$$

where the sum is over all S obtained from T by exchanging the top k elements in the $(j+1)^{\text{th}}$ column of T with k elements in the j^{th} column of T, preserving the vertical orders of each set of k elements. For example,

$$\pi_{1,3}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline 5 & 6 \\ \hline 8 & 7 \\ \hline\end{array}\right) = \left[\begin{array}{|c|c|} 2 & 1 \\ 3 & 4 \\ 6 & 5 \\ 8 & 7 \end{array}\right] + \left[\begin{array}{|c|c|} 2 & 1 \\ 3 & 4 \\ 5 & 8 \\ 6 & 7 \end{array}\right] + \left[\begin{array}{|c|c|} 2 & 1 \\ 4 & 5 \\ 3 & 8 \\ 6 & 7 \end{array}\right] + \left[\begin{array}{|c|c|} 1 & 4 \\ 2 & 5 \\ 3 & 8 \\ 6 & 7 \end{array}\right].$$

Now define Q^λ to be the subspace of $\tilde{M}^\lambda$ spanned by all elements of the form

$$[T] - \pi_{j,k}(T) \tag{3.9}$$

as T varies over all numberings T of λ, and j and k vary as above. The relations defined in (3.9) are called the *quadratic relations* which will manifest themselves in a few different ways in later sections. Note that, since $\pi_{j,k}$ commutes with the action of S_n, we have that Q^λ is an S_n-module.

Proposition 3.5.3. *For any partition λ we have*

$$S^\lambda \cong \frac{\tilde{M}^\lambda}{Q^\lambda}.$$

Sketch proof: We first show that $[T]$, as $[T]$ varies over the tableaux of shape λ, span $\tilde{M}^\lambda/Q^\lambda$. Define an ordering on fillings of λ whereby $T' \succ T$ if, in the right-most column in which they differ, the lowest different entry in said column has a larger entry in T'. It suffices to show that for any numbering T, that we can use the relations in Q^λ to express $[T]$ as a linear combination of classes $[S]$ with $S \succ T$. Indeed, $S \succ T$ means that S is "closer"[7] to being a tableau and so the fact that the $[T]$, as T varies over tableaux of shape λ, are spanning will thus follow. We assume that the columns of T are strictly increasing. If not then there is a numbering $T' \succ T$ that has strictly increasing columns and $[T] = \pm[T']$. Now suppose that T is not a tableau. Suppose that the k^{th} entry of the j^{th} column is strictly larger than the k^{th} entry of the $(j+1)^{\text{th}}$ column. Then each of the numberings S that appear in $\pi_{j,k}(T)$ have $S \succ T$ (after reordering the columns) and so the result follows.

Now we will have an isomorphism if Q^λ is in the kernel of α. Indeed, this would mean that α defines a surjection $\tilde{M}^\lambda/Q^\lambda \to S^\lambda$. By the first part of the proof, we know that the dimension of $\tilde{M}^\lambda/Q^\lambda$ is at most f^λ, the number of standard tableaux on λ. On the other hand, by Proposition 3.3.3, we have that $\dim(S^\lambda) = f^\lambda$. Therefore α determines an isomorphism $\tilde{M}^\lambda/Q^\lambda \to S^\lambda$. Proving that Q^λ is in the kernel of α is detail heavy and is thus omitted. □

Corollary 3.5.4. *The Specht module S^λ is the vector space with generators v_T, as T varies over numberings of λ, and with relations of the form $v_T - \sum v_S$, where the sum is over all S obtained from T by exchanging the top k elements of one column with any k elements of the preceding column, maintaining the vertical orders of each set exchanged. (There is one such relation for each numbering T, each choice of adjacent columns, and each k at most equal to the length of the shorter column).*

Proof: Note, in the proof of Proposition 3.5.3, that α maps $[T]$ to v_T. Then the result follows easily using the relations that define Q^λ. □

Remark 3.5.5. The first part of the proof of Proposition 3.5.3 shows that $[T]$, as T varies over tableaux of shape λ, span $\tilde{M}^\lambda/Q^\lambda$. In particular, the proof gives us a methodology, called the "straightening algorithm", for expressing an element of $\tilde{M}^\lambda/Q^\lambda$ as a linear combination of classes of tableaux. For example, consider the column tabloid $[T]$ below. As in the proof, the 2^{nd} entry in the 1^{st} column, namely 4, is strictly larger than its neighbour, namely 3. Hence, we change the representative of $[T]$ in $\tilde{M}^\lambda/Q^\lambda$ to $\pi_{1,2}(T)$ so that only the top row of the resulting tabloids fail to satisfy the increasing property of tableaux.

$$\left[\begin{array}{|c|c|} 1 & 2 \\ 4 & 3 \\ 5 & 6 \end{array}\right] = \left[\begin{array}{|c|c|} 2 & 1 \\ 3 & 4 \\ 5 & 6 \end{array}\right] - \left[\begin{array}{|c|c|} 2 & 1 \\ 3 & 5 \\ 4 & 6 \end{array}\right] + \left[\begin{array}{|c|c|} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{array}\right].$$

We then apply the same process to the first two tabloids above and cancel to obtain

$$\left[\begin{array}{|c|c|} 1 & 2 \\ 4 & 3 \\ 5 & 6 \end{array}\right] = \left[\begin{array}{|c|c|} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{array}\right] - \left[\begin{array}{|c|c|} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{array}\right] - \left[\begin{array}{|c|c|} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{array}\right] - \left[\begin{array}{|c|c|} 1 & 2 \\ 3 & 5 \\ 4 & 6 \end{array}\right] + \left[\begin{array}{|c|c|} 1 & 3 \\ 2 & 5 \\ 4 & 6 \end{array}\right].$$

[7] In the sense that S has fewer rows breaking the "increasing" property of tableau.

The straightening algorithm, using the isomorphism $S^\lambda \cong \tilde{M}^\lambda/Q^\lambda$, thus also gives us a procedure for expressing any generator v_T of S^λ in terms of generators $v_{T'}$ where T' is a tableau.

Remark 3.5.6. It can also be shown that $\tilde{S}^\lambda$ is isomorphic to S^λ (which gives another proof of Theorem 3.5.2 (2)) and we can dually construct an isomorphism between $\tilde{S}^\lambda$ and a quotient of M^λ.

3.6 Alternative Construction of Specht Modules

We now make one final construction of the Specht modules S^λ [1]. The goal of this section is to prove the following result:

Proposition 3.6.1. *Fix a tableau T_0 of shape $\lambda \vdash n$. Then*[8]

$$S^\lambda \cong \mathbb{C}[S_n] \cdot c_{T_0}$$

where $c_{T_0} = b_{T_0} \cdot a_{T_0}$ is as defined in Section 3.1.

Firstly, we note that there is a one-to-one correspondence between tabloids $\{T\}$ and sums of the form $a_T \cdot T = \sum_{\sigma \in R(T)} \sigma \cdot T$. Indeed, we have $\{T\} = \{T'\}$ if and only if $a_T \cdot T = a_{T'} \cdot T'$. Now recall that S^λ is the subspace of M^λ that is spanned by the elements v_T, where

$$v_T = b_T \cdot \{T\} = \sum_{q \in C(T)} \operatorname{sgn}(q)\{q \cdot T\} \tag{3.10}$$

as defined in equation (3.3). Thus, for each numbering T, we can associate v_T with

$$b_T \cdot (a_T \cdot T). \tag{3.11}$$

Furthermore, we can fix a tableau T_0 of shape λ and write every numbering T in the form $T = \pi \cdot T_0$ for some $\pi \in S_n$. Therefore, the elements in (3.11) can be identified with the elements of $\mathbb{C}[S_n] \cdot c_{T_0}$ given by

$$b_{\pi T_0} a_{\pi T_0} \pi = \pi b_{T_0} a_{T_0}$$

for $\pi \in S_n$, where we have used that $b_{\pi T} = \pi b_T \pi^{-1}$ and $a_{\pi T} = \pi a_T \pi^{-1}$. We have thus constructed a map $v_{\pi T_0} \mapsto \pi b_{T_0} a_{T_0}$ from S^λ to $\mathbb{C}[S_n] \cdot c_{T_0}$.

Proof of Proposition 3.6.1: We show that the map defined above is an isomorphism of S_n-modules. Firstly, it is clear that this map is surjective and a homomorphism of S_n-modules. It remains to show that S^λ and $\mathbb{C}[S_n] \cdot c_{T_0}$ have the same dimension. However, this follows from the *Hook length formula* [4, p.57] which states that the number of tableaux of shape λ is given by

$$f_\lambda = \frac{n!}{\prod_{i \le \lambda_j} h(i,j)}$$

where (i,j) is the box on the diagram of shape λ in the i^{th} row and j^{th} column, and $h(i,j)$, called the *hook length*, is the number of boxes (i',j') on λ where $i' \ge i,\ j' = j$ or $i' = i,\ j' \ge j$. However, it can also be shown that [2, p.119]

$$\dim(\mathbb{C}[S_n] \cdot c_{T_0}) = \frac{n!}{\prod_{i \le \lambda_j} h(i,j)}.$$

Therefore, using Proposition 3.3.3, the dimensions of the two spaces are the same and so

$$S^\lambda \cong \mathbb{C}[S_n] \cdot c_{T_0}.$$

This concludes the proof. □

Remark 3.6.2. Dually to Proposition 3.6.1 we can also construct an isomorphism between $\tilde{S}^\lambda$ and $\mathbb{C}[S_n] \cdot a_{T_0} \cdot b_{T_0}$.

[8] This construction is often referred to as *Zoë's construction.*

◇ **Remark 3.6.3.** It is intuitive to see that our four constructions of Specht modules[9] are essentially the same. Multiplying a numbering T by the Young symmetrizers a_T and b_T is the analogue of taking the column tabloid $[T]$ or the row tabloid $\{T\}$ respectively. Indeed, just under Proposition 3.6.1 we saw that $a_T \cdot T$ can be identified with $\{T\}$ and, similarly, we also have that $b_T \cdot T$ can be identified with $[T]$. Therefore our constructions can be viewed as four different ways of constructing the same equivalence classes in $\mathbb{C}[S_n]$, where each class can be represented by a tableau.

4 Representations of the General Linear Group

In this section we will construct all[10] irreducible representations of the general linear group $\mathrm{GL}(E)$. Furthermore, similarly to Section 3.4, we will define a representation ring that contains all the representations of $\mathrm{GL}(E)$ (for E of fixed dimension). This representation ring will allow us to construct homomorphisms that relate Specht modules S^λ with irreducible representations of $\mathrm{GL}(E)$.

4.1 Schur Module Construction

Let E be a finite-dimensional complex vector space of dimension m. For each partition λ of n we will construct an irreducible representation, denoted by E^λ, of $\mathrm{GL}(E)$. Let $E^{\times\lambda}$ denote the cartesian product of n copies of E so that an element of $E^{\times\lambda}$ can be thought of as a Young diagram where each box contains a vector in E. Now let F be a complex vector space, we will later define the representations E^λ to be the universal target module of maps $\phi : E^{\times\lambda} \to F$ satisfying the following properties

(1) ϕ is $\mathbb{C}$-multilinear;

(2) ϕ is alternating in the entries of any column of λ;

(3) for any $\mathbf{v}$ in $E^{\times\lambda}$, we have $\phi(\mathbf{v}) = \sum \phi(\mathbf{w})$, where the sum is over all $\mathbf{w}$ obtained from $\mathbf{v}$ by exchanging between two given columns, with a given subset of boxes in the right-hand chosen column.

Note that (1) and (2) imply that $\phi(\mathbf{v}) = -\phi(\mathbf{v}')$, where $\mathbf{v}'$ is obtained from $\mathbf{v}$ by interchanging two entries in a column. For example, with $\lambda = (2, 2, 2)$, we have

$$\phi\left(\begin{array}{|c|c|}\hline y & u \\ \hline x & v \\ \hline z & w \\ \hline\end{array}\right) + \phi\left(\begin{array}{|c|c|}\hline x & u \\ \hline y & v \\ \hline z & w \\ \hline\end{array}\right) \overset{(1)}{=} \phi\left(\begin{array}{|c|c|}\hline x+y & u \\ \hline x+y & v \\ \hline z & w \\ \hline\end{array}\right) \overset{(2)}{=} 0 \tag{4.1}$$

where the entries in the boxes are vectors in E. The presence of (1) and (2) also means that (3) can be altered so that the sum is only over exchanges where the boxes are chosen from the top of the column. For example, let our given subset of the right-hand column be the middle box in a tableau of shape $\lambda = (2, 2, 2)$. Then use (4.1) to swap the entry in the middle box with the entry in the top box, apply the relations (3), where the given subset of the right-hand column is the top box, and finally use (4.1) to put everything back. Explicitly,

$$\begin{aligned}\phi\left(\begin{array}{|c|c|}\hline x & u \\ \hline y & v \\ \hline z & w \\ \hline\end{array}\right) &\overset{u\leftrightarrow v}{=} -\phi\left(\begin{array}{|c|c|}\hline x & v \\ \hline y & u \\ \hline z & w \\ \hline\end{array}\right) \overset{(3)}{=} -\phi\left(\begin{array}{|c|c|}\hline v & x \\ \hline y & u \\ \hline z & w \\ \hline\end{array}\right) - \phi\left(\begin{array}{|c|c|}\hline x & y \\ \hline v & u \\ \hline z & w \\ \hline\end{array}\right) - \phi\left(\begin{array}{|c|c|}\hline x & z \\ \hline y & u \\ \hline v & w \\ \hline\end{array}\right) \\ &= \phi\left(\begin{array}{|c|c|}\hline v & u \\ \hline y & x \\ \hline z & w \\ \hline\end{array}\right) + \phi\left(\begin{array}{|c|c|}\hline x & u \\ \hline v & y \\ \hline z & w \\ \hline\end{array}\right) + \phi\left(\begin{array}{|c|c|}\hline x & u \\ \hline y & z \\ \hline v & w \\ \hline\end{array}\right).\end{aligned}$$

[9] S^λ in Section 3.1, $\tilde{S}^\lambda$ in Section 3.5, $\mathbb{C}[S_n] \cdot a_{T_0} \cdot b_{T_0}$ and $\mathbb{C}[S_n] \cdot b_{T_0} \cdot a_{T_0}$ in Section 3.6.

[10] All finite-dimensional, holomorphic representations. Holomorphic representations are defined in Section 4.3.

The properties $(1)-(3)$ appear in the following useful identity for the multiplication of determinants:

Lemma 4.1.1 (Sylvester)**.** *For any $p \times p$ matrices M and N, and $1 \leq k \leq p$,*

$$\det(M) \cdot \det(N) = \sum \det(M') \cdot \det(N'),$$

where the sum is over all pairs of matrices (M', N') obtained from M and N by exchanging a fixed set of k columns of N with any k columns of M, preserving the ordering of the columns.

The proof is omitted (see [3, pp.108-109]).

Definition 4.1.2 (Schur Module)**.** We define the *Schur module* E^λ to be the universal target module of such maps ϕ, i.e. this means that E^λ is a complex vector space, we have a map $E^{\times\lambda} \to E^\lambda$, denoted by $\mathbf{v} \mapsto \mathbf{v}^\lambda$, which satisfies $(1)-(3)$, and if F is another such complex vector space then there is a unique linear map $\tilde{\phi}$ which satisfies the following commuting diagram

$$\begin{tikzcd} E^{\times\lambda} \arrow[r, "\mathbf{v}\mapsto\mathbf{v}^\lambda"] \arrow[dr, "\phi"] & E^\lambda \arrow[d, "\tilde{\phi}"] \\ & F \end{tikzcd}$$

Examples 4.1.3. (1) Consider the case $\lambda = (n)$, which corresponds to a tableau with just one row. This means property (2) may be dropped and (3) says that all entries commute. Hence, $E^{(n)}$ is the symmetric power $\mathrm{Sym}^n(E)$.

(2) For $\lambda = (1^n)$ we have a single column and so property (3) may be dropped and (2) says that the entries are alternating. Thus $E^{(1^n)}$ is the exterior algebra $\bigwedge^n(E)$.

In order to show that E^λ exists for any λ, we construct it by considering each property $(1)-(3)$ in turn. Firstly, the universal module with property (1) is the tensor product $E^{\otimes\lambda}$ of n copies of E. We use this notation to emphasise that each copy of E corresponds to a box in λ. Then the universal module satisfying properties (1) and (2) is the quotient of $E^{\otimes\lambda}$ generated by all the elements which have equal entries in a column. If we let $\mu = (\mu_1, \ldots, \mu_\ell)$ denote the heights of the **columns** of λ, that is $\mu = \tilde{\lambda}$, this module can be identified with

$$\bigwedge^{\mu_1} E \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \bigwedge^{\mu_\ell} E.$$

The map from $E^{\times\lambda}$ to $\bigotimes \bigwedge^{\mu_i} E$ is intuitive to construct. For example,

$$\begin{array}{|c|c|c|c|} \hline x_1 & y_1 & z_1 & w_1 \\ \hline x_2 & y_2 & z_2 & \\ \cline{1-3} x_3 & y_3 & & \\ \cline{1-2} \end{array} \mapsto (x_1 \wedge x_2 \wedge x_3) \otimes (y_1 \wedge y_2 \wedge y_3) \otimes (z_1 \wedge z_2) \otimes w_1 \tag{4.2}$$

wherein this case $\bigotimes \bigwedge^{\mu_i} E = \bigwedge^3 E \otimes \bigwedge^3 E \otimes \bigwedge^2 E \otimes E$. This map is denoted by $\mathbf{v} \mapsto \wedge\mathbf{v}$.

Remark 4.1.4. Elements of $\bigotimes \bigwedge^{\mu_i} E$ are strongly linked to the column tabloids we saw in Section 3.5. Indeed, swapping two entries in the column of a column tabloid results in a change of sign. This mimics the behaviour of elements in the exterior algebra since swapping entries in a wedge product also results in a change of sign.

Finally, we use property (3) to give

$$E^\lambda = \frac{\bigwedge^{\mu_1} E \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \bigwedge^{\mu_\ell} E}{Q^\lambda(E)} \tag{4.3}$$

where $Q^\lambda(E)$ is the submodule generated by all the elements of the form $\wedge\mathbf{v} - \sum \wedge\mathbf{w}$ where the sum is over all $\mathbf{w}$ obtained from $\mathbf{v}$ as described in (3). It is clear to see that the map $E^{\times\lambda} \to E^\lambda$ induced by $\mathbf{v} \mapsto \wedge\mathbf{v}$ satisfies the properties (1) – (3). The relations given by $Q^\lambda(E)$ are another manifestation of the quadratic relations that we first saw in (3.9).

The fact that E^λ is unique up to canonical isomorphism follows since E^λ is the solution to a universal problem.

Example 4.1.5. Let $\lambda = (2,1,1)$ and so $\mu = (3,1)$. Then we have that E^λ is the quotient of $\bigwedge^3 E \otimes E$ generated by the submodule generated by all elements of the form $x \wedge y \wedge z \otimes u - u \wedge y \wedge z \otimes x - x \wedge u \wedge z \otimes y - x \wedge y \wedge u \otimes z$.

◇ **Remark 4.1.6.** Following on from Remark 4.1.4, taking the quotient of $\bigotimes \bigwedge^{\mu_i} E$ by $Q^\lambda(E)$ is entirely analogous to taking the quotient of $\tilde{M}^\lambda$ by Q^λ. Therefore, using Proposition 3.5.3, we can already see that there is a strong relationship between Schur modules E^λ and Specht modules S^λ.

4.2 Schur Module Generators

In this section we aim to show that if E has basis $e_1, \dots, e_m$, then E^λ has basis $\{e_T\}$, as T varies over the tableaux on λ with entries in $[m]$ where e_T is defined as follows: take a filling T of λ with elements in $[m]$, then we can build an element in E^λ by replacing any i in a box of T by the element e_i. Then e_T is the image of this element in E^λ.

We first need an alternative construction of E^λ.

Lemma 4.2.1. *If E has basis $e_1, \dots, e_m$, then $E^\lambda \cong E^{\otimes\lambda}/Q$, where $E^{\otimes\lambda}$ has basis comprising elements e_T for all fillings T of λ with entries in $[m]$, and Q is generated by the elements*

(i) e_T, where T has equal entries in any column;

(ii) $e_T + e_{T'}$, where T' is obtained from T by interchanging two entries in a column;

(iii) $e_T - \sum e_S$, where the sum is over all S obtained from T by an exchange as in (3).

Proof: This follows from the construction (4.3) and noting that the map $E^{\otimes\lambda} \to E^\lambda$ is multilinear. (For more details see [3, p.108].) □

Before proving that E^λ has basis $\{e_T\}$, as T varies over the tableaux with entries in $[m]$, we show that there is a linear map from E^λ to a subring of the polynomial ring $\mathbb{C}[Z_{1,1}, Z_{1,2}, \dots, Z_{n,m}]$ where $Z_{i,j}$ are indeterminates for $1 \le i \le n$, $1 \le j \le m$. This will allow us to work in $\mathbb{C}[Z_{1,1}, Z_{1,2}, \dots, Z_{n,m}]$ making said result easier to prove.

For each p-tuple $i_1, \dots, i_p$ of integers from $[m]$, with $p \le n$, set

$$D_{i_1,\dots,i_p} = \det \begin{bmatrix} Z_{1,i_1} & \cdots & Z_{1,i_p} \\ \vdots & & \vdots \\ Z_{p,i_1} & \cdots & Z_{p,i_p} \end{bmatrix}.$$

For an arbitrary filling T of λ with numbers in $[m]$, define

$$D_T = \prod_{j=1}^{\ell} D_{T(1,j),T(2,j),\dots,T(\mu_j,j)}, \tag{4.4}$$

where $\mu = (\mu_1, \dots, \mu_\ell)$ is the conjugate of λ and $T(i,j)$ is the entry of T in the i^{th} row and j^{th} column.

Example 4.2.2. Let $T = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \\ \cline{1-1}\end{array}$ and so we have

$$D_T = D_{3,1} D_2 = \begin{vmatrix} Z_{1,3} & Z_{1,1} \\ Z_{2,3} & Z_{2,1} \end{vmatrix} \cdot Z_{1,2} = Z_{1,2}Z_{1,3}Z_{2,1} - Z_{1,1}Z_{1,2}Z_{2,3}.$$

Lemma 4.2.3. *If E has basis $e_1, \dots, e_m$, then there is a homomorphism of $\mathbb{C}$-vector spaces*

$$E^\lambda \to \mathbb{C}[Z_{1,1}, Z_{1,2}, \dots, Z_{n,m}]$$

such that $e_T \mapsto D_T$.

Proof: Using Lemma 4.2.1 it suffices to show that the D_T satisfy $(i)-(iii)$. Indeed, we would then have that there is a unique linear map from E^λ to the span of D_T in $\mathbb{C}[Z_{1,1}, Z_{1,2}, \ldots, Z_{n,m}]$. Firstly, if T has two equal entries in column j, then $D_{T(1,j),T(2,j),\ldots,T(\mu_j,j)}$ is the determinant of a matrix with two equal columns and thus $D_{T(1,j),T(2,j),\ldots,T(\mu_j,j)}$, and hence D_T, is zero. This gives us (i). Secondly, suppose that T' is obtained from T by transposing two entries in the j^{th} column. Then $D_{T(1,j),T(2,j),\ldots,T(\mu_j,j)}$ and $D_{T'(1,j),T'(2,j),\ldots,T'(\mu_j,j)}$ are the determinant of the same matrix, except that two columns have been swapped. Therefore, by the properties of determinants, we have that $D_T = -D_{T'}$ so that (ii) holds.

Finally, consider an exchange as in (3). Suppose that the exchange takes place between the i^{th} and j^{th} columns of T. Let $i_1, \ldots, i_p$ and $j_1, \ldots, j_q$ be the entries in these columns respectively. Define $p \times p$ matrices

$$M = \begin{pmatrix} Z_{1,i_1} & \ldots & Z_{1,i_p} \\ \vdots & & \vdots \\ Z_{p,i_1} & \ldots & Z_{p,i_p} \end{pmatrix}, \qquad N = \begin{pmatrix} Z_{1,j_1} & \ldots & Z_{1,j_q} & 0 \\ \vdots & & \vdots & \\ Z_{p,j_1} & \ldots & Z_{p,j_q} & I_{p-q} \end{pmatrix}$$

and observe that $\det M = D_{T(1,i),T(2,i),\ldots,T(p,i)}$ and $\det N = D_{T(1,j),T(2,j),\ldots,T(q,j)}$. Hence,

$$D_T = (\det M \cdot \det N) \prod_{s \neq i,j} D_{T(1,s),T(2,s),\ldots,T(\mu_s,s)}.$$

From here we can apply Sylvester's Lemma (4.1.1) to M and N with the exchange between columns of M and the fixed subset of the columns of N that correspond to the same fixed subset of boxes in the j^{th} column of T. This gives us (iii) and so we're done. □

We can now prove the main result of Section 4.2.

Theorem 4.2.4. *If E has basis $e_1, \ldots, e_m$, then E^λ has basis e_T, as T varies over the tableaux on λ with entries in $[m]$. Moreover, the map from E^λ to $\mathbb{C}[Z_{1,1}, Z_{1,2}, \ldots, Z_{n,m}]$ is injective and its image D^λ has basis D_T, as T varies over the tableaux on λ with entries in $[m]$.*

Proof: Consider the ordering $T' \succ T$ as defined in the proof of Proposition 3.5.3. Let T be some filling of λ with entries in $[m]$. Using $E^\lambda \cong E^{\otimes\lambda}/Q$ we show that e_T can be written as a linear combination of e_S, where $S \succ T$, and elements of Q. Indeed, $S \succ T$ means that S is "closer" to being a tableau (as we saw in the proof of Proposition 3.5.3) and so the fact that the e_T, as T varies over tableaux, are spanning will thus follow. Now we may use the properties (i) and (ii) of Lemma 4.2.1 to assume that the columns of T are strictly increasing. If not then, either T has two equal entries in a column and is thus zero, or there is a filling $T' \succ T$ that has strictly increasing columns and is equivalent to T in this presentation. Now suppose that T is not a tableau. Suppose that the k^{th} entry of the j^{th} column is strictly larger than the k^{th} entry of the $(j+1)^{\text{th}}$ column. Then using sums as in (iii) we can write $e_T = \sum e_S$ where the sum is over all fillings S whereby the top k entries in the $(j+1)^{\text{th}}$ column are replaced with k entries in the j^{th} column. Then, since the columns of T are strictly increasing, this guarantees that each filling S (after reordering the columns) satisfies $S \succ T$.

We now show that the e_T are linearly independent. By Lemma 4.2.3 it suffices to show that the D_T are linearly independent (as T varies over tableaux). Firstly we define an ordering on monomials in the variables $Z_{i,j}$. We say that $Z_{i,j} < Z_{i',j'}$ if $i < i'$ or if we have $i = i'$ and $j < j'$. Using this we order the monomials lexicographically whereby $M_1 < M_2$ if the smallest $Z_{i,j}$ that occurs (in either) has a smaller power in M_1 than in M_2. Note that this ordering is preserved by multiplication. It follows that the smallest monomial in $D_{i_1,\ldots,i_p}$, if i_1 through i_p are strictly increasing, is the diagonal term $Z_{1,i_1}, \ldots, Z_{p,i_p}$. Hence, if T has strictly increasing columns, the smallest monomial in D_T is

$$\prod (Z_{i,j})^{m_T(i,j)},$$

where $m_T(i,j)$ is the number of times j occurs in the i^{th} row of T. Notice that this monomial occurs with a coefficient of 1.

Now suppose that $\sum r_T D_T = 0$ where the sum is over tableaux T. Now order the tableaux in this sum as follows. Say that T is smaller than T' if for the smallest i for which there exists a j with $m_T(i,j) \neq m_{T'}(i,j)$, and the smallest such j, then $m_T(i,j) < m_{T'}(i,j)$. Let T be the maximal tableau in this ordering. Then

$$r_T D_T + \sum_{T \neq T'} r_{T'} D_{T'} = 0$$

and the above ordering guarantees that the smallest monomial in D_T, namely $\prod (Z_{i,j})^{m_T(i,j)}$, is the smallest monomial in the entire sum. Thus if $r_T \neq 0$ (else ignore T and take next smallest tableau) the coefficient of $\prod (Z_{i,j})^{m_T(i,j)}$ in the sum must be r_T. This shows linear independence.

Finally, by the above, we have that the map $E^\lambda \to D^\lambda$ sends basis vectors to basis vectors and is therefore a linear isomorphism. Hence, $E^\lambda \to \mathbb{C}[Z_{1,1}, Z_{1,2}, \dots, Z_{n,m}]$ is injective. □

Before using Schur modules E^λ to construct all the irreducible representations of $\mathrm{GL}(E)$ we give one more lemma that will later allow us to decompose the E^λ (in a manner we will see later). This will consequently help us prove the results of Section 4.3 about irreducible representations of $\mathrm{GL}(E)$.

Lemma 4.2.5. *Let $g = (g_{i,j}) \in M_m(\mathbb{C})$ and let T be a filling with entries $j_1, \dots, j_n$ in its n boxes. Then*

$$g \cdot e_T = \sum g_{i_1,j_1} \cdot \dots \cdot g_{i_n,j_n} e_{T'},$$

where the sum is over the m^n fillings T' obtained from T by replacing the entries $(j_1, \dots, j_n)$ with $(i_1, \dots, i_n)$.

◇ **Example 4.2.6.** Here is an example of the formula given in Lemma 4.2.5. Suppose E has dimension 2 and basis e_1, e_2.

(1) Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{C})$ and let $T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$. Then

$$g \cdot e_T = g \cdot \begin{array}{|c|}\hline e_1 \\ \hline e_2 \\ \hline \end{array} = ab \cdot \begin{array}{|c|}\hline e_1 \\ \hline e_1 \\ \hline \end{array} + ad \cdot \begin{array}{|c|}\hline e_1 \\ \hline e_2 \\ \hline \end{array} + bc \cdot \begin{array}{|c|}\hline e_2 \\ \hline e_1 \\ \hline \end{array} + cd \cdot \begin{array}{|c|}\hline e_2 \\ \hline e_2 \\ \hline \end{array}$$

which is indeed a sum over $m^n = 2^2 = 4$ fillings of T.

(2) Let $g = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in M_2(\mathbb{C})$ and let $T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array}$. Then

$$g \cdot e_T = g \cdot \begin{array}{|c|c|}\hline e_1 & e_2 \\ \hline e_2 \\ \cline{1-1} \end{array} = \begin{array}{|c|c|}\hline e_1 & e_1 \\ \hline e_1 \\ \cline{1-1} \end{array} + \begin{array}{|c|c|}\hline e_1 & e_1 \\ \hline e_2 \\ \cline{1-1} \end{array} + \begin{array}{|c|c|}\hline e_1 & e_2 \\ \hline e_1 \\ \cline{1-1} \end{array} + \begin{array}{|c|c|}\hline e_1 & e_2 \\ \hline e_2 \\ \cline{1-1} \end{array}.$$

Note that in the above examples the structure of E^λ has been ignored. In particular, Young diagrams with equal entries in a column should vanish.

4.3 Irreducible Representations of GL(E)

We now show that the Schur modules E^λ are irreducible representations and that all finite-dimensional (holomorphic) representations of $\mathrm{GL}(E)$ can be described in terms of these representations.

Definition 4.3.1 (Polynomial/holomorphic Representation)**.** A representation V of $\mathrm{GL}(E)$ is said to be *polynomial/holomorphic* if the corresponding map $\rho : \mathrm{GL}(E) \to \mathrm{GL}(V)$ is given by polynomials/holomorphic functions.

Example 4.3.2. [8, p.1] We give a small example of a polynomial representation. Let $\lambda = (2)$ and let E be a 2-dimensional vector space with basis e_1, e_2. Then, using Examples 4.1.3 (1), we have that $E^{(2)} = \mathrm{Sym}^2(E)$ and is a representation of $\mathrm{GL}(E)$ with basis $e_1^2, e_1 e_2, e_2^2$. Now let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(E)$. Then g acts on e_1 and e_2 as

$$e_1 \mapsto a e_1 + c e_2$$

$$e_2 \mapsto be_1 + de_2$$

Thus g acts on the basis e_1^2, e_1e_2, e_2^2 in $E^{(2)}$ as

$$\begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}.$$

Thus, since all the entries are polynomial expressions, we have that $E^{(2)}$ is a polynomial representation (and therefore also a holomorphic representation). In fact, for all partitions λ of n, we have that E^λ is a polynomial representation.

We will show later in this section that all irreducible polynomial representations of $\mathrm{GL}(E)$ are exactly the Schur modules E^λ, where λ varies over all Young diagrams with at most $m = \dim E$ rows. To prove this we first need to look at weight spaces which will allow us to characterise finite-dimensional holomorphic representations.

Choose a basis for E and let $H \leq G$ denote the subgroup of G that consists of diagonal matrices $x = \mathrm{diag}(x_1, \dots, x_m)$. Then a vector v in a representation V is called a *weight vector* with *weight* $\alpha = (\alpha_1, \dots, \alpha_m)$, for integers α_i, if

$$x \cdot v = x_1^{\alpha_1} \cdots x_m^{\alpha_m} v$$

for all $x \in H$. It is a general fact that any representation of $\mathrm{GL}(E)$ is a direct sum of its *weight spaces*. That is, $V = \bigoplus V_\alpha$, where

$$V_\alpha := \left\{ v \in V \mid x \cdot v = \left(\prod x_i^{\alpha_i} \right) v \ \ \forall x \in H \right\}.$$

Lemma 4.3.3. *The Schur module E^λ decomposes into weight spaces*

$$E^\lambda = \bigoplus_\alpha V_\alpha$$

where, for $\alpha = (\alpha_1, \dots, \alpha_m)$, we have that V_α has basis e_T as T varies over all tableaux that have α_i entries i.

◇ *Proof:* Consider the basis $\{e_T\}$ for E^λ, as given in Theorem 4.2.4. It suffices to show that the e_T are weight vectors and have weight as given in the lemma. However, this all follows from Lemma 4.2.5 (For example, consider Examples 4.2.6 (1) with $b = c = 0$.) □

We also define the *highest weight vector*. Let $B \leq G$ be the subgroup of all upper triangular matrices. A weight vector v in V is called a highest weight vector if $B \cdot v = \mathbb{C}^* \cdot v$. Highest weight vectors uniquely determine irreducible finite-dimensional holomorphic representations (see Theorem 4.3.9 later) and, consequently, will be very useful for proving results about the irreducibility of E^λ.

Example 4.3.4. [7, p.33] We decompose $E^{(2)} = \mathrm{Sym}^2(E)$ from Example 4.3.2 into its weight spaces. By said example we see that $\left(\begin{smallmatrix} m & 0 \\ 0 & n \end{smallmatrix}\right) \in H \leq G$ acts on the basis e_1^2, e_1e_2, e_2^2 as

$$\begin{aligned} e_1^2 &\mapsto m^2 e_1^2 \\ e_1e_2 &\mapsto mne_1e_2 \\ e_2^2 &\mapsto n^2 e_2^2. \end{aligned}$$

Thus $E^{(2)} = V_{(2,0)} \oplus V_{(1,1)} \oplus V_{(0,2)}$ where $V_{(2,0)} = \mathrm{span}\{e_1^2\}$, $V_{(1,1)} = \mathrm{span}\{e_1e_2\}$ and $V_{(0,2)} = \mathrm{span}\{e_2^2\}$. Notice that this agrees with Lemma 4.3.3 since the tableaux with shape $\lambda = (2)$ with entries in $\{1, 2\}$ are $\boxed{1}\boxed{1}$, $\boxed{1}\boxed{2}$, $\boxed{2}\boxed{2}$ which correspond to e_1^2, e_1e_2, e_2^2 respectively.

From this example we can also see that all polynomial representations have non-negative weights. Indeed, a negative weight in a polynomial representation will only occur if we have negative exponents and this is absurd.

We can also find *the*[11] highest weight vector. We see that $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in H \leq G$ acts on e_1^2 as

$$e_1^2 \quad \mapsto \quad a^2 e_1^2.$$

Thus e_1^2 is the highest weight vector in $E^{(2)}$ with weight $(2,0)$.

The last result in the example above can be generalised.

Lemma 4.3.5. *The only highest weight vector in E^λ (up to scalar multiplication) is the vector e_T, where $T = U(\lambda)$ is the tableau on λ whose i^{th} row only contains the integer i.*

◇ **Example 4.3.6.** For example if $\lambda = (6,4,2,2,1)$ then $U(\lambda)$ is given by

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \cline{1-4} 3 & 3 \\ \cline{1-2} 4 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}\;.$$

Proof of Lemma 4.3.5: Using Lemma 4.2.5 it is straight forward to show that e_T, for $T = \lambda(U)$, is a highest weight vector. (Consider Example 4.2.6 (1) with $c = 0$.[12]) Indeed, for $g = (g_{i,j}) \in B$ we have

$$g \cdot e_T = \sum g_{i_1,j_1} \cdot \ldots \cdot g_{i_n,j_n} e_{T'},$$

where the sum is as described in Lemma 4.2.5. However, since B consists of upper unitriangular matrices we have $g_{i,j} = 0$ for $i > j$. Now consider $e_{T'}$ with $T \neq T'$. Then let j be the smallest integer such that the j^{th} row of T and T' differ and let i be the smallest entry in the j^{th} row of T' such that T and T' differ in the box corresponding to i. Now if $i > j$ then $g_{i,j} = 0$ and so $e_{T'}$ has zero coefficient in the sum that gives $g \cdot e_T$. On the other hand, if $i < j$, by the form of T we see there must be another entry i in that column. Therefore, by the alternating property of E^λ, we have that $e_{T'} = 0$ in E^λ. We conclude that the only possible non-zero entry in the sum will be a multiple of e_T and so e_T is a highest weight vector.

To conclude we show that $e_{U(\lambda)}$ is the only highest weight vector in E^λ. Consequently, suppose that $T \neq U(\lambda)$ is a tableau with entries in $[m]$ and e_T is a highest weight vector. (Note that we need only consider the case where T is a tableau by Lemma 4.3.3.) Let p be such that the p^{th} row of T is the first row that contains an element larger than p and let q be the smallest such entry. Now define $g = (g_{i,j}) \in B$ where

$$g_{i,j} = \begin{cases} 1 & i = j \text{ or } (i,j) = (p,q) \\ 0 & \text{otherwise.} \end{cases}$$

Then consider T' where T' is the filling obtained from T by changing every instance of q in the p^{th} row of T to a p. Then $e_{T'}$ has coefficient 1 in the sum $g \cdot e_T$ and so e_T is not a highest weight vector. (See Example 4.3.7 for a demonstration of this argument.) □

◇ **Example 4.3.7.** We give an example of a weight vector that is not a highest weight vector. Let E have basis e_1, e_2, e_3 and let $\lambda = (4,2)$. Now adopt the notation at the end of the proof of Lemma 4.3.5 and set

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$$

and thus we obtain $p = 2$, $q = 3$ and

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

[11] See Lemma 4.3.5.

[12] where term with coefficient ab in the sum vanishes by the alternating property of E^λ.

Now e_T corresponds to the vector $(e_1 \wedge e_2) \otimes (e_1 \wedge e_3) \otimes e_1 \otimes e_1$ in E^λ and so compute

$$\begin{aligned} g \cdot e_T &= g \cdot \big((e_1 \wedge e_2) \otimes (e_1 \wedge e_3) \otimes e_1 \otimes e_1\big) \\ &= (e_1 \wedge e_2) \otimes (e_1 \wedge (e_2 + e_3)) \otimes e_1 \otimes e_1 \\ &= (e_1 \wedge e_2) \otimes (e_1 \wedge e_3) \otimes e_1 \otimes e_1 + (e_1 \wedge e_2) \otimes (e_1 \wedge e_2) \otimes e_1 \otimes e_1 \\ &= e_T + (e_1 \wedge e_2) \otimes (e_1 \wedge e_2) \otimes e_1 \otimes e_1, \end{aligned}$$

which is not a multiple of e_T. Thus we see that e_T is not a highest weight vector.

Remark 4.3.8. Note that the highest weight vector is literally the weight vector with the highest weight. For example, using Lemma 4.3.3 we see that for E^λ the weight vector e_T with the highest possible weight α (in the lexicographic ordering) will be given by the tableaux $T = U(\lambda)$.

The final step before characterising all irreducible representations of $\mathrm{GL}(E)$ is to consider some basic facts about representation theory.

Theorem 4.3.9. *A finite-dimensional, holomorphic representation V of $GL(E)$ is irreducible if and only if it has unique highest weight vector, up to multiplication by a scalar.*

In addition, two representations are isomorphic if and only if their highest weight vectors have the same weight.

For a proof of this theorem see Fulton and Harris [4]. Now let $D^{\otimes k}$ denote the one-dimensional representation of E such that $g \mapsto \det g^k$. We are now in a position to give the theorem we have been working towards:

Theorem 4.3.10. *If λ has at most m rows, then the representation E^λ of $GL(E)$ is an irreducible representation with highest weight $\lambda = (\lambda_1, \ldots, \lambda_m)$. These are all the irreducible polynomial representations of $GL(E)$. Furthermore, for any $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_1 \geq \cdots \geq \alpha_m$ integers, there is a unique irreducible representation of $GL(E)$ with highest weight vector α, which can be realised as $E^\lambda \otimes D^{\otimes k}$, for $k \in \mathbb{Z}$ with $\lambda_i = \alpha_i - k \geq 0$ for all i.*

Proof: For the first part we have by Lemma 4.3.5 that each E^λ has unique highest weight vector e_T where $T = U(\lambda)$ and it is not difficult to show that e_T has weight λ. Thus, by Theorem 4.3.9, we have that E^λ is irreducible.

Conversely, by Theorem 4.3.9 again, any representation of $\mathrm{GL}(E)$ with weight $\lambda = (\lambda_1, \ldots, \lambda_m)$ (for $\lambda_1, \ldots, \lambda_m > 0$) must be isomorphic to E^λ. Hence, they define all the irreducible polynomial representations of $\mathrm{GL}(E)$.

Now for the second part of the theorem, since E^λ is irreducible, we have that $E^\lambda \otimes D^{\otimes k}$ is an irreducible representation with highest weight α where $\alpha_i = \lambda_i + k$. Indeed, for $e_T \in E^\lambda$ with $T = U(\lambda)$ and $g = \mathrm{diag}(g_1, \ldots, g_m) \in \mathrm{GL}(E)$ we have

$$g \cdot (e_T \otimes 1) = (g \cdot e_T) \otimes \det(g)^k = (g_1^{\lambda_1} \cdots g_m^{\lambda_m} e_T) \otimes (g_1^k \cdots g_m^k) = g_1^{\lambda_1 + k} \cdots g_m^{\lambda_m + k} (e_T \otimes 1).$$

This concludes the proof. □

4.4 Specht and Schur Module Relationship

We now construct an exact functor from the category of S_n-modules to the category of $\mathrm{GL}(E)$-modules. In particular, this will give us an alternative way to construct the Schur modules E^λ. Let M be a representation of S_n and define

$$E(M) := E^{\otimes n} \otimes_{\mathbb{C}[S_n]} M$$

where for $\sigma \in S_n$ we have $(u_1 \otimes \cdots \otimes u_m) \cdot \sigma = u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(m)}$. Now the natural action of $\mathrm{GL}(E)$ on $E^{\otimes n}$ commutes with the action of S_n, thus we obtain an action of $\mathrm{GL}(E)$ on $E(M)$

$$g \cdot (w \otimes v) = (g \cdot w) \otimes v.$$

This construction defines an exact functor from the category of S_n-modules to the category of $\mathrm{GL}(E)$ modules. That is, a representation M of S_n determines a representation $E(M)$ of $\mathrm{GL}(E)$ and a homomorphism of S_n-modules $\phi : M \to N$ determines a homomorphism of $\mathrm{GL}(E)$-modules $E(\phi) : E(M) \to E(N)$. Furthermore, E sends injective/surjective homomorphisms of S_n-modules to injective/surjective homomorphisms of $\mathrm{GL}(E)$-modules. In addition to being an exact functor, we also have that E preserves direct sum decompositions.

Examples 4.4.1. (1) Take M to be the trivial representation. Then for any $\sigma \in S_n$ we have

$$(u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(m)}) \otimes 1 = (u_1 \otimes \cdots \otimes u_m) \otimes \sigma \cdot 1 = (u_1 \otimes \cdots \otimes u_m) \otimes 1$$

and thus $E(M)$ is the symmetric power $\mathrm{Sym}^n(E)$.

(2) If $M = \mathbb{C}[S_n]$ then $E(M) = E^{\otimes n}$.

(3) If M^λ is as defined in Section 3.1, then

$$E(M^\lambda) \cong \mathrm{Sym}^{\lambda_1}(E) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_k}(E),$$

for $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$. This can be seen by noting that if T is a numbering of λ with distinct entries in $\{1, \ldots, n\}$ then $\{T\}$ is invariant under the action of the row group $R(T)$. More rigorously, the map $\mathbb{C}[S_n] \to M^\lambda$ given by $\sigma \mapsto \sigma \cdot \{T\}$ is a surjective homomorphism of $\mathbb{C}[S_n]$ modules and has kernel generated by elements $p - 1$ for $p \in R(T)$. Thus, since E is an exact functor, we obtain a surjection $E^{\otimes n} \to E(M^\lambda)$, whose kernel is generated by all elements

$$u_{p(1)} \otimes \cdots \otimes u_{p(n)} - u_1 \otimes \cdots \otimes u_n$$

over all $p \in R(T)$.

(4) Now take $M = \tilde{M}^\lambda$ as in Section 3.5. Similarly to above, we have a surjective homomorphism of $\mathbb{C}[S_n]$-modules given by $\mathbb{C}[S_n] \to \tilde{M}^\lambda$; $\sigma \mapsto \sigma \cdot [T]$. This map has kernel generated by elements $q - \mathrm{sgn}(q)1$ for $q \in C(T)$ and again this induces a surjection $E^{\otimes n} \to E(\tilde{M}^\lambda)$ whose kernel is generated by the elements

$$u_{q(1)} \otimes \cdots \otimes u_{q(n)} - \mathrm{sgn}(q) u_1 \otimes \cdots \otimes u_n$$

over all $q \in C(T)$. Thus

$$E(\tilde{M}^\lambda) \cong \bigwedge^{\mu_1}(E) \otimes \cdots \otimes \bigwedge^{\mu_\ell}(E),$$

where $\mu = (\mu_1, \ldots, \mu_\ell)$ is the conjugate of λ. This is analogous to (4.2) whereby each column of a tabloid (with entries being elements in E) is identified with wedge product.

We can now give an alternative construction of E^λ:

Proposition 4.4.2. *There is a canonical isomorphism $E^\lambda \cong E(S^\lambda)$.*

Proof: It is sufficient to show that

$$E^\lambda \cong \frac{E(\tilde{M}^\lambda)}{E(Q^\lambda)}. \tag{4.5}$$

Indeed, by Proposition 3.5.3, we know that $S^\lambda \cong \tilde{M}^\lambda / Q^\lambda$. Then, since E is an exact functor, it sends the short exact sequence

$$0 \to Q^\lambda \to \tilde{M}^\lambda \to S^\lambda \to 0$$

to the short exact sequence

$$0 \to E(Q^\lambda) \to E(\tilde{M}^\lambda) \to E(S^\lambda) \to 0.$$

In order to prove (4.5), we first use the construction of E^λ given in (4.3), which gives us that

$$E^\lambda = \frac{\bigwedge^{\mu_1} E \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \bigwedge^{\mu_\ell} E}{Q^\lambda(E)}.$$

Then, in Examples 4.4.1 (4), we saw that $E(\tilde{M}^\lambda) \cong \bigwedge^{\mu_1}(E) \otimes \cdots \otimes \bigwedge^{\mu_\ell}(E)$. Using the same identification, we also have that $Q^\lambda(E) \cong E(Q^\lambda)$. Indeed, $Q^\lambda(E)$ is the submodule of $\bigwedge^{\mu_1} E \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \bigwedge^{\mu_\ell} E$ generated by all the elements of the form $\wedge \mathbf{v} - \sum \wedge \mathbf{w}$ where the sum is as described in (3) at the start of Section 4.1. On the other hand, Q^λ is the subspace of $\tilde{M}^\lambda$ spanned by all elements of the form

$$[T] - \pi_{j,k}(T)$$

as defined in Section 3.5. Therefore we see that these generators correspond under the identification $E(\tilde{M}^\lambda) \cong \bigwedge^{\mu_1}(E) \otimes \cdots \otimes \bigwedge^{\mu_\ell}(E)$.

Finally, we now have that

$$E^\lambda \cong \frac{E(\tilde{M}^\lambda)}{E(Q^\lambda)} \cong E(S^\lambda)$$

and so this concludes the proof. □

4.5 Characters of Representations of GL(E)

In the next section we construct maps that relate the ring of symmetric functions (Section 2.3), representations of the symmetric group (Section 3.2), polynomial representations of the GL(E) and their characters. In light of this, we first need to discuss the characters of Schur modules.

Recall that for representations V and W it is known that

$$\mathrm{Char}(V \oplus W) = \mathrm{Char}(V) + \mathrm{Char}(W) \qquad \mathrm{Char}(V \otimes W) = \mathrm{Char}(V) \cdot \mathrm{Char}(W). \tag{4.6}$$

Recall also, from Section 4.3, that any representation V of GL(E) can be written as a direct sum of its weight spaces $V = \bigoplus V_\alpha$. Thus, letting χ_V denote that character of V, for $x = \mathrm{diag}(x_1, \ldots, x_m) \in H$ we see that

$$\chi_V(x) = \sum_\alpha \dim(V_\alpha) x^\alpha = \sum_\alpha \dim(V_\alpha) x_1^{\alpha_1} \cdot \ldots \cdot x_m^{\alpha_m},$$

which defines the character for any element in $G = \mathrm{GL}(E)$ using the additive property in (4.6). In particular, the weight space decomposition of E^λ in Lemma 4.3.3 gives us:

Lemma 4.5.1. *The Schur module E^λ has character given by*

$$\mathrm{Char}(E^\lambda) = \sum x^T = s_\lambda(x_1, \ldots, x_m)$$

where the sum is over all tableaux of shape λ with entries in $[m]$ and s_λ is the Schur polynomial defined in Section 2.2.

This result allows us to deduce some useful facts about Schur polynomials and, using that the character of a representation is unique, deduce some useful facts about Schur modules using Schur polynomials.

Firstly, by Proposition 3.3.3 and Theorem 3.2.1, we know that

$$\mathbb{C}[S_n] = \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus f^\lambda}$$

where f^λ is the number of standard tableaux on λ. This gives us

$$E^{\otimes n} = E(\mathbb{C}[S_n]) \cong \bigoplus_{\lambda \vdash n} (E(S^\lambda))^{\oplus f^\lambda} \cong \bigoplus_{\lambda \vdash n} (E^\lambda)^{\oplus f^\lambda} \tag{4.7}$$

where we have used Examples 4.4.1 (2) in the first equality and made use of the fact that E preserves direct sum decompositions. We can use (4.7) to give an alternative proof that the e_T, as T varies over the tableaux on λ with entries in $[m]$, are linearly independent (the second part of the proof of Theorem 4.2.4). Indeed, the first part of the proof of Theorem 4.2.4 gives us that $\dim(E) \leq d_\lambda(m)$, where $d_\lambda(m)$ is the number of tableaux of shape λ with entries in $[m]$. Now from the Robinson-Schensted correspondence [3, p.52] it follows that

$$m^n = \sum_{\lambda \vdash n} f^\lambda d_\lambda(m).$$

Therefore

$$\sum_{\lambda \vdash n} f^\lambda d_\lambda(m) = m^n = \dim(E^{\otimes n}) = \sum_{\lambda \vdash n} f^\lambda \dim(E^\lambda) \leq \sum_{\lambda \vdash n} f^\lambda d_\lambda(m)$$

and so E^λ has dimension $d_\lambda(m)$. Now there are $d_\lambda(m)$ many e_T and they span E^λ (by the first part of the proof of Theorem 4.2.4). Hence, the e_T must be linearly independent.

Now using equations (4.6) and Lemma 4.5.1 (and Theorem 4.3.10) we have that any polynomial representation $V = \bigoplus (E^\lambda)^{\oplus a_\lambda}$ of $\mathrm{GL}(E)$ has character $\sum a_\lambda s_\lambda(x_1, \ldots, x_m)$. This also means that we can determine the highest weights[13] using the character. Indeed, each s_λ has a monomial $x^{U(\lambda)}$, with $U(\lambda)$ as defined in Lemma 4.3.5, from which we can read off a highest weight.

Using the above we can find the decomposition of any polynomial representation of $\mathrm{GL}(E)$ by expressing its character as a sum of Schur polynomials. For example, using Examples 4.1.3 (1) and equation (2.9), since $s_{(\lambda_k)} = h_{(\lambda_k)}$, we're able to deduce that

$$\mathrm{Sym}^{\lambda_1} E \otimes \cdots \otimes \mathrm{Sym}^{\lambda_n} E = E^{(\lambda_1)} \otimes \cdots \otimes E^{(\lambda_n)} \cong E^\lambda \oplus \bigoplus_{\nu \rhd \lambda} (E^\nu)^{\oplus K_{\nu\lambda}}$$

where $K_{\nu\lambda}$ is the Kostka number. Similarly, using Examples 4.1.3 (2) and equation (2.10), since $s_{1^{(\mu_k)}} = e_{\mu_k}$, we have that

$$\bigwedge^{\mu_1} E \otimes \cdots \otimes \bigwedge^{\mu_\ell} E = E^{(1^{\mu_1})} \otimes \cdots \otimes E^{(1^{\mu_\ell})} \cong E^{\tilde{\mu}} \oplus \bigoplus_{\tilde{\nu} \rhd \mu} (E^\nu)^{\oplus K_{\tilde{\nu}\lambda}}.$$

Finally, using the Schur polynomial equivalent of equation (2.7), we have

$$E^\lambda \otimes E^\mu \cong \bigoplus_\nu (E^\nu)^{\oplus c^\nu_{\lambda\mu}}.$$

where $c^\nu_{\lambda\mu}$ is the Littlewood-Richardson number.

4.6 Representation Ring of GL(E)

As mentioned at the start of Section 4.5, we now aggregate the results of Section 2.3, Section 3.2 and Section 4.5. Firstly, we need an analogous construction to the ring of representations of S_n (defined in Section 3.4) for the general linear group $\mathrm{GL}(E)$. Define the *representation ring* of $\mathrm{GL}(E)$, denoted by $\mathcal{R}(m)$, to be the free abelian group on the isomorphism classes of irreducible polynomial representations of $\mathrm{GL}(E)$. This is a commutative ring with operations

$$[V] + [W] := [V \oplus W], \qquad [V] \cdot [W] := [V \otimes_{\mathbb{C}} W].$$

Recall that Λ is the ring of symmetric functions (Section 2.3), R is the ring of representations of the symmetric group (Section 3.4) and $\Lambda(m)$ is the ring of polynomials in the variables $x_1, \ldots, x_m$. We are therefore now in a position to give the main statement of this section (and perhaps the entire project):

Theorem 4.6.1. *There are ring homomorphisms*

$$\Lambda \xrightarrow{\sim} R \twoheadrightarrow \mathcal{R}(m) \xrightarrow{\sim} \Lambda(m) \tag{4.8}$$

where the first and last are isomorphisms and the middle is surjective where the kernel, denoted by W, is the subring of R spanned by $[S^\lambda]$ where $\lambda_{m+1} \neq 0$.

[13] We have *weights* not *weight* because these representations are not necessarily irreducible.

Proof: The first homomorphism is given by $\phi : \Lambda \to R$, which we defined in Section 3.4. Further, ϕ was shown to be an (isometric) isomorphism in Theorem 3.4.3 and the same theorem also showed that ϕ takes the Schur polynomial s_λ to the Specht module S^λ.

The second homomorphism is given by E, defined at the start Section 4.4, which sends a representation M of S_n to a representation $E(M)$ of $\mathrm{GL}(E)$. Indeed, we have already noted that E preserves direct sum decompositions, and, if M is an S_n-module and N an S_m-module, it can be shown that [3, p.118]

$$E(N \circ M) \cong E(N) \otimes E(M)$$

where $N \circ M$ is the representation of S_{n+m} defined in (3.5).

The last homomorphism sends representations of $\mathrm{GL}(E)$ to their characters. It follows from equations (4.6) that this defines a homomorphism $\mathcal{R}(m) \to \Lambda(m)$. Now, to show that this is an isomorphism, we first note that injectivity follows because representations are uniquely determined by their character. To prove surjectivity it suffices to show that the composition $\Lambda \to \Lambda(m)$ is surjective. To this end, we observe that the first of the maps in the composition $\Lambda \to \Lambda(m)$ takes the Schur polynomial s_λ to the Specht module class $[S^\lambda]$, the second takes $[S^\lambda]$ to $[E^\lambda]$ and the third takes E^λ to its character $s_\lambda(x_1, \ldots, x_m)$. By Proposition 2.3.2 the s_λ form a basis for Λ, hence the composition $\Lambda \to \Lambda(m)$ takes a symmetric function f to $f(x_1, \ldots, x_m, 0, \ldots, 0)$. It is clear to see that this is surjective.

The above shows that $E : R \to \mathcal{R}(m)$ is surjective and so it remains to compute the kernel of the middle homomorphism. Notice that the composition $\Lambda \to \Lambda(m)$ has kernel generated by the Schur polynomials s_λ for all λ with more than m rows. Indeed, if λ has more than m rows then a tableau of shape λ with entries in $[m]$ doesn't exist and so $s_\lambda(x_1, \ldots, x_m) = 0$. Therefore we deduce that the kernel of E is the subring of R spanned by $[S^\lambda]$ where $\lambda_{m+1} \neq 0$, namely W. $\square$

The conclusions of Theorem 4.6.1 are summarised in the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & W \arrow[d] & & \\
0 \arrow[r] & \Lambda \arrow[r] & R \arrow[r] \arrow[d] & 0 & \\
 & 0 \arrow[r] & \mathcal{R}(m) \arrow[r] \arrow[d] & \Lambda(m) \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

5 The Grassmannian

In this section we express some of the results of Chapter 4 in terms of symmetric algebras, express the Grassmannian as a subvariety of projective space using the quadratic relations and draw parallels between both of these.

5.1 The Ideal of Quadratic Relations

We firstly use the quadratic relations to define a symmetric algebra that is isomorphic to a direct sum of Schur modules E^λ. We do this because this symmetric algebra will also play a role in expressing the Grassmannian as a subvariety of projective space in Section 5.3.

For a complex vector space V define

$$\mathrm{Sym}\ V = \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n V$$

where $\mathrm{Sym}^0 V = \mathbb{C}$. We can equip Sym V with the natural map

$$\mathrm{Sym}^n V \otimes \mathrm{Sym}^m V \mapsto \mathrm{Sym}^{n+m}(V);\ v_1 \cdots v_n \otimes w_1 \cdots w_m \mapsto v_1 \cdots v_n w_1 \cdots v_m$$

which makes Sym V a commutative (graded) $\mathbb{C}$-algebra. Taking a basis $X_1, \ldots, X_r$ for V we can identify Sym V with the polynomial ring $\mathbb{C}[X_1, \ldots, X_r]$, which is then the space of polynomial functions on V^*. Let E be a vector space of dimension m and fix integers $m \geq d_1 > \cdots > d_s > 0$. We define the algebra

$$\mathrm{Sym}(E; d_1, \ldots, d_s) = \frac{\bigoplus_{(a_1,\ldots,a_s)\in\mathbb{N}^s} \mathrm{Sym}^{a_1}(\bigwedge^{d_1} E) \otimes \cdots \otimes \mathrm{Sym}^{a_s}(\bigwedge^{d_s} E)}{Q}$$

where $Q = Q(E; d_1, \ldots, d_s)$ is the ideal generated by all relations, for $p, q \in \{d_1, \ldots, d_s\}$ with $p \geq q$, of the form

$$\begin{aligned}(v_1 \wedge \cdots \wedge v_p) \cdot (w_1 \wedge \cdots \wedge w_q) - \\ \sum_{i_1 < \cdots < i_k} (v_1 \wedge \cdots \wedge w_1 \wedge \cdots \wedge w_k \wedge \cdots \wedge v_p) \cdot (v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge w_{k+1} \wedge \cdots \wedge w_q)\end{aligned} \tag{5.1}$$

where $v_1, \ldots, v_p, w_1, \ldots, w_q \in E$ and the vectors $w_1, \ldots, w_k$ are interchanged with $v_{i_1}, \ldots, v_{i_k}$ in the sum. Also, note that if $p > q$ then $\cdot$ in equation (5.1) represents $\otimes$ since the generator is in $\bigwedge^p E \otimes \bigwedge^q E$. If $p = q$ then $\cdot$ can be ignored since the generator is in $\mathrm{Sym}^2(\bigwedge^p E)$; this is because the $d_1, \ldots, d_s$ are distinct and so none of the tensor components in $\mathrm{Sym}(E; d_1, \ldots, d_s)$ are of the form $\bigwedge^p E \otimes \bigwedge^p E$.

This construction has connections with the Schur modules we discussed in Section 4. Let λ be a partition with conjugate $\tilde{\lambda} = (d_1^{a_1}, \ldots, d_s^{a_s})$ for non-negative integers $a_1, \ldots, a_s$. In Section 4.1 (explicitly equation (4.3)) we saw that

$$E^\lambda = \frac{\mathrm{Sym}^{a_1}\left(\bigwedge^{d_1} E\right) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathrm{Sym}^{a_s}\left(\bigwedge^{d_s} E\right)}{Q^\lambda(E)} \tag{5.2}$$

where we have used the fact that

$$\frac{\mathrm{Sym}^{a_k}\left(\bigwedge^{d_k} E\right)}{Q^\lambda(E)} = \frac{\overbrace{\left(\bigwedge^{d_k} E \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \bigwedge^{d_k} E\right)}^{a_k \text{ times}}}{Q^\lambda(E)}.$$

This follows from relation (3) in Section 4.1 since

$$(v_{i_1} \wedge \cdots \wedge v_{i_{d_k}}) \otimes (v_{j_1} \wedge \cdots \wedge v_{j_{d_k}}) = (v_{j_1} \wedge \cdots \wedge v_{j_{d_k}}) \otimes (v_{i_1} \wedge \cdots \wedge v_{i_{d_k}})$$

where we have exchanged all d_k of the indices. Thus by (5.2), we have that $\mathrm{Sym}(E; d_1, \ldots, d_s)$ is a direct sum of copies of the Schur modules E^λ, one for each λ with columns of heights in $\{d_1, \ldots, d_s\}$.

We can produce an analogous result for the identification of E^λ with the subalgebra D^λ of $\mathbb{C}[Z_{1,1}, Z_{1,2}, \ldots, Z_{n,m}]$ (which we saw in Theorem 4.2.4). Choose a basis $e_1, \ldots, e_m$ for E so that we can identify $\mathrm{Sym}(E; d_1, \ldots, d_s)$ with a quotient of a polynomial ring. Indeed, we saw above that $\mathrm{Sym}\left(\bigoplus \bigwedge^{d_i} E\right)$ can be identified with the polynomial ring

$$\mathbb{C}[X_{i_1,\ldots,i_p}; p \in \{d_1, \ldots, d_s\}]$$

where $i_1, \ldots, i_p \in [m]$ and $X_{i_1,\ldots,i_p}$ is the indeterminant corresponding to the basis vector $e_{i_1} \wedge \cdots \wedge e_{i_p} \in \bigwedge^p E$. Then in this setting we can define

$$\mathrm{Sym}(m; d_1, \ldots, d_s) = \frac{\mathbb{C}[X_{i_1,\ldots,i_p}; p \in \{d_1, \ldots, d_s\}]}{Q} \tag{5.3}$$

where Q is the ideal generated by all the quadratic relations

$$X_{i_1,\ldots,i_p} X_{j_1,\ldots,j_q} - \sum X_{i'_1,\ldots,i'_p} X_{j'_1,\ldots,j'_q} \tag{5.4}$$

where the sum is over all ways in which we can exchange $j_1, \ldots, j_k$ with k of the indices $i_1, \ldots, i_p$, with $p \geq q \geq k \geq 1$ and $p, q \in \{d_1, \ldots, d_s\}$.

Now recall that $D^\lambda (\cong E^\lambda)$ is generated by the elements D_T, as T varies over the tableaux on λ with entries in $[m]$, where D_T was defined in (4.4). Hence, if $n \geq d_1$, then $\mathrm{Sym}(m; d_1, \ldots, d_s)$ is isomorphic to the subalgebra of $\mathbb{C}[Z_{1,1}, Z_{1,2}, \ldots, Z_{n,m}]$ generated by all D_T, where T varies over all the tableaux with entries in $[m]$ on Young diagrams whose columns have heights in $\{d_1, \ldots, d_s\}$.[14] Indeed, we have already seen that each E^λ is isomorphic to D^λ and so the result follows since it can be shown that the sum of D^λ in $\mathbb{C}[Z_{1,1}, Z_{1,2}, \ldots, Z_{n,m}]$ is direct.

The identification of $\mathrm{Sym}(m; d_1, \ldots, d_s)$ with a subalgebra of $\mathbb{C}[Z_{1,1}, Z_{1,2}, \ldots, Z_{n,m}]$ allows us to deduce an important result that will be useful for Section 5.3. Notice, since subrings of integral domains are integral domains, that $\mathrm{Sym}(E; d_1, \ldots, d_s)$ is an integral domain. Then, using that $\mathrm{Sym}(E; d_1, \ldots, d_s)$ can also be defined as the symmetric algebra on $\bigwedge^{d_1} E \oplus \cdots \oplus \bigwedge^{d_s} E$, we obtain:

Proposition 5.1.1. *The ideal in* $\mathrm{Sym}\left(\bigwedge^{d_1} E\right) \bigotimes \cdots \bigotimes \mathrm{Sym}\left(\bigwedge^{d_d} E\right)$ *generated by the quadratic relations is a prime ideal.*

5.2 Projective Embedding of the Grassmannian

In order to express the Grassmannian as a subvariety of projective space we first need to embed it in projective space. This section, therefore, introduces the Plücker embedding, which does just that.

Let E be a vector space of dimension m over $\mathbb{C}$. For $0 < d \leq m$ define $\mathrm{Gr}^d E$ to be the collection of subspaces of E of codimension d, we call $\mathrm{Gr}^d E$ the *Grassmannian*. We start by giving an explicit construction of $\mathrm{Gr}^d E$. Let $e_1, \ldots, e_m$ be a basis for E and consider a k-dimensional subspace V of E where $k := m - d$. Let $v_1, \ldots, v_k \in E$ be a basis for V and associate V to the matrix with rows $v_1, \ldots, v_k$. This does not give us a well-defined map $\mathrm{Gr}^d E \to M_{k \times m}(\mathbb{C})$. Indeed, left multiplication by an element $\mathrm{GL}(E)$ can produce a different matrix with the same row span.

To assign a unique matrix to each element in $\mathrm{Gr}^d E$ we associate the matrix with rows $v_1, \ldots, v_k$ to the vector that contains all the $k \times k$ matrix minors of the $k \times m$ matrix. For example, consider the case $k = 3$ and $m = 4$. The matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 4 & 0 & 1 \\ 3 & 0 & 0 & 0 \end{pmatrix}$$

is associated to the vector $(0, -18, 0, 0)$. This association is unique for an element of $\mathrm{Gr}^d E$ (up to scalar). Indeed, suppose that the vector described above, for two subspaces V_1, V_2 of E (with dimension k), is exactly the same. Let A_1, A_2 be the matrices with row span equal to V_1, V_2 respectively. Then, since A_1 and A_2 have rank k, we can apply Echelon row operations so that the $k \times k$ matrix minor in the first $m - d$ columns is the identity for both A_1 and A_2. Then observe that the other entries, not in the first k columns, are maximal minors of this resulting matrix. Thus the two original matrices must be equal. [6, pp.1-3]

It follows that, since there are $\binom{m}{k}$ possible matrix minors, we can associate each element in $\mathrm{Gr}^d E$ with a unique element in $\mathbb{P}(E^{\binom{m}{k}})$. This is called the *Plücker embedding*, which by the above discussion is injective.

[14] Notice the subtle difference; T is not fixed to a specific shape λ as for E^λ.

The Plücker embedding is sometimes defined alternatively in terms of exterior powers. Indeed, adopting the notation above, we associate the subspace spanned by $v_1, \ldots, v_k$ with $[v_1 \wedge \cdots \wedge v_k] \in \mathbb{P}(\bigwedge^k E)$. Expanding $v_1 \wedge \cdots \wedge v_k$ with respect to a basis for E then gives us the $\binom{m}{k}$ matrix minors of the matrix with rows $v_1, \ldots, v_k$.

Example 5.2.1. [6, p.3] Consider $\mathrm{Gr}^2\mathbb{C}^4$. Let $v_1 = (a,b,c,d)^T$ and $v_2 = (e,f,g,h)^T$. Then the two-dimensional subspace of $\mathbb{C}^4$ spanned by v_1, v_2 is mapped to $\mathbb{P}^5$ as follows

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \mapsto [af - be : ag - ce : ah - de : bg - cf : bh - df : ch - dg] \in \mathbb{P}^5.$$

Alternatively, let e_1, e_2, e_3, e_4 be the standard basis of $\mathbb{C}^4$. Then

$$\begin{aligned}(ae_1 + be_2 + ce_3 + de_4) \wedge (ee_1 + fe_2 + ge_3 + ge_4) \\ = (af - be)e_1 \wedge e_2 + (ag - ce)e_1 \wedge e_3 + (ah - de)e_1 \wedge e_4 \\ + (bg - cf)e_2 \wedge e_3 + (bh - df)e_2 \wedge e_4 + (ch - dg)e_3 \wedge e_4\end{aligned}$$

and so we see that our two definitions of the Plücker embedding are indeed the same for this example.

Now, for reasons we will see in Section 5.3, it will be more helpful to work with the dual projective space $\mathbb{P}^*(\bigwedge^d E)$ since then for any $v_1, \ldots, v_d$ in E, we have that $v_1 \wedge \cdots \wedge v_d$ is a linear form on $\mathbb{P}^*(\bigwedge^d E)$. In order to do this we construct the Plücker embedding in the dual space $\mathbb{P}^*(\bigwedge^d E)$ as follows. For a subspace V of dimension k (codimension d), the kernel of the map

$$\bigwedge^d(E) \to \bigwedge^d\left(\frac{E}{V}\right)$$

is a hyperplane in $\bigwedge^d(E)$. Hence, if we associate this hyperplane to V we obtain an embedding $\mathrm{Gr}^d E \to \mathbb{P}^*(\bigwedge^d E)$ which we also call the *Plücker embedding*.

Using the result [5, p.7] from projective geometry that $(\bigwedge^d E)^* = \bigwedge^d(E^*) \cong \bigwedge^{m-d}(E)$ we have that $\mathbb{P}^*(\bigwedge^d E) \cong \mathbb{P}(\bigwedge^{m-d} E)$.[15] In particular, if $v_1, \ldots, v_d$ are in E and we extend to a basis $v_1, \ldots, v_m$ of E then this isomorphism takes $[v^*_{m-d} \wedge \cdots \wedge v^*_m]$ to $[v_1 \wedge \cdots \wedge v_d]$, where $v^*_1, \ldots, v^*_m$ is the dual basis for E^*. We claim:

◇ **Proposition 5.2.2.** *Our two constructions of the Plücker embedding are equivalent.*

Proof: Let E be an m-dimensional vector space. Suppose that V is a k-dimensional subspace of E, where $k = m-d$, with basis $v_1, \ldots, v_k$ and extend this to a basis $v_1, \ldots, v_m$ for E. Now, under our first definition of the Plücker embedding, V corresponds to the point $[v_1 \wedge \cdots \wedge v_k] \in \mathbb{P}(\bigwedge^{m-d} E)$. Therefore, by the above discussion, it suffices to show that the corresponding point $[v^*_{k+1} \wedge \cdots \wedge v^*_m] \in \mathbb{P}^*(\bigwedge^d E)$ defines the kernel of the map

$$\bigwedge^d(E) \to \bigwedge^d\left(\frac{E}{V}\right). \tag{5.5}$$

Now the quotient map E to E/V is spanned by $v_1, \ldots, v_k$ and so the kernel of the map in (5.5) is spanned by the set $\{v_{i_1} \wedge \cdots \wedge v_{i_d} \mid 1 \le i_1 < \cdots < i_d \le m\} \setminus \{v_{k+1} \wedge \cdots \wedge v_m\}$. By duality this corresponds to the point $[v^*_{k+1} \wedge \cdots \wedge v^*_m] \in \mathbb{P}^*(\bigwedge^d E)$ and so we're done. □

◇ **Examples 5.2.3.** (1) Consider Example 5.2.1 in the specific case $v_1 = (0,2,0,1)^T$, $v_2 = (1,0,1,0)^T$ and let V have basis v_1, v_2. Then, as before, we compute the matrix minors of the matrix with rows given by v_1, v_2 to obtain the Plücker coordinates of V

$$\begin{pmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \mapsto [-2 : 0 : -1 : 2 : 0 : -1] = [2 : 0 : 1 : -2 : 0 : 1] \in \mathbb{P}^5,$$

[15] Note that $\mathbb{P}^*(\bigwedge^d E) = \mathbb{P}((\bigwedge^d E)^*)$.

where V can also be associated with the point $[v_1 \wedge v_2] \in \mathbb{P}(\bigwedge^2 \mathbb{C}^4)$. Now extend v_1, v_2 to a basis v_1, v_2, v_3, v_4 where $v_3 = (0, 1, 0, -2)^T$ and $v_4 = (-1, 0, 1, 0)^T$. Therefore, using the isomorphism $\mathbb{P}^*(\bigwedge^d E) \cong \mathbb{P}(\bigwedge^{m-d} E)$, we have that $[v_1 \wedge v_2] \in \mathbb{P}(\bigwedge^2 \mathbb{C}^4)$ identifies with $[v_3^* \wedge v_4^*] \in \mathbb{P}^*(\bigwedge^2 \mathbb{C}^4)$, which has coordinates $[1, 0, -2, 1, 0, 2]$ with respect to the basis $e_1^* \wedge e_2^*, e_1^* \wedge e_3^*, e_1^* \wedge e_4^*, e_2^* \wedge e_3^*, e_2^* \wedge e_4^*, e_3^* \wedge e_4^*$. Now we can verify this using the matrix with rows v_3, v_4, namely

$$A = \begin{pmatrix} 0 & 1 & 0 & -2 \\ -1 & 0 & 1 & 0 \end{pmatrix},$$

which has kernel spanned by v_1, v_2. Thus we can also find the Plücker coordinates of V in $\mathbb{P}^*(\bigwedge^2 \mathbb{C}^4)$ by computing the kernel of $\bigwedge^2(A) : \bigwedge^2 \mathbb{C}^4 \to \bigwedge^2(\mathbb{C}^4/V)$. Now $\bigwedge^2 \mathbb{C}^4$ has basis $e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4$ and each such basis vector $e_i \wedge e_j$ is mapped by $\bigwedge^2(A)$ to the determinant $x_{i,j}$ of the matrix minor using the i^{th} and j^{th} columns of A, so that

$$\begin{aligned} \bigwedge^2(A)(a_{1,2}e_1 \wedge e_2 + a_{1,3}e_1 \wedge e_3 + a_{1,4}e_1 \wedge e_4 + a_{2,3}e_2 \wedge e_3 + a_{2,4}e_2 \wedge e_4 + a_{3,4}e_3 \wedge e_4) \\ = a_{1,2}x_{1,2} + a_{1,3}x_{1,3} + a_{1,4}x_{1,4} + a_{2,3}x_{2,3} + a_{2,4}x_{2,4} + a_{3,4}x_{3,4} \\ = a_{1,2} - 2a_{1,4} + a_{2,3} + 2a_{3,4}. \end{aligned}$$

Hence, as expected, the hyperplane that is the kernel of $\bigwedge^2(A)$ corresponds to the point $[1 : 0 : -2 : 1 : 0 : 2] \in \mathbb{P}^*(\bigwedge^2 \mathbb{C}^4)$.

(2) Consider $[1 : 2 : 1 : 1 : 2 : 3] \in \mathbb{P}^*(\bigwedge^2 \mathbb{C}^4)$. We seek the subspace of $\mathbb{C}^4$ with these Plücker coordinates. Hence we set the maximal minors of

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$$

equal to $1, 2, 1, 1, 2, 3$ and solve for $a, b, c, d \in \mathbb{C}$ to obtain

$$A = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$

Then we compute the kernel of A to find that the subspace of $\mathbb{C}^4$ with these Plücker coordinates is spanned by the vectors $(1, -2, 1, 0)^T$ and $(2, -1, 0, 1)^T$.

5.3 The Grassmannian as a Subvariety of Projective Space

We are now in a position to show that the Grassmannian can be realised as a subvariety of projective space.

Theorem 5.3.1. *The Plücker embedding is a bijection from $Gr^d E$ to the subvariety of $\mathbb{P}^*(\bigwedge^d E)$ defined by the quadratic equations*

$$\begin{aligned} &(v_1 \wedge \cdots \wedge v_d) \cdot (w_1 \wedge \cdots \wedge w_d) - \\ &\qquad \sum_{i_1 < \cdots < i_k} (v_1 \wedge \cdots \wedge w_1 \wedge \cdots \wedge w_k \wedge \cdots \wedge v_d) \cdot (v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge w_{k+1} \wedge \cdots \wedge w_d) = 0, \end{aligned}$$

for $v_1, \dots, v_d, w_1, \dots, w_d$ in E. Any polynomial vanishing on the image of $Gr^d E$ is in the ideal generated by these quadratic equations.

Remark 5.3.2. Note that the relations given in Theorem 5.3.1 are a special case of the quadratic relations in (5.1) with $p = q = d$.

Before embarking on the proof we make life easier for ourselves by first reframing the result using coordinates. Let $e_1, \dots, e_m$ be a basis for E and write $X_{i_1,\dots,i_d} = e_{i_1} \wedge \cdots \wedge e_{i_d}$ so that $X_{i_1,\dots,i_d}$ is a linear form on $\mathbb{P}^*(\bigwedge^d E)$. Now for a subspace V of E the homogeneous coordinates of V given by the Plücker embedding in $\mathbb{P}^*(\bigwedge^d E)$ are determined similarly to Examples 5.2.3 (1). Explicitly, let A be a $d \times m$ matrix with kernel V and consider the map $\bigwedge^d(A) : \bigwedge^d(\mathbb{C}^m) \to \bigwedge^d(\mathbb{C}^d) = \mathbb{C}$ which maps $e_{i_1} \wedge \cdots \wedge e_{i_d}$ to the determinant of the matrix minor of A using the columns numbered $i_1, \ldots, i_d$. Thus the hyperplane that is the kernel of $\bigwedge^d(A)$ corresponds to the point in $\mathbb{P}^*(\bigwedge^d E)$ whereby the Plücker coordinate $x_{i_1,\ldots,i_d}$ is this determinant.

In particular, the relations in Theorem 5.3.1 can be written in the form

$$X_{i_1,\ldots,i_d} \cdot X_{j_1,\ldots,j_d} = \sum X_{i'_1,\ldots,i'_d} \cdot X_{j'_1,\ldots,j'_d} \tag{5.6}$$

where the sum is over all pairs obtained by interchanging a fixed set of k of the subscripts $j_1, \ldots, j_d$ with k of the subscripts $i_1, \ldots, i_d$, maintaining the order in each. Since the wedge product is skew-commutative, it suffices to use the first k subscripts $j_1, \ldots, j_k$ in such an exchange.

Proof of Theorem 5.3.1: By the discussion at the start of Section 5.2 and Proposition 5.2.2 we have that the Plücker embedding is injective. Therefore, for the first statement of the theorem, it suffices to show that the Plücker coordinates of any linear subspace of codimension d satisfies (5.6) and that any point in $\mathbb{P}^*(\bigwedge^d E)$ satisfying (5.6) corresponds to a subspace of codimension d.

Firstly, if a subspace V of codimension d is the kernel of a matrix $A : E \to E/V$ then the corresponding Plücker coordinates satisfy (5.6) by Sylvester's Lemma (4.1.1). This is because the Plücker coordinates are given by the matrix minors of A.

Conversely, suppose that the coordinates $x_{i_1,\ldots,i_d}$ in $\mathbb{P}^*(\bigwedge^d E)$ satisfy (5.6). Now choose a particular coordinate $0 \neq x_{i_1,\ldots,i_d}$ and set $x_{i_1,\ldots,i_d} = 1$. Then define $A : \mathbb{C}^m \to \mathbb{C}^d$ by writing down its matrix $A = (a_{s,t})$ where

$$a_{s,t} = x_{i_1,\ldots,i_{s-1},t,i_{s+1},\ldots,i_d}$$

where $1 \leq s \leq d, 1 \leq t \leq m$. We need to show that the kernel of A is a subspace of codimension d whose Plücker coordinates are the given $x_{j_1,\ldots,j_d}$. Firstly, the columns $i_1, \ldots, i_d$ of A give the identity which means that A has full rank and so its kernel is of codimension d. Moreover, the determinant of this matrix minor is $1 = x_{i_1,\ldots,i_d}$. Now let $I = (i_1, \ldots, i_d)$ and $J = (j_1, \ldots, j_d)$. Consider the case where I and J have $d-1$ entries in common so that J is obtained from I by replacing i_s with t. The corresponding minor is then the identity but where the s^{th} column has diagonal entry $a_{s,t}$. Thus the determinant of this minor is $a_{st} = x_{i_1,\ldots,i_{s-1},t,i_{s+1},\ldots,i_d}$, which is precisely the Plücker coordinate we desire. Finally, suppose that I and J differ so that $j_r \in J$ is not in I. We argue by induction. Therefore suppose that the determinant of any minor corresponding to $J' = (j'_1, \ldots, j'_d)$, where J' has more in common with I than J, has determinant $x_{j'_1,\ldots,j'_d}$. Then (5.6) applied to I and J with $k = 1$, making the exchanges with j_r, gives $x_{j_1,\ldots,j_d}$ as a linear combination of products whereby the subscripts have more in common with I than J does. Thus the result follows by Sylvester's lemma.

It remains to show that any polynomial vanishing on the image of $\mathrm{Gr}^d E$ is in the ideal generated by these quadratic relations. We saw in Proposition 5.1.1 that the ideal of quadratic relations Q is a prime ideal. Thus, by the Nullstellansatz, the ideal of polynomials that vanish on the set of zeros of Q is Q itself. Finally, as $\mathrm{Gr}^d E$ is this set of zeros, the result follows. □

The relations in Theorem 5.3.1 are the same as the relations used to define $\mathrm{Sym}(E; d_1, \ldots, d_s)$ in the specific case $s = 1$ and $d_1 = d$ (see equations (5.1)). Hence, we have identified the homogeneous coordinate ring of $\mathrm{Gr}^d E \subseteq \mathbb{P}^*(\bigwedge^d E)$ with the ring

$$\mathrm{Sym}(m; d) = \frac{\mathrm{Sym}\left(\bigwedge^d E\right)}{Q} = \frac{\mathbb{C}[X_{i_1,\ldots,i_d}]}{Q}$$

where Q is the ideal generated by the quadratic relations given in Theorem 5.3.1.

Example 5.3.3. Consider $\mathrm{Gr}^2\mathbb{C}^4$ of 2-dimensional subspaces of a 4-dimensional vector space embedded in $\mathbb{P}^*(\bigwedge^2 \mathbb{C}^4)$. The discussion after Theorem 5.3.1 states that $\mathrm{Gr}^2\mathbb{C}^4$ is the subvariety defined by the quadratic relations

$$X_{i_1,i_2} X_{j_1,j_2} = \sum X_{i'_2,i'_2} X_{j'_1,j'_2} \tag{5.7}$$

where the sum is as described after equation (5.6). However, if the sets $\{i_1, i_2\}$ and $\{j_1, j_2\}$ overlap, $i_1 = i_2$, $j_1 = j_2$ or the sum is over the exchange of 2 indices then these relations are trivial. Therefore the only such relation is

$$X_{1,2}X_{3,4} - X_{1,3}X_{2,4} + X_{2,3}X_{1,4} = 0.$$

In the case of $\mathrm{Gr}^2\mathbb{C}^4$, the Grassmannian is called the *Klein quadric.*

We finish by briefly discussing how the contents of Theorem 5.3.1 can be generalised to flag varieties. Fix a sequence of integers $m \geq d_1 > \cdots > d_s \geq 0$. We define the *partial flag variety* $F\ell^{d_1,\ldots,d_s}(E)$ to be the set of nested subspaces

$$\{E_1 \subseteq E_2 \subseteq \cdots \subseteq E_s \subseteq E \mid \mathrm{codim}(E_i) = d_i,\ 1 \leq i \leq s\}$$

which can be viewed as a subset of the product of Grassmannians $\mathrm{Gr}^{d_1}E \times \cdots \times \mathrm{Gr}^{d_s}E$. Therefore, using the Plücker embedding on each factor, the partial flag variety $F\ell^{d_1,\ldots,d_s}(E)$ can be viewed as subvariety of

$$\prod_{i=1}^{s} \mathbb{P}^*\left(\bigwedge^{d_i} E\right) = \mathbb{P}^*\left(\bigwedge^{d_1} E\right) \times \cdots \times \mathbb{P}^*\left(\bigwedge^{d_s} E\right).$$

This leads us to an analogous result to Theorem 5.3.1.

Theorem 5.3.4. *The flag variety $F\ell^{d_1,\ldots,d_s}(E) \subseteq \prod_{i=1}^{s} \mathbb{P}^*(\bigwedge^{d_i} E)$ is the locus of zeros of the quadratic equations*

$$X_{i_1,\ldots,i_p} \cdot X_{j_1,\ldots,j_q} = \sum X_{i'_1,\ldots,i'_p} \cdot X_{j'_1,\ldots,j'_q}$$

the sum over all pairs obtained by interchanging the first k of the j subscripts with k of the i subscripts, maintaining the order in each, for all $p \geq q$ in the set $\{d_1, \ldots, d_s\}$.

Theorem 5.3.4 identifies the multihomogeneous coordinate ring of $F\ell^{d_1,\ldots,d_s}(E) \subseteq \prod_{i=1}^{s} \mathbb{P}^*(\bigwedge^{d_i} E)$ with

$$\mathrm{Sym}(m; d_1, \ldots, d_s) = \frac{\mathbb{C}[X_{i_1,\ldots,i_p}; p \in \{d_1, \ldots, d_s\}]}{Q}.$$

◇ **Remark 5.3.5.** Theorem 5.3.4 tells us that flag varieties can give geometric realisations of all irreducible representations of $\mathrm{GL}(E)$. Indeed, let λ be a partition of n with columns that take heights from $\{d_1, \ldots, d_s\}$. Then at the end of Section 5.1 we saw that $\mathrm{Sym}(m; d_1, \ldots, d_s)$ identifies with a direct sum of Schur modules, which in this case includes E^λ. Conversely, $\mathrm{Sym}(m; d_1, \ldots, d_s)$ is the multihomogeneous coordinate ring of the flag variety $F\ell^{d_1,\ldots,d_s}(E)$.

References

[1] J.-F. Arbour. Young symmetrizer, specht modules and examples., Jan 2013.

[2] P. I. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, D. Vaintrob, and E. Yudovina. *Introduction to representation theory*, volume 59. American Mathematical Soc., 2011.

[3] W. Fulton. *Young tableaux: with applications to representation theory and geometry*, volume 35. Cambridge University Press, 1997.

[4] W. Fulton and J. Harris. *Representation theory: a first course*, volume 129. Springer Science & Business Media, 2013.

[5] D. A. Hudec. The grassmannian as a projective variety.

[6] M. Michalek. Linear spaces and grassmannians, May 2018.

[7] M. F. Steinmetz. *Polynomial Representations of the General Linear Group*. PhD thesis, Imperial College London, 2014.

[8] M. Wildon. Notes on polynomial representations of general linear groups, 2008.

[9] M. Zabrocki. Introduction to symmetric functions. *unpublished work http://garsia. math. yorku. ca/ghana03/mainfile. pdf*, 2015.