The McKay correspondence and representations of the McKay quiver

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A thesis submitted to the University of Warwick
for the degree of Doctor of Philosophy

Mathematics Institute
University of Warwick
June 2001

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Acknowledgements

Firstly, I wish to thank my research supervisor, Miles Reid, for his advice and encouragement, particularly during our collaboration which forms Chapter 3 of this thesis. His boundless enthusiasm and sheer love of mathematics has been a constant source of inspiration, while his patience and understanding have eased me through one or two difficult times. He sets the most remarkable example to aspiring young mathematicians.

I am grateful to Michel Brion and Gérard Gonzalez-Sprinberg for pointing out a gap in an earlier version of the article [Cra00] that forms Chapter 4 of this thesis. I am also indepted to Alastair King whose questions following my recent Calf talk at the University of Bath [Cra01] forced me to reassess and subsequently improve my understanding of the topic that forms Chapter 5 of this thesis.

Thanks to Dimitrios Dais, Akira Ishii, Yukari Ito, Rebecca Leng, Daniel Ryder, Balázs Szendrői and Willem Veys for useful comments and fruitful discussions.

Finally, I wish to thank my family – mathematicians all – for letting me discover the beauty of mathematics in my own time.

Declaration

Chapters 1 and 2 are largely expository, with the exception of the worked examples of Section 2.4. Chapter 3 is a partially rewritten version of a joint preprint with Miles Reid [CR99]. Otherwise I declare that, to the best of my knowledge, the material in this thesis is the original work of the author except where stated explicitly in the text.

Introduction

Twenty years ago McKay [McK80] observed that the graph of ADE type associated to a Kleinian singularity $\mathbb{C}^2/G$ can be constructed using only the representation theory of the finite subgroup $G \subset \mathrm{SL}(2,\mathbb{C})$. This establishes a one-to-one correspondence between the nontrivial irreducible representations of G and the exceptional prime divisors of the minimal resolution Y of $\mathbb{C}^2/G$. Gonzalez-Sprinberg and Verdier [GSV83] provided a geometric explanation by associating a locally free sheaf $\mathcal{F}_i$ on Y to each irreducible representation ρ_i of G. Case by case analysis of the finite subgroups $G \subset \mathrm{SL}(2,\mathbb{C})$ revealed that the classes $c_1(\mathcal{F}_i)$ corresponding to the nontrivial representations ρ_i form a basis of $H^2(Y,\mathbb{Z})$ dual to the exceptional divisor classes. The bijection

$$\Big\{\text{irreducible representations of } G\Big\} \longleftrightarrow \text{ basis of } H^*(Y,\mathbb{Z}) \tag{1}$$

follows immediately. This is the *McKay correspondence*. This thesis studies several approaches to the problem of generalising the McKay correspondence to higher dimensions.

In Chapter 2 we provide an elementary introduction to the theory of motivic integration developed by Kontsevich [Kon95]. Our primary goal is to calculate the motivic integral in several nontrivial examples. We conclude with a discussion of how Batyrev [Bat99b, Bat00] used the theory of motivic integration to prove the following strong version of the generalised McKay correspondence conjecture of Reid [Rei92].

Theorem 0.1 (strong McKay correspondence) *Let $G \subset \mathrm{SL}(n,\mathbb{C})$ be a finite subgroup and suppose that the quotient $X = \mathbb{C}^n/G$ admits a crepant resolution $\varphi\colon Y \to X$. The nonzero Betti numbers of Y are*

$$\dim_{\mathbb{C}} H^{2k}(Y,\mathbb{C}) = \#\Big\{\textit{age } k \textit{ conjugacy classes of } G\Big\},$$

for $k = 0, \dots, n-1$. In particular, the topological Euler number $e(Y)$ is equal to the number of conjugacy classes of G.

The first step is to construct the motivic integral of a pair (Y, D), for a complex manifold Y and an effective divisor D on Y with simple normal crossings. We define the *space of formal arcs* $J_\infty(Y)$ of Y and associate a function F_D defined on $J_\infty(Y)$ to the divisor D. The *motivic integral* of the pair (Y, D) is the integral of F_D over $J_\infty(Y)$ with respect to a certain measure μ on $J_\infty(Y)$. This measure is not real-valued; the subtlety in the construction is in defining the ring in which μ takes values. We adopt the structure of the proof of Theorem 6.28 from Batyrev [Bat98] to establish the following user-friendly formula:

Theorem 0.2 (formula for the motivic integral) *Let Y be a complex manifold of dimension n and $D = \sum_{i=1}^r a_i D_i$ an effective divisor on Y with simple normal crossings. The motivic integral of the pair (Y, D) is*

$$\int_{J_\infty(Y)} F_D \, \mathrm{d}\mu = \sum_{J \subseteq \{1,\dots,r\}} [D_J^\circ] \cdot \left(\prod_{j \in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_j+1} - 1} \right) \cdot \mathbb{L}^{-n} \tag{2}$$

where we sum over all subsets $J \subseteq \{1, \dots, r\}$ including $J = \emptyset$.

The motivic integral of a complex algebraic variety X with Gorenstein canonical singularities is defined to be the motivic integral of a pair (Y, D), where $Y \to X$ is a resolution of singularities for which the discrepancy divisor D has simple normal crossings. Crucially, this is well defined independent of the choice of resolution. The motivic integral induces a *stringy E-function*

$$E_{\mathrm{st}}(X) := \sum_{J \subseteq \{1,\dots,r\}} E(D_J^\circ) \cdot \left(\prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1} \right) \tag{3}$$

which is also independent of the choice of resolution. The E-polynomials $E(D_J^\circ)$ encode the Hodge–Deligne numbers of open strata $D_J^\circ \subset Y$, and the stringy E-function records these numbers with certain 'correction terms' written in parentheses in formula (3).

When $Y \to X$ is a crepant resolution the correction terms disappear leaving simply the terms $E(D_J^\circ)$ whose sum is $E(Y)$. In this way the function $E_{\mathrm{st}}(X)$ encodes the Hodge numbers of a crepant resolution $Y \to X$. However, crepant resolutions do not exist in general. To get a better feeling for the stringy E-function of varieties admiting no crepant resolution we calculate $E_{\mathrm{st}}(X)$ for several 4- and 6-dimensional Gorenstein terminal cyclic quotient singularities.

We conclude Chapter 2 with a discussion of Batyrev's calculation of the motivic integral of the quotient singularity $X = \mathbb{C}^n/G$, for a finite subgroup $G \subset \mathrm{SL}(n, \mathbb{C})$, leading to a proof of Theorem 0.1.

Chapter 3 deals with another approach to the McKay correspondence, namely Nakamura's G-Hilbert scheme. In 1995, Ito and Nakamura [IN99] showed that for a finite subgroup $G \subset \mathrm{SL}(2, \mathbb{C})$, the scheme $G\text{-Hilb}\,\mathbb{C}^2$ which parameterises G-clusters is the minimal resolution of the quotient $\mathbb{C}^2/G$; here, a *G-cluster* is a G-invariant, zero-dimensional subscheme $Z \subset \mathbb{C}^2$ with $H^0(Z, \mathcal{O}_Z)$ isomorphic to the regular representation of G. In 1996, Nakamura [Nak00] introduced the scheme $G\text{-Hilb}\,\mathbb{C}^3$ for a finite subgroup $G \subset \mathrm{SL}(3, \mathbb{C})$, and conjectured that it is a crepant resolution of the quotient $\mathbb{C}^3/G$. He proved this for an Abelian subgroup $A \subset \mathrm{SL}(3, \mathbb{C})$ by introducing an explicit algorithm that calculates $A\text{-Hilb}\,\mathbb{C}^3$. Chapter 3 is a partially rewritten version of a joint preprint with Miles Reid [CR99] which calculates $A\text{-Hilb}\,\mathbb{C}^3$ by a much more efficient and user-friendly procedure.

Let $A \subset \mathrm{SL}(3, \mathbb{C})$ be a finite Abelian subgroup. The quotient singularity $\mathbb{C}^3/A$ is the toric variety $X_{L,\sigma}$ associated to the cone $\sigma = \langle e_1, e_2, e_3\rangle$ inside the vector space $L \otimes \mathbb{R} \cong \mathbb{R}^3$, where e_1, e_2, e_3 is the standard basis of a lattice $\mathbb{Z}^3$ and $L \supset \mathbb{Z}^3$ is the overlattice generated by the elements of A written in the form $\frac{1}{n}(\alpha_1, \alpha_2, \alpha_3)$ with $n = |A|$ and $0 \leq \alpha_i < n$. The junior simplex $\Delta \subset L \otimes \mathbb{R}$ is the triangle with vertices e_1, e_2, e_3, containing the lattice points $\frac{1}{n}(\alpha_1, \alpha_2, \alpha_3)$ for which $\alpha_1 + \alpha_2 + \alpha_3 = n$.

Theorem 0.3 *The junior simplex Δ is partitioned by regular triangles.*

Here, a *regular triangle of side* $r \in \mathbb{N}$, or simply a *regular triangle*, is a lattice triangle in Δ with $r + 1$ lattice points spaced evenly along each edge. Join these lattice points by drawing $r - 1$ lines parallel to the sides of the regular triangle to produce its *regular tesselation* into r^2 basic triangles (see Figure 3.1(b)).

The partition of Theorem 0.3 is determined by a combinatorial procedure involving continued fractions. The locus $(x = 0)$ inside $\mathbb{C}^3$ cuts out a surface singularity $\mathbb{C}^2_{(x=0)}/A$ whose minimal resolution is determined by a Jung–Hirzebruch continued fraction. Similarly, the loci $(y = 0)$ and $(z = 0)$

give rise to continued fractions and by concatenating all three we produce a 'cyclic continued fraction', i.e., a list of integers without a preferred starting point. A *contraction* of the continued fraction determines a regular triangle, and a *chain of contractions* determines the partition of Δ into regular triangles.

Theorem 0.4 *Let Σ denote the toric fan determined by the regular tesselation of all regular triangles in the junior simplex Δ. The toric variety $X_{L,\Sigma}$ is Nakamura's A-Hilbert scheme A-Hilb $\mathbb{C}^3$.*

Corollary 0.5 ([Nak00]) *A-Hilb $\mathbb{C}^3 \to \mathbb{C}^3/A$ is a crepant resolution.*

The standard construction of toric geometry says that $X_{L,\Sigma}$ is the union of the affine pieces $X_{L,\tau} = \text{Spec } \mathbb{C}[\tau^\vee \cap M]$ taken over all 3-dimensional cones $\tau \in \Sigma$, where $M = \text{Hom}(L, \mathbb{Z})$ is the dual lattice. Having constructed Σ, it is straightforward to calculate explicit coordinates on this open cover of $X_{L,\Sigma}$. To prove Theorem 0.4 we calculate explicit coordinates on an open cover of A-Hilb $\mathbb{C}^3$ and observe that the calculations agree.

In Chapter 4 we use the calculation of $Y = A$-Hilb $\mathbb{C}^3$ introduced in Chapter 3 to establish a geometric construction of the McKay correspondence for finite Abelian subgroups $A \subset \text{SL}(3, \mathbb{C})$. In fact, for a finite Abelian subgroup $A \subset \text{SL}(3, \mathbb{C})$ we prove part (ii) of the following conjecture of Reid [Rei97]:

Conjecture 0.6 (Reid's second McKay conjecture) *Let $G \subset \text{SL}(n, \mathbb{C})$ be a finite subgroup and suppose that $Y = G$-Hilb $\mathbb{C}^n$ is a crepant resolution of the quotient $X = \mathbb{C}^n/G$. Then*

(i) *the Gonzalez-Sprinberg and Verdier sheaves $\mathcal{F}_i$ on Y are locally free and form a $\mathbb{Z}$-basis of the K-theory of Y.*

(ii) *a certain cookery with the Chern classes of the sheaves $\mathcal{F}_i$ leads to a $\mathbb{Z}$-basis of the cohomology $H^*(Y, \mathbb{Z})$ for which the bijection (1) holds. This is the McKay correspondence for $Y = G$-Hilb $\mathbb{C}^n$.*

Ito and Nakajima [IN00] proved part (i) for a finite Abelian subgroup $A \subset \text{SL}(3, \mathbb{C})$. By applying the Chern character they therefore established a basis of $H^*(Y, \mathbb{Q})$ in one-to-one correspondence with the irreducible representations of A, a rational version of the McKay correspondence (1). The main result of Chapter 4 establishes part (ii) of Conjecture 0.6 for a finite Abelian subgroup $A \subset \text{SL}(3, \mathbb{C})$. We begin by constructing a basis of $H^4(Y, \mathbb{Z})$:

Theorem 0.7 *There are virtual bundles $\mathcal{V}_m$ on $Y = A$-Hilb $\mathbb{C}^3$ indexed by certain characters χ_m of the group A such that the classes $c_2(\mathcal{V}_m)$ form a basis of $H^4(Y,\mathbb{Z})$ dual to the basis $[S] \in H_4(Y,\mathbb{Z})$ determined by the compact exceptional surfaces S of the resolution $\varphi\colon Y \to X$.*

The proof of Theorem 4.2 uncovers certain relations between tautological bundles of the form[1], say, $\mathcal{F}_m = \mathcal{F}_k \otimes \mathcal{F}_l$ for characters $\chi_m = \chi_k \otimes \chi_l$. In fact, one such relation arises for each compact exceptional surface S of the map φ and, following a recipe introduced by Reid [Rei97], we use each relation to cook up a virtual bundle $\mathcal{V}_m$ on Y with trivial rank and trivial first Chern class. We illustrate this construction by drawing the toric picture of Σ and marking each line with a character χ_i corresponding to the line bundle $\mathcal{F}_i$ and each vertex with a character χ_m (or a pair of characters χ_l, χ_m) corresponding to $\mathcal{V}_m$ (respectively to $\mathcal{F}_l$, $\mathcal{V}_m$). Case by case analysis of the compact exceptional surfaces $S \subset Y$ establishes Theorem 0.7.

The first Chern classes $c_1(\mathcal{F}_i)$ of the tautological bundles span $H^2(Y,\mathbb{Z})$, but they do not form a $\mathbb{Z}$-basis in general. However, we determine a subset which does base $H^2(Y,\mathbb{Z})$:

Theorem 0.8 *Given the first Chern classes of all nontrivial tautological bundles, discard those classes $c_1(\mathcal{F}_m)$ determined by characters χ_m which form the indexing set of the basis $c_2(\mathcal{V}_m)$ of $H^4(Y,\mathbb{Z})$. The remaining classes form a $\mathbb{Z}$-basis of $H^2(Y,\mathbb{Z})$.*

Corollary 0.9 *The McKay correspondence bijection (1) holds (replace G by A) for all finite Abelian subgroups $A \subset \mathrm{SL}(3,\mathbb{C})$.*

There are two key points here: first, it is not clear a priori that characters corresponding to different elements $c_2(\mathcal{V}_m)$ basing $H^4(Y,\mathbb{Z})$ are distinct; also, we must ensure that the relations between the tautological bundles are independent in $\mathrm{Pic}(Y)$. The proof of Theorem 0.8 uses the toric picture of Σ which is marked with characters of A. We plot each character χ marking either a line or a vertex in our picture on the *McKay quiver* and conclude that these characters correspond one-to-one with the characters of the group A. Using this observation, together with equality $e(Y) = |A|$ from Theorem 0.1, it is straightforward to deduce Theorem 0.8, and hence Corollary 0.9.

[1]Note however that the map $\chi_i \to \mathcal{F}_i$ is not multiplicative in general.

The final chapter introduces a procedure to calculate toric minimal models of $\mathbb{C}^3/G$ as moduli spaces $\mathcal{M}_\theta$ of representations of the McKay quiver associated to a finite Abelian[2] subgroup $G \subset \mathrm{SL}(3, \mathbb{C})$. Ito and Nakajima [IN00] observed that the minimal model G-Hilb $\mathbb{C}^3$ of $\mathbb{C}^3/G$ is of the form $\mathcal{M}_\theta$ for some parameter θ. Our goal in Chapter 5 is to provide examples where every toric minimal model of $\mathbb{C}^3/G$, or equivalently every toric flop of G-Hilb $\mathbb{C}^3$, is of this form.

The moduli $\mathcal{M}_\theta$ are GIT quotients of an affine variety X (parametrising certain representations of the McKay quiver) by the action of a reductive algebraic group $\mathrm{PGL}(\underline{r})$ with respect to a $\mathrm{PGL}(\underline{r})$-linearisation of the trivial line bundle. Sardo Infirri constructs $\mathcal{M}_\theta$ explicitly by calculating all θ-stable points of X using a computer program. However, the computations are hard to carry out in practise.

Let Σ' be a basic triangulation of the junior simplex Δ of $\mathbb{C}^3/G$ so that the toric variety $Y = X_{L,\Sigma'}$ is a minimal model of $\mathbb{C}^3/G$. If the procedure introduced below successfully calculates Y as a moduli space $\mathcal{M}_{\theta'}$ for some parameter θ' then the flop linking Y and G-Hilb $\mathbb{C}^3$ can be regarded as a variation of GIT quotient:

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ Y = \mathcal{M}_{\theta'} \arrow[rr, dashed, leftrightarrow] \arrow[dr] & & \mathcal{M}_\theta = G\text{-Hilb } \mathbb{C}^3 \arrow[dl] \\ & \mathbb{C}^3/G & \end{tikzcd}$$

The procedure generalises Nakamura's calculation of G-Hilb $\mathbb{C}^3 = X_{L,\Sigma}$ and has several steps. The first step is to find a $(\mathbb{C}^*)^3$-invariant point of the toric variety $Y = X_{L,\Sigma'}$. To do this, we take a G-cluster $H^0(Z, \mathcal{O}_Z)$ and, while maintaining the isomorphism to the regular representation R, we alter the $\mathbb{C}[x,y,z]$-module structure. The resulting G-equivariant $\mathbb{C}[x,y,z]$-module M' is called a *G-constellation*. We do not yet have a clear understanding of this alteration process and for this reason we label this the *mysterious first step*. Nevertheless, if this step can be performed then the module M' defines (by construction) a $(\mathbb{C}^*)^3$-invariant point of $X_{L,\Sigma'}$. The generators m_i' of M' are indexed by some subset I_{gen} of the set $I := \{0, 1, \ldots, N\}$ indexing the irreducible representations $\rho_0, \ldots, \rho_N$ of G.

[2] In the final chapter, G (rather than A) denotes the finite Abelian subgroup of $\mathrm{SL}(3, \mathbb{C})$. You'll see why shortly.

Now, a G-constellation may also be regarded as a representation of the McKay quiver (satisfying certain relations) and hence defines a point $x \in X$. The GIT notion of stability for points $x \in X$ translates directly into the language of G-constellations and $\theta(M')$ denotes the parameters $\theta \in \mathbb{Q}^{N+1}$ with respect to which M' is θ-stable. A theorem of Ishii states that M' may be deformed with the moduli space $\mathcal{M}_\theta$ to produce deformation parameters $v_1, v_2, v_3 \in \mathrm{Def}(M')$ cutting out a 3-dimensional cone $\sigma' := \sigma(M') \subset L \otimes \mathbb{R}$ such that $X_{L,\sigma'} \cong \mathbb{C}^3$. Calculating the cone σ' is the second step.

The third step simulates a '*G-igsaw transformation*'[3] in the v-direction for some $v \in \mathrm{Def}(M')$ with $v \notin \mathbb{C}[x,y,z]^G$ by running an algorithm (this is Algorithm 5.22 in the main text). This algorithm is set up to ensure that the module generators of the resulting G-constellation N are indexed by a subset of I_{gen}. Then N may also be deformed to produce a cone $\sigma(N)$ and we repeat. Thus we have:

Procedure 0.10 *Let Σ' be a basic triangulation of the junior simplex Δ of $\mathbb{C}^3/G$ for a finite Abelian subgroup $G \subset \mathrm{SL}(3, \mathbb{C})$.*

STEP 1 *Perform the* mysterious first step *to get a G-constellation M' and a set I_{gen} indexing the generators of M'. Relabel $M := M'$.*

STEP 2 *Deform M according to Theorem 5.19 to produce deformation parameters $v_1, v_2, v_3 \in \mathrm{Def}(M)$ cutting out a cone $\sigma(M)$.*

STEP 3 *Run Algorithm 5.22 on M in the v_k-direction (for some $k = 1, 2, 3$) to produce a G-constellation N whose generators are indexed by a subset of I_{gen}.*

STEP 4 *Set $M := N$ and return to* STEP 2.

It is not yet clear whether the first step of this procedure can be carried out for every toric minimal model of $\mathbb{C}^3/G$. Moreover, even if the procedure does begin it is not clear a priori that it stabilises after finitely many steps. Nevertheless, in §5.7 and §5.8 we apply the procedure to every basic triangulation Σ' of the junior simplices of the cyclic quotient singularities $\frac{1}{6}(1, 2, 3)$ and $\frac{1}{11}(1, 2, 8)$. In each case, the procedure produces precisely $|G|$ modules M_i such that the corresponding cones $\sigma_i = \sigma(M_i)$ define the fan Σ'.

Thus far then, beginning with the fan Σ' we have found an unusual method to calculate the fan Σ'(!) We now come to the key point:

[3] You now see why G (rather than A) denotes the Abelian subgroup of $\mathrm{SL}(3, \mathbb{C})$.

Conjecture 0.11 *For Σ' as in Procedure 0.10, suppose that the procedure has been successfully implemented to produce $|G|$ modules M_i such that the fan Σ' is determined by the 3-dimensional cones $\sigma_i = \sigma(M_i)$. Then there exists an open subset of parameters θ in the orthant*

$$\mathrm{Orth}(M') := \{\theta \in \mathbb{Q}^{N+1} \bigm| \theta_i < 0 \textit{ if } i \in I_{\mathrm{gen}};\ \theta_i > 0 \textit{ otherwise}\}$$

such that every M_i is θ-stable.

Theorem 0.12 *When Conjecture 5.27 holds, $X_{L,\Sigma'} = \mathcal{M}_\theta$ for each of the parameters $\theta \in \mathrm{Orth}(M')$ given by the conjecture.*

When the conjecture holds, every application of Step 3 in Procedure 5.24 is called a *θ-stable G-igsaw transformation.*

We verify that the conjecture holds for every basic triangulation Σ' of the junior simplices of the cyclic quotient singularities $\frac{1}{6}(1,2,3)$ and $\frac{1}{11}(1,2,8)$. Thus, every minimal model $Y = X_{L,\Sigma'}$ of the quotient singularities of type $\frac{1}{6}(1,2,3)$ and $\frac{1}{11}(1,2,8)$ can be constructed as moduli $\mathcal{M}_{\theta'}$ for some θ'. As a result, the minimal models of these singularities can be regarded as a variation of GIT quotient as shown in the diagram on page xv.

We now describe the structure of this thesis. In Chapter 1 we gather some well known results that are used in subsequent chapters, focusing on the toric geometry construction of Abelian quotient singularities, the history of the McKay correspondence and the construction of the moduli spaces $\mathcal{M}_\theta$ of representations of the McKay quiver. Chapter 2 is an elementary introduction to motivic integration, containing a detailed calculation of the motivic integral of several Gorenstein terminal cyclic quotient singularities. In Chapter 3 we provide a simple calculation of A-Hilb $\mathbb{C}^3$ for finite Abelian subgroups $A \subset \mathrm{SL}(3,\mathbb{C})$ by proving Theorems 0.3 and 0.4. In Chapter 4 we establish Theorems 0.7 and 0.8, leading to an explicit basis for the integral cohomology of A-Hilb $\mathbb{C}^3$ satisfying Corollary 0.9. Chapter 5 reviews Nakamura's calculation of G-Hilb $\mathbb{C}^3$, introduces Procedure 0.10 and provides several worked examples. Appendix A consists of a short discussion about the motivic nature of motivic integration.

Chapter 1

Preliminaries

This chapter presents some well known results. Section 1.1 focuses on Gorenstein canonical singularities and the toric geometry construction of Abelian quotient singularities (see Reid [Rei80, Rei83, Rei87]). Section 1.2 reviews the history of the McKay correspondence: from McKay's result [McK80] and the work of Gonzalez-Sprinberg and Verdier [GSV83] to the conjectures of Reid [Rei92] and Batyrev–Dais [BD96] generalising the McKay correspondence to higher dimensions. Section 1.3 recalls the construction of moduli of representations of the McKay quiver for a finite subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$.

1.1 Background material

1.1.1 Kleinian singularities

Finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ are classified up to conjugacy as either the *cyclic* group of order $n \geq 2$ generated by the transformations

$$(x, y) \to (\varepsilon x, \varepsilon^{n-1} y) \quad \text{for} \quad \varepsilon^n = 1;$$

the *binary dihedral* group of order $4n$ ($n \geq 2$) generated by the pair

$$(x, y) \to (-y, x) \quad \text{and} \quad (x, y) \to (\varepsilon x, \varepsilon^{2n-1} y) \quad \text{for} \quad \varepsilon^{2n} = 1;$$

or one of three exceptional cases: the *binary tetrahedral*, *binary octahedral* or *binary icosahedral* groups of order 24, 48 and 120 respectively. Each exceptional case is the lift under the double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$ of the symmetry group of the corresponding Platonic solid.

Definition 1.1 For $G \subset \mathrm{SL}(2,\mathbb{C})$ a finite subgroup, the quotient variety $X = \mathbb{C}^2/G := \mathrm{Spec}\ \mathbb{C}[x,y]^G$ is called a *Kleinian* singularity (also known as a *Du Val* singularity, a *simple surface* singularity or a *rational double point*).

The quotient can be embedded as a hypersurface $X \subset \mathbb{C}^3$ with an isolated singularity at the origin. The defining equation for the singularity is determined by the conjugacy class of the group G as shown in Table 1.1. For the cyclic and binary dihedral cases we have $n \geq 2$.

Conjugacy class of G	Defining equation of X	Dynkin graph
cyclic $\mathbb{Z}/n\mathbb{Z}$	$x^2 + y^2 + z^n = 0$	A_{n-1}
binary dihedral $\mathbb{D}_{4n}$	$x^2 + y^2z + z^{n+1} = 0$	D_{n+2}
binary tetrahedral $\mathbb{T}_{24}$	$x^2 + y^3 + z^4 = 0$	E_6
binary octahedral $\mathbb{O}_{48}$	$x^2 + y^3 + yz^3 = 0$	E_7
binary icosahedral $\mathbb{I}_{120}$	$x^2 + y^3 + z^5 = 0$	E_8

Table 1.1: Classification of Kleinian singularities.

We now describe the ADE classification. Write $\varphi\colon Y \to X$ for the minimal resolution; that is, make an arbitrary resolution of singularities and successively contract (-1)-curves to produce Y. The exceptional locus of φ consists of (-2)-curves D_i intersecting transversally. The *resolution graph* of X is constructed as follows: each curve D_i is a vertex, and two vertices are joined by an edge if and only if the corresponding curves intersect in Y. For Kleinian singularities, the resolution graph is one of the Dynkin graphs of ADE type shown in Figure 1.1. Note that the graphs of type A_n and D_n each have n vertices.

1.1.2 Canonical and Gorenstein quotient singularities

The notion of canonical singularities, introduced by Reid [Rei80], generalises the adjunction-theoretic properties of a surface with Kleinian singularities to higher dimensions.

Definition 1.2 A normal variety X is said to have *canonical* (respectively *terminal*) singularities if it satisfies the following two conditions:

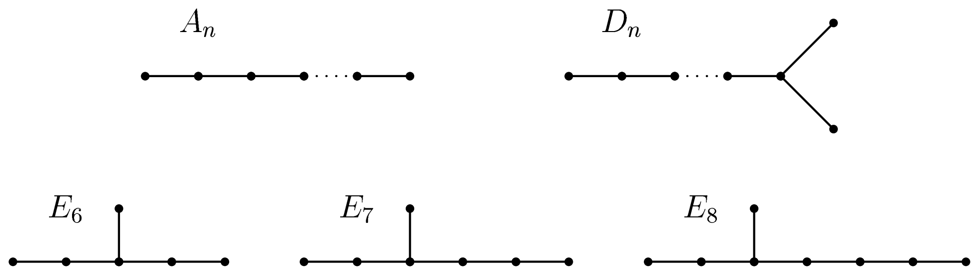


Figure 1.1: Dynkin graphs of type A_n, D_n, E_6, E_7 and E_8

(i) the Weil divisor rK_X is Cartier for some $r \in \mathbb{N}$.

(ii) for any resolution of singularities $\varphi \colon Y \to X$ with exceptional prime divisors $\{D_i\}$, the numbers $a_i \in \mathbb{Q}$ determined by the formula

$$K_Y = \varphi^* K_X + \sum a_i D_i$$

satisfy $a_i \geq 0$ (respectively $a_i > 0$).

The $\mathbb{Q}$-divisor $D = \sum a_i D_i$ is the *discrepancy divisor* of the resolution, and the rational number a_i, which is independent of the choice of Y, is the *discrepancy* of the divisor D_i. The resolution is said to be *crepant* if the discrepancy divisor is zero, in which case $K_Y = \varphi^* K_X$.

Kleinian singularities are canonical. Conversely, every canonical surface singularity is analytically isomorphic to a Kleinian singularity $X \subset \mathbb{C}^3$ (see [Rei87, §4.9] for a proof).

While searching for a 3-dimensional analogue of the minimal resolution of a Kleinian singularity Reid discovered the correct definition of minimal model for a 3-fold with canonical singularities:

Theorem 1.3 ([Rei83]) *Let X denote a 3-fold with canonical singularities. There exists a proper birational morphism $\varphi \colon Y \to X$ which is crepant, where Y has only $\mathbb{Q}$-factorial terminal singularities. The variety Y is a* minimal model *of X.*

Minimal models are nonunique in general. Different minimal models of a given 3-fold are isomorphic in codimension 1 and are related by a finite sequence of *flops* (see [KM96, §6]). That is, if $Y_1, \ldots, Y_k$ are minimal models of X then there are birational maps $\phi_i \colon Y_i \dashrightarrow Y_{i+1}$.

Canonical singularities are *rational* [Rei80, §3.8]. That is, for a normal variety X with canonical singularities, there is a resolution $\varphi \colon Y \to X$ such that $R^i \varphi_* \mathcal{O}_Y = 0$ for all $0 < i < \dim X$. As a consequence [Har77, III,§8], when X is affine the groups $H^i(Y, \mathcal{O}_Y)$ vanish for all $0 < i < \dim Y$, so the exponential exact sequence induces an isomorphism $\operatorname{Pic}(Y) \cong H^2(Y, \mathbb{Z})$ determined by sending a line bundle $\mathcal{L}$ to its first Chern class $c_1(\mathcal{L})$.

Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite subgroup acting on $\mathbb{C}^n$. The quotient variety is $\mathbb{C}^n/G := \operatorname{Spec}\ \mathbb{C}[x_1, \ldots, x_n]^G$.

Definition 1.4 An element $g \in \mathrm{GL}(n, \mathbb{C})$ of finite order is called a *quasi-reflection* if the matrix $g - I$ has rank 1, where I denotes the $n \times n$ identity matrix. A finite subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ is called *small* if it contains no quasireflections.

A theorem of Shephard and Todd [ST54] and Chevalley [Che55] states that if $G \subset \mathrm{GL}(n, \mathbb{C})$ is a finite group and $H \subset G$ is the maximal subgroup generated by quasireflections then $\mathbb{C}^n/H \cong \mathbb{C}^n$. This reduces the study of quotients $\mathbb{C}^n/G$ to the case where $G \subset \mathrm{GL}(n, \mathbb{C})$ is small.

Definition 1.5 A variety is *Gorenstein* if it is Cohen–Macaulay and the canonical sheaf ω_X is invertible.

Proposition 1.6 ([Wat74]) *Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a small subgroup. Then*

$$\mathbb{C}^n/G \text{ is Gorenstein} \iff G \subset \mathrm{SL}(n, \mathbb{C}).$$

In particular, Kleinian singularities are Gorenstein. We observed in the previous section that Kleinian singularities are canonical. More generally we have the following result:

Proposition 1.7 ([Rei80]) *Gorenstein quotient singularities are canonical.*

Proof. Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be small. The action is free in codimension 1 so the quotient map $\pi \colon \mathbb{C}^n \to \mathbb{C}^n/G$ is étale in codimension 1 on $\mathbb{C}^n$. The canonical sheaf of $\mathbb{C}^n/G$ is locally free by assumption and it follows from [Rei80, §1.7] that $\mathbb{C}^n/G$ is canonical. □

1.1.3 Toric geometry

Let $L \cong \mathbb{Z}^n$ be a lattice and write $M := \mathrm{Hom}(L, \mathbb{Z})$ for the dual lattice. A cone σ in $L_{\mathbb{R}} := L \otimes \mathbb{R} \cong \mathbb{R}^n$ with vertex at the origin is said to be *strongly convex rational polyhedral* if σ can be generated over $\mathbb{R}_{\geq 0}$ by finitely many points of the lattice L. The dual cone $\sigma^\vee \subset M_{\mathbb{R}} := M \otimes \mathbb{R}$ determines a finitely generated $\mathbb{C}$-algebra $\mathbb{C}[\sigma^\vee \cap M]$ and hence an *affine toric variety*

$$X_{L,\sigma} := \mathrm{Spec}\ \mathbb{C}[\sigma^\vee \cap M].$$

The intersection of σ with a supporting hyperplane is called a *face* of σ. The variety $X_{L,\tau}$ is a principal open subset in $X_{L,\sigma}$ when τ is a face of σ.

Definition 1.8 A *fan* is a finite collection of strongly convex rational polyhedral cones $\Sigma = \{\sigma_i\}_{i \in I}$ in $L_{\mathbb{R}}$ such that (i) the faces of each cone in Σ are also cones in Σ and (ii) any two cones in Σ meet in a common face. The *toric variety* $X_{L,\Sigma}$ associated to the fan Σ in $L_{\mathbb{R}}$ is constructed by taking the disjoint union of the varieties X_{L,σ_i}, one for each $i \in I$, and gluing X_{L,σ_i} and X_{L,σ_j} along the principal open subset $X_{L,\sigma_i \cap \sigma_j}$.

Geometric properties of toric varieties can be deduced by studying the corresponding fan. For instance, $X_{L,\Sigma}$ is a normal variety of dimension $\dim_{\mathbb{Z}} L = n$. Also, the affine toric variety $X_{L,\sigma}$ is smooth if and only if σ is generated by part of a basis of the lattice $L \cong \mathbb{Z}^n$, in which case $X_{L,\sigma} = \mathbb{C}^{\dim \sigma} \times (\mathbb{C}^*)^{n - \dim \sigma}$.

The algebraic torus $\mathbb{T}^n \simeq (\mathbb{C}^*)^n$ is a dense open subset of $X_{L,\Sigma}$ and the natural action of the torus on itself extends to an action on $X_{L,\Sigma}$, inducing a stratification of $X_{L,\Sigma}$ into orbits of the torus action $O_{L,\tau} \cong (\mathbb{C}^*)^{n - \dim \tau}$, one for each cone $\tau \in \Sigma$. The cones containing τ as a face define a fan $\mathrm{Star}(\tau)$ in $L(\tau) \otimes \mathbb{R}$ where $L(\tau) := L/(\tau \cap L)$. The toric variety $X_{L(\tau),\mathrm{Star}(\tau)}$ is the closure of the orbit $O_{L,\tau}$. These closed subvarieties are the *toric strata* of $X_{L,\Sigma}$. The correspondence between toric strata $X_{L(\tau),\mathrm{Star}(\tau)} \subset X_{L,\Sigma}$ and cones $\tau \in \Sigma$ is inclusion-reversing.

Definition 1.9 Let L and L' be lattices, and consider fans Σ and Σ' in $L_{\mathbb{R}}$ and $L'_{\mathbb{R}}$ respectively. If there is a homomorphism of lattices $\varphi\colon L' \to L$ such that, for each cone $\sigma' \in \Sigma'$, there is a cone $\sigma \in \Sigma$ for which $\varphi(\sigma') \subset \sigma$, then the *toric morphism* $\varphi\colon X_{L',\Sigma'} \to X_{L,\Sigma}$ is induced in the obvious way.

The toric morphism φ is proper and birational when the homomorphism φ is an isomorphism and Σ' is a subdivision of Σ. If every cone in Σ' is generated by part of a basis of the lattice L' then the corresponding morphism $\varphi\colon X_{L',\Sigma'} \to X_{L,\Sigma}$ is a resolution of singularities. The exceptional locus consists of the toric strata $X_{L'(\tau),\mathrm{Star}(\tau)}$ determined by the cones τ in Σ' which do not lie in Σ. In particular, the exceptional divisors are the toric strata corresponding to the 1-dimensional cones which lie in $\Sigma' \setminus \Sigma$.

Examples 1.10 In Figure 1.2 we draw two fans in $L_{\mathbb{R}} \cong \mathbb{R}^2$ for which the underlying lattice L has the standard basis e_1, e_2. It is well known that

Figure 1.2: Fans defining the toric varieties $\mathbb{P}^2$ and dP_6

$X_{L,\Sigma_1} \cong \mathbb{P}^2$. Indeed, for $x_1 = z_1/z_0$ and $x_2 = z_2/z_0$, the 2-dimensional cones in Σ_1 define the standard affine cover of $\mathbb{P}^2$ with coordinates z_0, z_1, z_2; for instance, if σ is the cone generated by e_1 and $-e_1 - e_2$ then $X_{L,\sigma} = \mathrm{Spec}\ \mathbb{C}\left[x_2^{-1}, x_1 x_2^{-1}\right] = \mathrm{Spec}\ \mathbb{C}[z_0/z_2, z_1/z_2] \cong \mathbb{C}^2$.

The fan Σ_2 is obtained from Σ_1 by subdividing the cones with rays generated by $-e_1$, $-e_2$ and $e_1 + e_2$. The corresponding toric morphism $X_{L,\Sigma_2} \to X_{L,\Sigma_1}$ is the blow-up of $\mathbb{P}^2$ at the points $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$. The resulting toric variety $X_{L,\Sigma_3} \cong \mathrm{dP}_6$ is the del Pezzo surface of degree 6 familiar from the plane Cremona transformation [Har77, V,§4].

1.1.4 Abelian quotients are toric varieties

Our treatment of cyclic quotient singularities follows Reid [Rei87, §4]. Write $\overline{M} \cong \mathbb{Z}^n$ for the lattice of Laurent monomials in $x_1, \ldots, x_n$, and $\overline{L}$ for the dual lattice with basis $e_1, \ldots, e_n$. For $r \in \mathbb{N}$ and integers $\alpha_1, \ldots, \alpha_n$ in the range $0 \leq \alpha_j < r$, consider the overlattice

$$L = \overline{L} + \mathbb{Z} \cdot \tfrac{1}{r}(\alpha_1, \dots, \alpha_n), \tag{1.1}$$

and let $M \subset \overline{M}$ denote the dual sublattice. A Laurent monomial in $x_1, \dots, x_n$ lies in the sublattice $M \subset \overline{M}$ if and only if it is invariant under the action of the small group $G = \mathbb{Z}/r \subset GL(n, \mathbb{C})$ generated by the diagonal matrix

$$g = \operatorname{diag}\left(\varepsilon^{\alpha_1}, \dots, \varepsilon^{\alpha_n}\right) \quad \text{with} \quad 0 \leq \alpha_j < r,$$

where r is the order of g and ε is a primitive rth root of unity. The assumption that G is small implies that $\gcd(r, \alpha_1, \dots, \widehat{\alpha_j}, \dots, \alpha_n) = 1$ for all $j = 1, \dots, n$, where the notation $\widehat{\alpha_j}$ means that α_j is omitted.

Definition 1.11 For $\sigma = \langle e_1, \dots, e_n \rangle$ in $L_{\mathbb{R}}$ with L as in (1.1) above, set

$$X_{L,\sigma} = \operatorname{Spec}\ \mathbb{C}[\sigma^\vee \cap M] = \operatorname{Spec}\ \mathbb{C}[x_1, \dots, x_n]^G = \mathbb{C}^n/G.$$

This is the *cyclic quotient singularity of type* $\frac{1}{r}(\alpha_1, \dots, \alpha_n)$.

More generally, let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a small Abelian subgroup. Choose coordinates $x_1, \dots, x_n$ on $\mathbb{C}^n$ to diagonalise the action and write $\overline{M}$ and $\overline{L}$ as above. To each group element

$$g = \operatorname{diag}\left(\varepsilon^{\alpha(g)_1}, \dots, \varepsilon^{\alpha(g)_n}\right) \quad \text{with} \quad 0 \leq \alpha(g)_j < r(g),$$

where $r(g)$ is the order of g and ε is a primitive $r(g)$th root of unity, we associate the vector

$$v_g = \frac{1}{r(g)}\big(\alpha(g)_1, \dots, \alpha(g)_n\big).$$

Definition 1.12 Consider the cone $\sigma = \langle e_1, \dots, e_n \rangle$ in $L_{\mathbb{R}}$ with

$$L := \overline{L} + \sum_{g \in G} \mathbb{Z} \cdot v_g.$$

The associated toric variety $X_{L,\sigma}$ is the *Abelian quotient singularity* $\mathbb{C}^n/G$.

For $\overline{L} \cong \mathbb{Z}^n$ generated by $e_1, \ldots, e_n$, consider $\overline{L} \otimes \mathbb{R} \cong \mathbb{R}^n$. Write

$$\square = \Big\{(r_1, \ldots, r_n) \in \mathbb{R}^n \;\Big|\; 0 \leq r_i < 1\Big\}$$

for the unit box. Each element $g \in G \cong L/\overline{L}$ has a unique representative $v_g \in \square$ so when drawing the toric picture of σ in $L_{\mathbb{R}}$ we need only draw $L \cap \square$ (see Examples 1.14). The notation v_g denotes both the vector in $\mathbb{R}^n$ and the lattice point in L. The criteria for determining whether an Abelian quotient singularity is canonical or Gorenstein are due to Reid:

Theorem 1.13 ([Rei80]) *Write $X_{L,\sigma}$ for the Abelian quotient $\mathbb{C}^n/G$. Then*

(i) *$X_{L,\sigma}$ is canonical (resp. terminal) if and only if*

$$\sum_{j=1}^{n} \alpha(g)_j \geq r(g) \tag{1.2}$$

(resp. $>$) for each $v_g = \frac{1}{r(g)}\big(\alpha(g)_1, \ldots, \alpha(g)_n\big) \in L \cap \square$.

(ii) *the vectors v_g for which (1.2) is equality correspond to the crepant exceptional divisors of a resolution of $\mathbb{C}^n/G$.*

(iii) *$X_{L,\sigma}$ is Gorenstein if and only if*

$$\sum_{j=1}^{n} \alpha(g)_j \equiv 0 \mod r(g)$$

for every $v_g = \frac{1}{r(g)}\big(\alpha(g)_1, \ldots, \alpha(g)_n\big) \in L \cap \square$.

Proof. For each primitive vector $v_g \in L \cap \square$, the barycentric subdivision of σ at v_g determines a toric blow-up $\varphi \colon B \to A$ of the Abelian quotient $A = X_{L,\sigma}$. The exceptional divisor is $\Gamma = X_{L(\tau),\mathrm{Star}(\tau)}$, where τ is the ray with primitive generator v_g. Adjunction for the toric blow-up φ is

$$K_B = \varphi^* K_A + \left(\frac{1}{r(g)} \sum_{j=1}^{n} \alpha(g)_j - 1\right) \Gamma; \tag{1.3}$$

see [Rei87, §4.8] for a proof. Statements (i) and (ii) of the theorem follow directly from the discrepancy calculation (1.3). For part (iii), recall that the element $g \in G$ of order $r(g)$ acts on $\mathbb{C}^n$ as

$$g = \operatorname{diag}\left(\varepsilon^{\alpha(g)_1}, \dots, \varepsilon^{\alpha(g)_n}\right) \quad \text{with } 0 \leq \alpha(g)_j < r(g),$$

for $\varepsilon^{r(g)} = 1$. Thus $G \subset \mathrm{SL}(n, \mathbb{C})$ if and only if $r(g)$ divides $\sum_{j=1}^{n} \alpha(g)_j$ for every $g \in G$. Part (iii) now follows from Proposition 1.6. □

The process of resolving toric singularities is straightforward and, moreover, the exceptional locus is easy to determine. To illustrate this process we now describe the well known procedure to determine the minimal *Jung–Hirzebruch resolution* of cyclic surface quotients $\mathbb{C}^2/G$.

For a cyclic group $G \subset \mathrm{GL}(2, \mathbb{C})$ it is enough (by killing quasireflections) to consider only cyclic quotients of type $\frac{1}{r}(1, a)$ with a coprime to r. The lattice is $L = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{r}(1, a)$ and the cone σ is the positive quadrant $\langle e_1, e_2 \rangle$. Let $v_0 = (0, 1), v_1 = \frac{1}{r}(1, a), \dots, v_{k+1} = (1, 0)$ denote the points which form the convex hull of lattice points in the positive quadrant (not including the origin). One can determine these points by a simple algorithm: expand the rational number r/a as a 'Hirzebruch continued fraction'

$$\frac{r}{a} = c_1 - \cfrac{1}{c_2 - \frac{1}{\dots - \frac{1}{c_k}}}$$

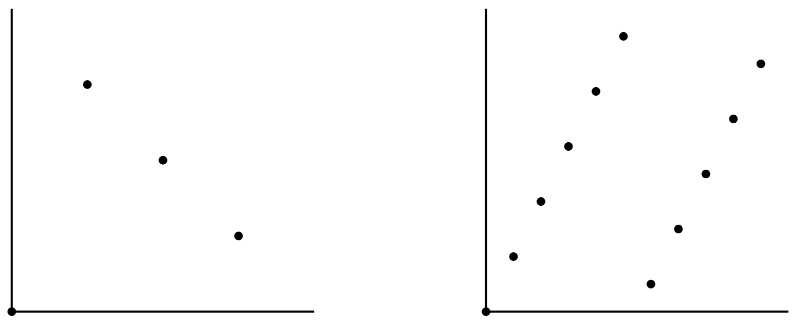
to produce $c_1, \dots, c_k \in \mathbb{Z}_{\geq 2}$, then set $v_0 := (0, 1)$, $v_1 := \frac{1}{r}(1, a)$ and define

$$v_{i+1} := c_i \cdot v_i - v_{i-1} \quad \text{for } i = 1, \dots k.$$

Each v_i lies in L and consecutive pairs generate L over $\mathbb{Z}$. To perform the resolution, subdivide the positive quadrant with rays generated by $v_0, v_1, \dots, v_{k+1}$ and write Σ for the resulting fan. The variety $X_{L,\Sigma}$ is smooth by construction and the toric morphism $\varphi \colon X_{L,\Sigma} \to X_{L,\sigma}$ corresponding to the subdivision is the Jung–Hirzebruch resolution.

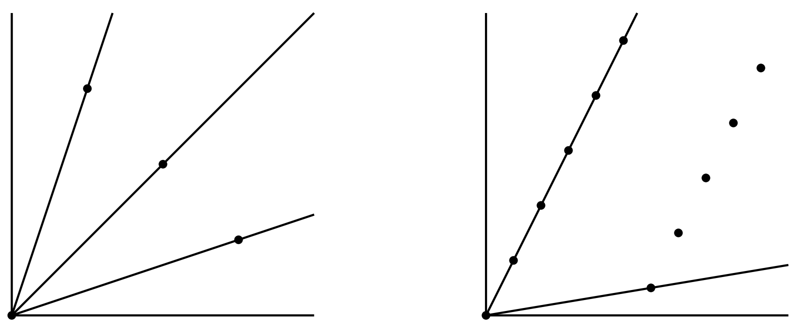
Examples 1.14 The singularity $\frac{1}{4}(1, 3)$ is determined by Figure 1.3(a). We draw the unit square □ in the lattice $L = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{4}(1, 3)$. Figure 1.3(b) illustrates $L \cap □$ for the singularity of type $\frac{1}{11}(1, 2)$.

The fan Σ of the minimal resolution of $\frac{1}{4}(1, 3)$ is shown in Figure 1.4(a). The exceptional locus consists of three toric strata $X_{L(\tau),\mathrm{Star}(\tau)} \cong \mathbb{P}^1$ which



(a) (b)

Figure 1.3: (a) The singularity $\frac{1}{4}(1,3)$; (b) The singularity $\frac{1}{11}(1,2)$



(a) (b)

Figure 1.4: (a) Resolution of $\frac{1}{4}(1,3)$; (b) Resolution of $\frac{1}{11}(1,2)$

correspond to the 1-dimensional cones $\tau \in \Sigma \setminus \sigma$. The resolution is crepant by Theorem 1.13(ii).

The minimal resolution of $\frac{1}{11}(1,2)$ illustrated in Figure 1.4(b) requires only two new rays through $\frac{1}{11}(1,2)$ and $\frac{1}{11}(6,1)$ corresponding to toric strata $X_{L(\tau),\mathrm{Star}(\tau)} \cong \mathbb{P}^1$. These divisors have discrepancy $-\frac{7}{11}$ and $-\frac{5}{11}$ respectively.

1.1.5 Hodge–Deligne numbers of toric varieties

For a smooth projective variety X over $\mathbb{C}$, the cohomology $H^k(X,\mathbb{Q})$ carries a *pure Hodge structure of weight k*; that is, a decreasing Hodge filtration on the complex cohomology

$$H^k(X,\mathbb{C}) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^k \supseteq F^{k+1} = 0$$

such that

$$H^k(X,\mathbb{C}) = F^p \oplus \overline{F^{k-p+1}}. \tag{1.4}$$

Here F^p is the subspace of cohomology classes which can be represented by forms with p or more terms of type dz^i. The integers $h^{p,q}(X) := \dim_{\mathbb{C}} F^p \cap \overline{F^q}$ are the classical *Hodge numbers* of X.

More generally, Deligne has shown [Del71, Del74] that the cohomology groups $H^k(X,\mathbb{Q})$ of a complex algebraic variety X carry a natural *mixed Hodge structure*. This consists of an increasing weight filtration

$$0 = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_{2k} = H^k(X,\mathbb{Q})$$

on the rational cohomology of X and a decreasing Hodge filtration

$$H^k(X,\mathbb{C}) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^k \supseteq F^{k+1} = 0$$

on the complex cohomology of X such that the filtration induced by $F^\bullet$ on the graded quotient $\mathrm{Gr}_l^W H^k(X) := W_l/W_{l-1}$ is a pure Hodge structure of weight l. Thus

$$\mathrm{Gr}_l H^k(X) \otimes \mathbb{C} = F^p \mathrm{Gr}_l^W H^k(X) \oplus \overline{F^{l-p+1} \mathrm{Gr}_l^W H^k(X)} \tag{1.5}$$

where $F^p \mathrm{Gr}_l^W H^k(X)$ denotes the complexified image of $F^p \cap W_l$ in the quotient $W_l/W_{l-1} \otimes \mathbb{C}$. The integers

$$h^{p,q}\big(H^k(X,\mathbb{C})\big) := \dim_{\mathbb{C}} \Big(F^p \mathrm{Gr}_{p+q}^W H^k(X) \cap \overline{F^q \mathrm{Gr}_{p+q}^W H^k(X)}\Big)$$

are called the *Hodge–Deligne numbers* of X. For a smooth projective variety X over $\mathbb{C}$, $\mathrm{Gr}_l^W H^k(X,\mathbb{Q}) = 0$ unless $l = k$ in which case (1.5) coincides with (1.4) and the Hodge–Deligne numbers are the classical Hodge numbers.

Danilov and Khovanskii observe [DK87] that cohomology with compact support $H_c^k(X,\mathbb{Q})$ also admits a mixed Hodge structure and they encode the corresponding Hodge–Deligne numbers in a single polynomial:

Definition 1.15 The *E-polynomial* $E(X) \in \mathbb{Z}[u,v]$ of a complex algebraic variety X of dimension n is defined to be

$$E(X) := \sum_{0 \leq p,q \leq n} \sum_{0 \leq k \leq 2n} (-1)^k h^{p,q}\left(H_c^k(X,\mathbb{C})\right) u^p\, v^q.$$

Evaluating $E(X)$ at $u = v = 1$ produces the standard topological Euler number $e_c(X) = e(X)$.

Theorem 1.16 ([DK87]) *Let X, Y be complex algebraic varieties. Then*

(i) *if $X = \bigsqcup X_i$ is stratified by a disjoint union of locally closed subvarieties then the E-polynomial is additive, i.e., $E(X) = \sum E(X_i)$.*

(ii) *the E-polynomial is multiplicative, i.e., $E(X \times Y) = E(X) \cdot E(Y)$.*

(iii) *if $f\colon Y \to X$ is a locally trivial fibration w.r.t. the Zariski topology and F is the fibre over a closed point then $E(Y) = E(F) \cdot E(X)$.*

Proof. The key to (i) is the existence of the following long exact sequence: for $Z \subset X$ a closed subvariety, the exact sequence

$$\cdots \to H_c^k(X \backslash Z, \mathbb{Q}) \to H_c^k(X, \mathbb{Q}) \to H_c^k(Z, \mathbb{Q}) \to \dots$$

consists of morphisms preserving the mixed Hodge structure. Part (ii) follows as the Künneth isomorphism preserves the mixed Hodge structure. Part (iii) is an application of (i) and (ii) to a locally trivial covering of the fibration f. See [DK87] for more details. □

Proposition 1.17 *The E-polynomial of a toric variety of dimension n is*

$$E\big(X_{L,\Sigma}\big) = \sum_{k=0}^{n} d_k \cdot \big(uv - 1\big)^{n-k} \tag{1.6}$$

where d_k is the number of cones of dimension k in Σ.

Proof. The Hodge numbers of $\mathbb{P}^1$ are well known and, by Theorem 1.16, we compute $E(\mathbb{C}^*) = E(\mathbb{P}^1) - E(0) - E(\infty) = uv - 1$, so

$$E\big((\mathbb{C}^*)^{n-k}\big) = E\big(\mathbb{C}^*\big)^{n-k} = \big(uv - 1\big)^{n-k}.$$

Recall from §1.1.3 that a toric variety $X_{L,\Sigma}$ of dimension n is stratified by orbits of the torus action $O_{L,\tau} \cong (\mathbb{C}^*)^{n-\dim\tau}$, one for each cone $\tau \in \Sigma$. The result follows from Theorem 1.16(i). □

1.2 History of the McKay correspondence

1.2.1 The classical McKay correspondence

McKay [McK80] observed that the Dynkin graph of ADE type associated to a Kleinian singularity $\mathbb{C}^2/G$ (as described in §1.1.1) can be recovered using only the representation theory of the finite subgroup $G \subset \mathrm{SL}(2,\mathbb{C})$. We now explain McKay's construction.

Let $G \subset \mathrm{SL}(2,\mathbb{C})$ be a finite subgroup and let $Q = \mathbb{C}^2$ be the representation induced by the inclusion $G \subset \mathrm{SL}(2,\mathbb{C})$. Let $\rho_0, \dots, \rho_N$ be the irreducible representations of G, with ρ_0 trivial, and set $I := \{0, \dots, N\}$. For any $j \in I$, the representation $Q \otimes \rho_j$ decomposes as a sum of irreducibles

$$Q \otimes \rho_j = \bigoplus_{i \in I} a_{ij}\rho_i \quad \text{for} \quad a_{ij} = \dim_{\mathbb{C}} \mathrm{Hom}_G(\rho_i, Q \otimes \rho_j).$$

Definition 1.18 The *McKay quiver* of $G \subset \mathrm{SL}(2,\mathbb{C})$ is the directed graph with vertex set I, and a_{ij} arrows from vertex i to vertex j.

The orientation of the McKay quiver is reversed when we replace Q by its dual. For $G \subset \mathrm{SL}(2,\mathbb{C})$ the representation Q is self-dual, so every arrow $i \to j$ has an opposite arrow $j \to i$; this pair of opposite arrows is usually referred to as an *edge* and the quiver is called the *McKay graph*, denoted $\widetilde{\Gamma}_G$. Write Γ_G for the subgraph consisting of the vertices ρ_i corresponding to the nontrivial representations, and the edges between them.

Theorem 1.19 ([McK80]) *For $G \subset \mathrm{SL}(2,\mathbb{C})$ a finite subgroup, the McKay graph $\widetilde{\Gamma}_G$ is an extended Dynkin graph of $\widetilde{A}\widetilde{D}\widetilde{E}$ type. Moreover, the subgraph Γ_G consisting of nontrivial representations is the graph A_n, D_n, E_6, E_7 or E_8 which arises as the resolution graph of the singularity $\mathbb{C}^2/G$.*

Discussion of proof. Case by case analysis of the subgroups listed in Table 1.1. For example, the nontrivial representations of the cyclic group action generated by $Q(x,y) = (\varepsilon x, \varepsilon^{n-1}y)$ for $\varepsilon^n = 1$ are the 1-dimensional representations $\rho_i(x) = \varepsilon^i x$, for $i = 1, \dots, n-1$. Clearly $Q = \rho_1 \oplus \rho_{n-1}$ so that $Q \otimes \rho_j = \rho_{j+1} \oplus \rho_{j-1}$. The vertex ρ_j of the McKay graph is therefore joined by an edge to the vertices ρ_{j+1} and ρ_{j-1}. This defines the extended Dynkin graph $\widetilde{A}_{n-1}$. The subgraph Γ_G is the Dynkin graph A_{n-1} which is the resolution graph of the Kleinian singularity $\mathbb{C}^2/G$ for the cyclic group $G = \mathbb{Z}/n$. $\square$

This result uncovers a one-to-one correspondence between the irreducible components D_i of the exceptional locus of the crepant resolution $Y \to \mathbb{C}^2/G$ and the nontrivial irreducible representations ρ_i of $G \subset \mathrm{SL}(2,\mathbb{C})$.

It is well known that the exceptional divisor classes $[D_i]$ define a basis for the homology $H_2(Y,\mathbb{Z})$ and, by adding the homology class of a point, we have a basis of $H_*(Y,\mathbb{Z})$. By adding the trivial representation to the nontrivial representations ρ_i we uncover a bijection

$$\Big\{\text{irreducible representations of } G\Big\} \longleftrightarrow \text{ basis of } H_*(Y,\mathbb{Z}). \tag{1.7}$$

This is the classical *McKay correspondence* for $G \subset \mathrm{SL}(2,\mathbb{C})$.

1.2.2 Geometric interpretation via K-theory

A geometric construction of the McKay correspondence (1.7) was provided by Gonzalez-Sprinberg and Verdier [GSV83]. The key lies in the construction of 'tautological vector bundles' on Y, one for each irreducible representation ρ of the group G.

Definition 1.20 For an irreducible representation $\rho_i \colon G \to \mathrm{GL}(V_i)$, let M_i denote the $\mathcal{O}_X$-module defined by

$$M_i := \mathrm{Hom}_{\mathbb{C}[G]}\big(V_i, \mathbb{C}[x,y]\big).$$

Define $\mathcal{F}_i := \varphi^* M_i/\mathrm{Tors}_{\mathcal{O}_Y}$, where $\mathrm{Tors}_{\mathcal{O}_Y}$ denotes the $\mathcal{O}_Y$-torsion of $\varphi^* M_i$. The sheaf $\mathcal{F}_i$ is locally free of rank equal to the dimension of ρ_i [GSV83, §2.10] and is the *tautological bundle* on Y associated to ρ_i.

Write $\{D_j\}$ for the exceptional prime divisors of φ. Gonzalez-Sprinberg and Verdier [GSV83, §6.2.1] showed, by analysing the subgroups in Table 1.1 case by case, that the nontrivial tautological bundles $\{\mathcal{F}_i\}$ satisfy

$$c_1(\mathcal{F}_i) \cdot [D_j] = \deg\big(\mathcal{F}_i|_{D_j}\big) = \delta_{ij}, \tag{1.8}$$

where δ_{ij} is the Kronecker delta symbol. The classes $c_1(\mathcal{F}_i) \in H^2(Y,\mathbb{Z})$ are therefore dual to the basis $[D_j] \in H_2(Y,\mathbb{Z})$, leading to (1.7).

Example 1.21 Let $X := \mathbb{C}^2/G$ be the Du Val singularity of type A_{n-1}. The minimal resolution $\varphi\colon Y \to X$ is covered by affine sets $U_0, \dots, U_{n-1}$, where $U_j \cong \mathbb{C}^2$ has coordinates

$$u_j = \frac{x^{j+1}}{y^{n-(j+1)}} \quad \text{and} \quad v_j = \frac{y^{n-j}}{x^j};$$

see Figure 1.4(a) for the toric picture of the case $n = 4$. Let D_j denote the exceptional curve defined by the equation $u_{j-1} = 0$ in U_{j-1} and by $v_j = 0$ in U_j, so $D_j \cong \mathbb{P}^1$ is parametrised by the ratio $x^j : y^{n-j}$. Let ρ_i denote the 1-dimensional representation with character $\chi_i = \varepsilon^i$. The monomials x^i and y^{n-i} lie in the χ_i-eigenspace of the G-action and generate the module M_i. The restriction of $\mathcal{F}_i$ to the affine set U_j has generator

$$\mathcal{F}_i|_{U_j} = \begin{cases} \langle x^i \rangle & \text{when } j \geq i, \\ \langle y^{n-i} \rangle & \text{when } j < i. \end{cases}$$

Thus the transition function of $\mathcal{F}_i$ on $U_{j-1} \cap U_j$ is trivial for $i \neq j$, whereas for $i = j$ we have $y^{n-i} = v_i \cdot x^i$. The relation (1.8) follows immediately.

1.2.3 The physicists' Euler number conjecture

For a finite group G acting on a compact manifold M, the quotient M/G is an *orbifold*. Motivated by the physics of string theory, Dixon et al. [DHVW85] introduced the *orbifold Euler number*

$$e(M, G) := \frac{1}{|G|} \sum_{gh=hg} e\left(M^g \cap M^h\right).$$

The sum runs over commuting pairs of elements $g, h \in G$, $|G|$ is the order of G, M^g is the fixed point set of g and e denotes the topological Euler number. The physicists were interested in the case where G is a finite group of automorphisms of a simply connected Kähler manifold M acting in such a way that the stabiliser subgroup of a point $m \in M$ forms a subgroup of $\mathrm{SL}(T_m M)$. The Kähler form is then G-invariant so M/G has trivial canonical sheaf $\omega_{M/G} \cong \mathcal{O}_{M/G}$; that is, M/G is *Calabi–Yau*.

Under these circumstances it was conjectured by Dixon et al. [DHVW86] that the following result holds:

Conjecture 1.22 (physicists' Euler number conjecture) *Suppose that the quotient M/G admits a crepant resolution $Y \to M/G$. Then the orbifold Euler number $e(M,G)$ equals the topological Euler number $e(Y)$.*

The relation between this conjecture and the McKay correspondence was observed by Hirzebruch and Höfer [HH90] (this observation was made independently in a different context by Atiyah and Segal [AS89]). They rewrote the orbifold Euler number as the sum

$$e(M,G) = \sum_{[g]\in\mathrm{Conj}(G)} e\big(M^g/C(g)\big) \tag{1.9}$$

over the conjugacy classes of G, where $C(g)$ denotes the centralizer of $g \in G$. They observe that for a finite subgroup $G \subset \mathrm{U}(n)$ acting on $\mathbb{C}^n$, every fixed point set is contractible. Thus

$$\begin{aligned} e(\mathbb{C}^n, G) &= \#\Big\{\text{conjugacy classes of } G\Big\} && (1.10)\\ &= \#\Big\{\text{irreducible representations of } G\Big\}. && (1.11)\end{aligned}$$

In particular, for $G \subset \mathrm{SU}(2,\mathbb{C})$ write $\varphi\colon Y \to \mathbb{C}^2/G$ for the minimal crepant resolution of the corresponding Kleinian singularity (see §1.1.1). It follows from the classical McKay correspondence that the number $b_2(Y)$ of exceptional divisors of φ coincides with the number of nontrivial irreducible representations. Hence the equality

$$e(\mathbb{C}^2, G) = b_2(Y) + 1 = e(Y) \tag{1.12}$$

can be viewed as a version of the McKay correspondence. As a result, the Physicists' Euler number conjecture may be regarded as a generalisation of the McKay correspondence to dimension three, albeit on a fairly weak cohomological level.

Inspired by the observation of Hirzebruch and Höfer, Reid [Rei92] proposed that a local version of the physicists' Euler number conjecture should hold in arbitrary dimension:

Conjecture 1.23 (generalised McKay conjecture) *For $G \subset \mathrm{SL}(n,\mathbb{C})$ a finite subgroup, suppose that the quotient variety $X := \mathbb{C}^n/G$ admits a crepant resolution $\varphi\colon Y \to X$. Then $H^*(Y,\mathbb{Q})$ has a basis consisting of* *algebraic cycles which correspond one-to-one with conjugacy classes of G. In particular the relation*

$$e(Y) = \#\Big\{\textit{conjugacy classes of } G\Big\} = e(\mathbb{C}^n, G)$$

follows from the observation (1.10) of Hirzebruch and Höfer.

The assumption that X admits a crepant resolution is nontrivial a priori even in dimension 3 (however, see Theorem 1.24(i) below). Indeed, it follows from Propositions 1.6–1.7 and Theorem 1.3 that there is a crepant *partial* resolution $Y \to X$, where Y has at worst $\mathbb{Q}$-factorial terminal singularities. It is not clear a priori that Y is smooth.

Moreover, if a crepant resolution exists it is not necessarily unique. As we remarked following Theorem 1.3, different minimal models of a given 3-fold are related by a finite sequence of flops.

1.2.4 Generalised McKay for 3-folds

Case by case analysis of the finite subgroups of $\mathrm{SL}(3, \mathbb{C})$ by several authors [MOP87, Roa89, Mar92, Roa94, Ito94, Roa96] led to a verification of the generalised McKay conjecture for 3-folds:

Theorem 1.24 *Let $G \subset \mathrm{SL}(3, \mathbb{C})$ be a finite subgroup. Then*

(i) *there is a crepant resolution $Y \to \mathbb{C}^3/G$; that is, Gorenstein 3-fold quotients admit smooth minimal models*[1]*.*

(ii) *Conjecture 1.23 holds, so $e(Y) = \#\{\textit{conjugacy classes of } G\}$.*

We choose not to discuss the original proof of Theorem 1.24 because an elegant proof of Theorem 1.24(ii) was subsequently provided by Ito and Reid [IR96]. The first step in their construction is the introduction of an age grading on the group[2] G:

[1] In dimension 4 and higher this is false – see the examples in §2.4.

[2] Definition 1.25 relies on the choice of root of unity $e^{2\pi i/|G|}$. However there is a canonical age grading on the conjugacy classes of $\Gamma := \mathrm{Hom}(\mu_{|G|}, G)$, for $\mu_{|G|}$ the group of complex $|G|$th roots of unity. A choice of root of unity induces an isomorphism $\Gamma \cong G$; see [IR96, p. 2-3] for more details.

Definition 1.25 For $G \subset \mathrm{SL}(n, \mathbb{C})$ a finite subgroup, each $g \in G$ is conjugate to a diagonal matrix

$$g = \mathrm{diag}\left(e^{2\pi i \alpha(g)_1 / r(g)}, \ldots, e^{2\pi i \alpha(g)_n / r(g)}\right) \quad \text{with} \quad 0 \leq \alpha(g)_j < r(g),$$

where $r(g)$ is the order of g and $i = \sqrt{-1}$. To each conjugacy class $[g]$ of the group G we associate an integer in the range $0 \leq \mathrm{age}[g] \leq n-1$ defined by

$$\mathrm{age}[g] := \frac{1}{r(g)} \sum_{j=1}^{n} \alpha(g)_j.$$

Conjugacy classes of age 1 are called *junior* classes.

Definition 1.26 Let $X_{L,\sigma} = \mathbb{C}^n/G$ be an Abelian quotient singularity. In the notation of §1.1.4, the age grading on G corresponds to the slicing of the unit box $\square \subset \overline{L}_{\mathbb{R}} \cong \mathbb{R}^n$ into polytopes

$$\Delta_k := \left\{(r_1, \ldots, r_n) \in \square \;\middle|\; \sum r_i = k\right\}$$

for $k = 0, \ldots, n-1$; see [IR96, p. 4] for a picture. The simplex $\Delta := \Delta_1$ is called the *junior simplex* of the Abelian quotient $X_{L,\sigma} = \mathbb{C}^n/G$. It contains the lattice points v_g corresponding to junior elements $g \in G$.

Theorem 1.13(ii) can be rephrased in terms of age as follows: the points $v_g \in L$ for which $g \in G$ is junior correspond to the crepant exceptional divisors of a resolution of $\mathbb{C}^n/G$. The main result of [IR96] is a generalisation of this theorem to finite (not necessarily Abelian) subgroups $G \subset \mathrm{SL}(n, \mathbb{C})$:

Theorem 1.27 ([IR96]) *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite subgroup. There is a canonical one-to-one correspondence between junior conjugacy classes of G and crepant discrete valuations of $X = \mathbb{C}^n/G$.*

Discussion of proof. Let $E \subset V \to X$ be a prime divisor on a partial resolution. Ramification theory of discrete valuations in the Galois field extension $k(X) \subset k(\mathbb{C}^n)$ reduces the calculation of the discrepancy of E to the case of a cyclic subgroup in the Galois group $G = \mathrm{Gal}\left(k(\mathbb{C}^n)/k(X)\right)$. The result follows from the Abelian case, Theorem 1.13(ii). See [IR96] for more details. $\square$

Corollary 1.28 *Suppose $Y \to \mathbb{C}^n/G$ is a crepant resolution. Then there is a canonical one-to-one correspondence*

$$\Big\{\textit{junior conjugacy classes of } G\Big\} \longleftrightarrow \textit{basis of } H^2(Y,\mathbb{Q}). \tag{1.13}$$

Proof. The quotient $\mathbb{C}^n/G$ is an affine rational singularity (see §1.1.2) so $\mathrm{Pic}(Y) \cong H^2(Y,\mathbb{Z})$. Thus, $H^2(Y,\mathbb{Z})$ is generated by classes $c_1\big(\mathcal{O}_Y(D_i)\big)$ corresponding to divisors $D_i \subset Y$. Some multiple of every D_i is linearly equivalent to an exceptional divisor and it remains to note that exceptional divisors correspond one-to-one with crepant discrete valuations. $\square$

Using this result, Ito and Reid refined Theorem 1.24(ii) without resorting to case by case analysis of the finite subgroups of $\mathrm{SL}(3,\mathbb{C})$:

Theorem 1.29 ([IR96]) *Let $G \subset \mathrm{SL}(3,\mathbb{C})$ be a finite subgroup and let $Y \to \mathbb{C}^3/G$ be a crepant resolution. Then*

$$\dim_{\mathbb{Q}} H^{2k}(Y,\mathbb{Q}) = \#\Big\{\textit{age } k \textit{ conjugacy classes of } G\Big\}$$

for $k = 0, 1, 2$, so $e(Y) = \#\{$conjugacy classes of $G\}$.

Proof. A crepant resolution $Y \to \mathbb{C}^3/G$ exists by Theorem 1.24(i). Poincaré duality for noncompact manifolds induces a bijection between a basis of $H^4(Y,\mathbb{Q})$ and a basis of $H^2_c(Y,\mathbb{Q})$. Corollary 1.28 establishes that classes in $H^2_c(Y,\mathbb{Q})$ correspond one-to-one with junior classes which fix only the origin in $\mathbb{C}^3$. Now, for $g \in G$ the fixed locus is the linear subspace of $\mathbb{C}^3$ corresponding to the coordinates x_j for which $\alpha(g)_j = 0$, so a basis of $H^2_c(Y,\mathbb{Q})$ corresponds one-to-one with junior classes $[g]$ for which every $\alpha(g)_j > 0$. Now, $\alpha(g)_j > 0$ for $j = 1, 2, 3$ if and only if $\mathrm{age}[g^{-1}] = 2$, leading to a bijection

$$\Big\{\text{age } 2 \text{ classes of } G\Big\} \longleftrightarrow \text{basis of } H^4(Y,\mathbb{Q}). \tag{1.14}$$

The result follows from bijections (1.13) and (1.14), plus the observation that the trivial cohomology class corresponds to the trivial conjugacy class of age zero. $\square$

1.2.5 The strong McKay correspondence

Using the notion of age, Batyrev and Dais [BD96] proposed that the following generalisation of Theorem 1.29 should hold in all dimensions:

Conjecture 1.30 (strong McKay conjecture) *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite subgroup and suppose that the quotient $X = \mathbb{C}^n/G$ admits a crepant resolution $\varphi\colon Y \to X$. The nonzero Betti numbers of Y are*

$$\dim_{\mathbb{C}} H^{2k}(Y, \mathbb{C}) = \#\Big\{\text{age } k \text{ conjugacy classes of } G\Big\}$$

for $k = 0, \ldots, n-1$, so $e(Y) = \#\{\text{conjugacy classes of } G\}$.

The case $n = 2$ is the classical McKay correspondence, while $n = 3$ is, of course, Theorem 1.29 with $\mathbb{Q}$ replaced by $\mathbb{C}$ (see [BD96] for an alternative proof). Batyrev and Dais provided further evidence for their conjecture by proving the toric case:

Theorem 1.31 ([BD96]) *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite Abelian subgroup. Suppose that $\mathbb{C}^n/G$ admits a crepant toric resolution $\varphi\colon Y \to \mathbb{C}^n/G$. Then Conjecture 1.30 holds.*

Proof. Write $\pi\colon \mathbb{C}^n \to \mathbb{C}^n/G$ for the quotient map and let $F := \varphi^{-1}(\pi(0))$ denote the exceptional fibre over $\pi(0)$. There is a deformation retraction of Y onto $F \subset Y$ inducing an isomorphism $H^*(Y, \mathbb{C}) \cong H^*(F, \mathbb{C})$. Moreover, F is compact so $H^{2k}(F, \mathbb{C}) = H^{2k}_c(F, \mathbb{C})$. Thus, to compute the Betti numbers (in fact Hodge–Deligne numbers) of Y we need only compute the E-polynomial of F.

In the notation of §1.1.4, the quotient $\mathbb{C}^n/G$ is the toric variety $X_{L,\sigma}$ and the crepant toric resolution φ corresponds to a basic subdivision Σ of the junior simplex Δ (see Definition 1.26). Moreover, the locus $F = \varphi^{-1}(\pi(0))$ is a toric variety determined by the cones in Σ which do not lie in the boundary of Δ, so the E-polynomial $E(F)$ can be computed using Proposition 1.17. The key observation is that this polynomial can be written in the form

$$E(F) = P_\Delta(uv)(1 - uv)^n = \psi_0(\Delta) + \psi_1(\Delta)uv + \cdots + \psi_{n-1}(\Delta)(uv)^{n-1},$$

where $P_\Delta(uv) = \sum_{k \geq 0} \#\Big\{\text{lattice points in } k\Delta\Big\}(uv)^k$ is the Ehrhart power series. Now, for every $k = 0, \ldots, n-1$ the simplex $k\Delta$ contains the lattice points v_g such that $\mathrm{age}(g) = k$, as well as appropriate combinations of the vectors e_i. It is easy to see that

$$\psi_k(\Delta) = \#\Big\{\text{age } k \text{ conjugacy classes of } G\Big\}.$$

This proves the theorem. □

1.3 Moduli spaces of representations of the McKay quiver

1.3.1 Geometric invariant theory

Let X be a complex affine algebraic variety equipped with an action of a reductive algebraic group G. For each character $\chi \in \mathrm{Hom}(G, \mathbb{C}^*)$, let $\mathbb{C}[X]^\chi$ be the space of functions $f \in \mathbb{C}[X]$ satisfying $f(g \cdot x) = \chi(g) \cdot f(x)$ for all $x \in X$, $g \in G$. These spaces define a graded ring $\bigoplus_{n\geq 0} \mathbb{C}[X]^{n\chi}$ and a variety

$$X /\!\!/_\chi G := \mathrm{Proj}\left(\bigoplus_{n\geq 0} \mathbb{C}[X]^{n\chi}\right)$$

which is projective over $X/G = \mathrm{Spec}\ \mathbb{C}[X]^G$.

The variety $X /\!\!/_\chi G$ admits an alternative construction as a Geometric Invariant Theory (GIT) quotient of X by G with respect to a G-linearisation L_χ of the trivial line bundle; our treatment follows King [Kin94] (see also Newstead [New78] or Mumford et al. [MFK92]). Let L_χ denote the linearisation of the trivial line bundle $L = X \times \mathbb{C}$ with nontrivial action $g(x, \lambda) = (g \cdot x, \chi(g) \cdot \lambda)$ for $g \in G$. A point $x \in X$ is said to be χ-*semistable* if there exists $n \geq 1$ and $f \in \mathbb{C}[X]^{n\chi}$ such that $f(x) \neq 0$. Write $X^{\mathrm{ss}}(L_\chi)$ for the set of χ-semistable points of X. Then

$$X /\!\!/_\chi G = X^{\mathrm{ss}}(L_\chi)/\sim,$$

where $x \sim y$ if and only if the G-orbit closures of x and y intersect in $X^{\mathrm{ss}}(L_\chi)$. In fact, the set $X /\!\!/_\chi G$ parameterises the closed G-orbits in $X^{\mathrm{ss}}(L_\chi)$ because every orbit closure contains a unique closed orbit.

A χ-semistable point $x \in X$ is said to be χ-*stable* if the G-orbit of x is closed in $X^{\mathrm{ss}}(L_\chi)$ and the stabiliser G_x is finite. The set $X^{\mathrm{s}}(L_\chi)$ of χ-stable points is open in $X^{\mathrm{ss}}(L_\chi)$, and the orbit space

$$\mathcal{M}_\chi := X^{\mathrm{s}}(L_\chi)/G \tag{1.15}$$

of χ-stable points forms an open subset of the GIT quotient $X /\!\!/_\chi G$. There is a projective morphism

$$\varphi_\chi\colon X /\!\!/_\chi G \to X /\!\!/_0 G = X/G \tag{1.16}$$

which induces an isomorphism $\varphi_\chi^{-1}(\mathcal{M}_0) \cong \mathcal{M}_0$. When X contains a 0-stable point, the open subset $\mathcal{M}_0 \subset X /\!\!/_0 G$ is dense so φ_χ is birational when X is irreducible.

Thaddeus [Tha96] (see also Dolgachev–Hu [DH98] for the projective case) investigated how $X /\!\!/_\chi G$ depends upon the choice of linearisation L_χ. They showed that the space of linearisations (see Thaddeus [Tha96, §3] for the construction) is a locally polyhedral cone in a rational vector space containing finitely many codimension 1 *walls* which divide the cone into finitely many *chambers*. By definition L_χ lies inside a chamber if and only if $X^{\mathrm{s}}(L_\chi) = X^{\mathrm{ss}}(L_\chi)$, in which case

$$\mathcal{M}_\chi = X /\!\!/_\chi G. \tag{1.17}$$

As L_χ varies within a fixed chamber the variety $\mathcal{M}_\chi$ remains fixed, whereas when L_χ crosses a wall the variety $\mathcal{M}_\chi$ may change; when X is irreducible, wall crossings determine birational transformations of $\mathcal{M}_\chi$.

1.3.2 Moduli of representations of the McKay quiver

Let $G \subset \mathrm{GL}(n,\mathbb{C})$ be a finite subgroup and let $Q = \mathbb{C}^n$ be the *given* representation induced by the inclusion $G \subset \mathrm{GL}(n,\mathbb{C})$. Let $\rho_0, \dots, \rho_N$ be the irreducible representations of G, with ρ_0 trivial, and set $I := \{0, \dots, N\}$. For any $j \in I$, the representation $Q \otimes \rho_j$ decomposes as a sum of irreducibles

$$Q \otimes \rho_j = \bigoplus_{i \in I} a_{ij}\rho_i \quad \text{for} \quad a_{ij} = \dim_{\mathbb{C}} \mathrm{Hom}_G(\rho_i, Q \otimes \rho_j).$$

Definition 1.32 The *McKay quiver* of $G \subset \mathrm{GL}(n,\mathbb{C})$ is the directed graph $\mathcal{Q}$ with vertex set I, and a_{ij} arrows from vertex i to vertex j. A (complex) *representation of* $\mathcal{Q}$ is an I-graded $\mathbb{C}$-vector space $V = \bigoplus V_i$ together with $\mathbb{C}$-linear maps $V_i \to V_j$ indexed by the arrows of $\mathcal{Q}$. Every such representation has a *dimension vector* $\underline{\dim}(V) \in \mathbb{Z}^{N+1}$ whose ith component is $\dim_{\mathbb{C}} V_i$.

Let R denote the regular representation of G. Here we consider only representations of $\mathcal{Q}$ with fixed dimension vector $\underline{r} = (r_0, \dots, r_N) \in \mathbb{Z}^{N+1}$, for $r_i := \dim \rho_i = \dim_{\mathbb{C}} \operatorname{Hom}_G(\rho_i, R)$. The set of all such representations forms an affine space

$$\operatorname{Rep}(\mathcal{Q}, \underline{r}) = \bigoplus_{\{i \to j\} \in \mathcal{Q}} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{r_i}, \mathbb{C}^{r_j}),$$

where we sum over the arrows of $\mathcal{Q}$. The isomorphism classes of $\operatorname{Rep}(\mathcal{Q}, \underline{r})$ correspond to orbits under the action of the group $\operatorname{GL}(\underline{r}) := \prod_{i \in I} \operatorname{GL}(r_i, \mathbb{C})$ of G-equivariant automorphisms of V. The diagonal scalar subgroup $\mathbb{C}^* \subset \operatorname{GL}(\underline{r})$ acts trivially, leaving a faithful action of $\operatorname{PGL}(\underline{r}) := \operatorname{GL}(\underline{r})/\mathbb{C}^*$.

Our interest lies not with the space $\operatorname{Rep}(\mathcal{Q}, \underline{r})$, but with an affine subset $X \subset \operatorname{Rep}(\mathcal{Q}, \underline{r})$ consisting of representations V whose linear maps $V_i \to V_j$ are subject to certain commutativity relations, i.e., certain 'paths' of arrows in the quiver $\mathcal{Q}$ are required to be equal. In order to define X, note first that representations of $\mathcal{Q}$ of dimension vector $\underline{r}$ correspond one-to-one with G-equivariant maps $R \to Q \otimes R$. Indeed, since the number of arrows in $\mathcal{Q}$ from vertex i to vertex j is $\dim_{\mathbb{C}} \operatorname{Hom}_G(\rho_i, Q \otimes \rho_j)$, we have

$$\begin{aligned} \operatorname{Rep}(\mathcal{Q}, \underline{r}) &\cong \bigoplus_{i,j \in I} \operatorname{Hom}_G(\rho_i, Q \otimes \rho_j) \otimes \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{r_i}, \mathbb{C}^{r_j}) \\ &= \operatorname{Hom}_G\Big(\bigoplus_{i \in I} \rho_i \otimes \mathbb{C}^{r_i}, \bigoplus_{j \in I} Q \otimes \rho_j \otimes \mathbb{C}^{r_j} \Big) \\ &\cong \operatorname{Hom}_G(R, Q \otimes R). \end{aligned}$$

Let $X \subset \operatorname{Rep}(\mathcal{Q}, \underline{r})$ denote the affine subset consisting of representations for which the corresponding G-equivariant map $B \in \operatorname{Hom}_G(R, Q \otimes R)$ satisfies the equation $B \wedge B = 0 \in \operatorname{Hom}_G(R, R \otimes \bigwedge^2 Q)$. Thus

$$X \cong \big\{ B \in \operatorname{Hom}_G(R, Q \otimes R) \bigm| B \wedge B = 0 \big\}. \tag{1.18}$$

The reductive algebraic group $\operatorname{PGL}(\underline{r})$ acts on X and moduli are constructed using GIT: the quotient $X /\!\!/_{\chi} \operatorname{PGL}(\underline{r})$ parameterises the closed $\operatorname{PGL}(\underline{r})$-orbits of χ-semistable points in X, and $\mathcal{M}_\chi$ denotes the open subset corresponding to χ-stable points.

There is a direct interpretation of stability for representations of a quiver due to King [Kin94]. Every character of $\operatorname{PGL}(\underline{r})$ is of the form

$$\chi_\theta = \textstyle\prod_{i \in I} \det(g_i)^{\theta_i}$$

for some $\theta = (\theta_0, \theta_1, \ldots, \theta_N) \in \mathbb{Z}^{N+1}$. Given $\theta \in \mathbb{Z}^{N+1}$ and a representation V of the McKay quiver $\mathcal{Q}$ of dimension vector $\underline{\dim}(V) = (v_0, \ldots, v_N)$, let

$$\theta(V) := \theta \cdot \underline{\dim}(V) = \sum v_i \theta_i.$$

Definition 1.33 A representation V of $\mathcal{Q}$ is said to be *θ-stable* if $\theta(V) = 0$ and every proper subrepresentation $0 \subsetneq W \subsetneq V$ has $\theta(W) > 0$. The notion of *θ-semistable* is the same with $\geq$ replacing $>$.

Theorem 1.34 ([Kin94]) *A representation of $\mathcal{Q}$ of dimension vector $\underline{r}$ is θ-stable if and only if the corresponding point of the affine variety* $\mathrm{Rep}(\mathcal{Q}, \underline{r})$ *is χ_θ-stable. The same holds for semistability.*

Definition 1.35 Let $\mathcal{M}_\theta$ denote the orbit space of θ-stable points of the affine variety $X \subset \mathrm{Rep}(\mathcal{Q}, \underline{r})$. Theorem 1.34 gives $\mathcal{M}_\theta = \mathcal{M}_\chi$ for $\chi = \chi_\theta$.

There is an isomorphism $\mathcal{M}_{k\theta} \cong \mathcal{M}_\theta$ for $k \in \mathbb{Z}_{>0}$, so $\mathcal{M}_\theta$ can be defined even for $\theta \in \mathbb{Q}^{N+1}$ (see Sardo Infirri [SI96b, Remark 2.4]). This observation enables us to describe the space of linearisations of the GIT quotient in terms of the parameter space $\mathbb{Q}^{N+1}$ as follows. For a representation V of dimension vector $\underline{r} = (r_0, \ldots, r_N)$ to be θ-stable we require $0 = \theta(V) = \sum r_i \theta_i$. The chamber decomposition on the space of linearisations described in §1.3.1 defines a chamber decomposition on the hyperplane

$$\Pi := \{\theta \in \mathbb{Q}^{N+1} \mid \textstyle\sum_{i \in I} r_i \theta_i = 0\} \tag{1.19}$$

in the rational vector space $\mathbb{Q}^{N+1}$. For a parameter $\theta \in \Pi$ lying inside a chamber, $\mathcal{M}_\theta = X /\!\!/_\chi \mathrm{PGL}(\underline{r})$ holds by (1.17) for the character $\chi = \chi_\theta$ of $\mathrm{PGL}(\underline{r})$. Moreover, the variety $\mathcal{M}_\theta$ remains fixed as we vary θ within a fixed chamber, while an irreducible component of $\mathcal{M}_\theta$ may undergo a birational transformation as θ crosses a wall.

The observation that these moduli spaces form resolutions of a quotient singularity goes back to Kronheimer and subsequent work of Sardo Infirri:

Theorem 1.36 ([Kro86]) *Let $G \subset \mathrm{SL}(2, \mathbb{C})$ be a finite subgroup. Then*

$$\varphi_\theta \colon \mathcal{M}_\theta \longrightarrow \mathbb{C}^2/G$$

is the minimal resolution for generic $\theta \in \Pi$. Moreover, the chambers in Π are Weyl chambers in the weight space of an affine root system of the same ADE-type as $\mathbb{C}^2/G$ (see Table 1.1).

The original proof (see Kronheimer [Kro89]) uses hyper-Kähler reduction. For a proof in the spirit of the GIT construction described here, see Cassens and Slodowy [CS98].

Theorem 1.37 ([SI94]) *Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite subgroup.*

1. *There is an inclusion $\mathbb{C}^n/G \hookrightarrow X /\!\!/_0 \mathrm{PGL}(\underline{r})$ which is an isomorphism if and only if G acts freely on $\mathbb{C}^n$ outside the origin.*
2. *If G acts freely on $\mathbb{C}^n$ outside the origin then $\varphi_\theta \colon \mathcal{M}_\theta \to \mathbb{C}^n/G$ is a (partial) resolution.*

Remark 1.38 Even if $G \subset \mathrm{GL}(n, \mathbb{C})$ fails to act freely outside the origin, one expects there to be a birational morphism from $\mathcal{M}_\theta$ to the component $\mathbb{C}^n/G$ of $X /\!\!/_0 \mathrm{PGL}(\underline{r})$ (see Sardo Infirri [SI96a]). Ishii [Ish00] observed that the method of Bridgeland, King and Reid [BKR99] proves this for finite subgroups $G \subset \mathrm{SL}(3, \mathbb{C})$.

Chapter 2

Motivic integration

This chapter provides an elementary introduction to Kontsevich's theory of motivic integration and discusses how Batyrev [Bat99b, Bat00] used motivic integration to prove the strong McKay conjecture (Conjecture 1.30). Also, in Section 2.4 we provide a detailed calculation of the motivic integral of several Gorenstein terminal cyclic quotient singularities. The original references on this topic are Batyrev [Bat98, §6] and Denef and Loeser [DL99b]. This chapter is an expanded version of the preprint Craw [Cra99]. A recent article by Looijenga [Loo00] provides a more detailed survey of the subject.

2.1 The Hodge number conjecture

In the course of proving the Abelian case of the strong McKay conjecture (see Theorem 1.31), Batyrev and Dais established that the E-polynomial of a crepant resolution of $\mathbb{C}^n/G$ is determined by the Ehrhart power series and hence by the lattice L of the toric variety $\mathbb{C}^n/G = X_{L,\sigma}$. In particular, the Hodge–Deligne numbers of a crepant resolution of $\mathbb{C}^n/G$ do not depend upon the simplicial subdivision of Δ so they are independent of the choice of crepant resolution. This observation led Batyrev and Dais [BD96] to formulate the following conjecture:

Conjecture 2.1 (Hodge number conjecture) *For a complex projective variety with only mild Gorenstein singularities, the Hodge numbers of a crepant resolution do not depend upon the choice of crepant resolution.*

In a subsequent paper Batyrev [Bat99a] used methods of p-adic integration to prove that the Betti numbers of a crepant resolution do not depend upon the choice of crepant resolution. Kontsevich later proved the conjecture by associating a 'motivic integral' to every complex projective variety X with Gorenstein canonical singularities as we now describe.

2.2 Construction of the motivic integral

Let X be a complex algebraic variety with Gorenstein canonical singularities and $\varphi\colon Y \to X$ a resolution of singularities for which the discrepancy divisor D has simple normal crossings. Following Batyrev [Bat98, §6] and Denef and Loeser [DL99b] we associate an integral to X in five steps:

1. Construct the *space of formal arcs* $J_\infty(Y)$ of the complex manifold Y.
2. Associate to the divisor D a function F_D defined on $J_\infty(Y)$.
3. Introduce a measure μ on $J_\infty(Y)$ with respect to which F_D is measurable. This measure is not real-valued; the subtlety in the construction of the motivic integral is in defining the ring in which μ takes values.
4. Define the *motivic integral* of the pair (Y, D) to be the integral of F_D over $J_\infty(Y)$ with respect to μ.
5. Set the motivic integral of X to be the motivic integral of the pair (Y, D). We prove that this is independent of the choice of resolution.

2.2.1 The space of formal arcs of a complex manifold

Definition 2.2 Let Y be a complex manifold of dimension n, and $y \in Y$ a point. A *k-jet* over y is a morphism

$$\gamma_y\colon \text{Spec } \mathbb{C}[z]/\langle z^{k+1}\rangle \longrightarrow Y \quad \text{with} \quad \gamma_y(\text{Spec } \mathbb{C}) = y.$$

Once local co-ordinates are chosen the space of k-jets over y can be viewed as the space of n-tuples of polynomials of degree k whose constant terms are zero. Let $J_k(Y)$ denote the bundle over Y whose fibre over $y \in Y$ is the space of k-jets over y. A *formal arc* over y is a morphism

$$\gamma_y\colon \text{Spec } \mathbb{C}[\![z]\!] \longrightarrow Y \quad \text{with} \quad \gamma_y(\text{Spec } \mathbb{C}) = y.$$

Once local co-ordinates are chosen the space of formal arcs over y can be viewed as the space of n-tuples of power series whose constant terms are zero.

Let $\pi_0 \colon J_\infty(Y) \to Y$ denote the bundle whose fibre over $y \in Y$ is the space of formal arcs over y. For each $k \in \mathbb{Z}_{\geq 0}$ the inclusion $\mathbb{C}[z]/\langle z^{k+1}\rangle \hookrightarrow \mathbb{C}[\![z]\!]$ induces a surjective map

$$\pi_k \colon J_\infty(Y) \longrightarrow J_k(Y).$$

Definition 2.3 A subset $C \subseteq J_\infty(Y)$ of the space of formal arcs is called a *cylinder set* if $C = \pi_k^{-1}(B_k)$ for $k \in \mathbb{Z}_{\geq 0}$ and $B_k \subseteq J_k(Y)$ a constructible subset. Recall that a subset of a variety is *constructible* if it is a finite, disjoint union of (Zariski) locally closed subvarieties.

It's clear that the collection of cylinder sets forms an algebra of sets (see [Rud87, p. 10]); that is, $J_\infty(Y) = \pi_0^{-1}(Y)$ is a cylinder set, as are finite unions and complements (and hence finite intersections) of cylinder sets.

2.2.2 The function F_D associated to an effective divisor

Definition 2.4 Let D be an effective divisor on Y, $y \in Y$ a point, and g a local defining equation for D on a neighbourhood U of y. For an arc γ_u over a point $u \in U$, define the *intersection number* $\gamma_u \cdot D$ to be the order of vanishing of the formal power series $g(\gamma_u(z))$ at $z = 0$. The function

$$F_D \colon J_\infty(Y) \to \mathbb{Z}_{\geq 0} \cup \infty$$

associated to the divisor D on Y is given by $F_D(\gamma_u) = \gamma_u \cdot D$. If we write $D = \sum_{i=1}^r a_i D_i$ as a linear combination of prime divisors then g decomposes as a product $g = \prod_{i=1}^r g_i^{a_i}$ of defining equations for D_i, hence $F_D = \sum_{i=1}^r a_i F_{D_i}$. Furthermore

$$F_{D_i}(\gamma_u) = 0 \iff u \notin D_i \quad \text{and} \quad F_{D_i}(\gamma_u) = \infty \iff \gamma_u \subseteq D_i. \tag{2.1}$$

Our ultimate goal is to integrate the function F_D over $J_\infty(Y)$, so we must understand the nature of the level set $F_D^{-1}(s) \subseteq J_\infty(Y)$ for each $s \in \mathbb{Z}_{\geq 0} \cup \infty$. With this goal in mind, we introduce a partition of $F_D^{-1}(s)$.

Definition 2.5 For $D = \sum_{i=1}^r a_i D_i$ and $J \subseteq \{1, \ldots, r\}$ any subset, define

$$D_J := \begin{cases} \bigcap_{j \in J} D_j & \text{if } J \neq \emptyset \\ Y & \text{if } J = \emptyset \end{cases} \quad \text{and} \quad D_J^\circ := D_J \setminus \bigcup_{i \in \{1,\ldots,r\} \setminus J} D_i.$$

These subvarieties stratify Y and define a partition of the space of arcs into cylinder sets:

$$Y = \bigsqcup_{J\subseteq\{1,\ldots,r\}} D_J^\circ \quad\text{and}\quad J_\infty(Y) = \bigsqcup_{J\subseteq\{1,\ldots,r\}} \pi_0^{-1}(D_J^\circ).$$

For any $s \in \mathbb{Z}_{\geq 0}$ and $J \subseteq \{1,\ldots,r\}$, define

$$M_{J,s} := \left\{(m_1,\ldots,m_r) \in \mathbb{Z}_{\geq 0}^r \mid \sum a_i m_i = s \text{ with } m_j > 0 \Leftrightarrow j \in J\right\}.$$

It now follows from (2.1) that

$$\gamma_u \in \pi_0^{-1}(D_J^\circ) \cap F_D^{-1}(s) \iff (F_{D_1}(\gamma_u),\ldots,F_{D_r}(\gamma_u)) \in M_{J,s}.$$

As a result we produce a finite partition of the level set

$$F_D^{-1}(s) = \bigsqcup_{J\subseteq\{1,\ldots,r\}} \bigsqcup_{(m_1,\ldots,m_r)\in M_{J,s}} \left(\bigcap\nolimits_{i=1,\ldots r} F_{D_i}^{-1}(m_i)\right). \tag{2.2}$$

Proposition 2.6 *If D is an effective divisor with simple normal crossings then $F_D^{-1}(s)$ is a cylinder set (see Definition 2.3) for each $s \in \mathbb{Z}_{\geq 0}$.*

Recall (see [KMM87, p. 25]) that a divisor $D = \sum_{i=1}^r a_i D_i$ on Y has only simple normal crossings if, at each point $y \in Y$, there is a neighbourhood U of y with coordinates $z_1,\ldots,z_n$ for which a local defining equation for D is

$$g = z_1^{a_1}\cdots z_{j_y}^{a_{j_y}} \quad\text{for some } j_y \leq n. \tag{2.3}$$

Proof of Proposition 2.6. A finite union of cylinder sets is cylinder and we have a partition (2.2) of $F_D^{-1}(s)$, so it is enough to prove, for some $J \subseteq \{1,\ldots,r\}$ and $(m_1,\ldots,m_r) \in M_{J,s}$, that $\bigcap_{i=1,\ldots r} F_{D_i}^{-1}(m_i)$ is a cylinder set[1]. Cover $Y = \bigcup U$ by finitely many charts on which D has a local equation of the form (2.3), and lift to cover $J_\infty(Y) = \bigcup \pi_0^{-1}(U)$. Clearly we need only prove that the set

$$U_{m_1,\ldots,m_r} := \bigcap\nolimits_{i=1,\ldots r} F_{D_i}^{-1}(m_i) \cap \pi_0^{-1}(U)$$

[1]Finite intersections of cylinder sets are cylinder, so we could reduce to proving the result for $F_{D_i}^{-1}(m_i)$. However we require (2.4) in §2.2.4.

is cylinder. In the notation of (2.3), if $J \not\subseteq \{1, \ldots, j_y\}$ then $D_J^\circ \cap U = \emptyset$ which forces $U_{m_1,\ldots,m_r} \subset \pi_0^{-1}(D_J^\circ \cap U)$ to be empty, and hence a cylinder set. We suppose therefore that $J \subseteq \{1, \ldots, j_y\}$, thus $|J| \leq n$ holds by (2.3).

The key observation is that when we regard each arc γ_u as an n-tuple $(p_1(z), \ldots, p_n(z))$ of formal power series with zero constant term, each condition $F_{D_i}(\gamma_u) = m_i$ is equivalent to a condition on the truncation of the power series $p_i(z)$ to degree m_i. Indeed, since D_i is cut out by $z_i = 0$ on U, it follows that $F_{D_i}(\gamma_u) = \{\text{order of } p_i(z) \text{ at } z = 0\}$. Thus $\gamma_u \in F_{D_i}^{-1}(m_i)$ if and only if the truncation of $p_i(z)$ to degree m_i is of the form $c_{m_i} z^{m_i}$, with $c_{m_i} \neq 0$. Truncating all n of the power series to degree $t := \max\{m_j | j \in J\}$ produces $n - |J|$ polynomials of degree t with zero constant term, and, for each $j \in J$, a polynomial of the form

$$\pi_t(p_j(z)) = 0 + \cdots + 0 + c_{m_j} z^{m_j} + c_{(m_j+1)} z^{m_j+1} + \cdots + c_t z^t$$

for $c_{m_j} \in \mathbb{C}^*$ and $c_k \in \mathbb{C} \ \ \forall\, k > m_j$. The space of all such n-tuples is isomorphic to $\mathbb{C}^{t(n-|J|)} \times (\mathbb{C}^*)^{|J|} \times \mathbb{C}^{t|J| - \sum_{j\in J} m_j}$, hence

$$U_{m_1,\ldots,m_r} = \pi_t^{-1}\left((U \cap D_J^\circ) \times \mathbb{C}^{tn - \sum_{j\in J} m_j} \times (\mathbb{C}^*)^{|J|}\right). \tag{2.4}$$

The set $(U \cap D_J^\circ) \times \mathbb{C}^{tn - \sum_{j\in J} m_j} \times (\mathbb{C}^*)^{|J|}$ is constructible, so $U_{m_1,\ldots,m_r}$ is a cylinder set. This completes the proof of the proposition. $\square$

It is worth noting that $F_D^{-1}(\infty)$ is not a cylinder set. Indeed, suppose otherwise, so there exists a constructible subset $B_k \subseteq J_k(Y)$ for which $F_D^{-1}(\infty) = \pi_k^{-1}(B_k)$. Each arc $\gamma_y \in F_D^{-1}(\infty)$ is an n-tuple of power series, at least one of which is identically zero, whereas each $\gamma_y \in \pi_k^{-1}(B_k)$ is an n-tuple of power series whose terms of degree higher than k may take any complex value; clearly this is absurd.

Proposition 2.7 $F_D^{-1}(\infty)$ *is a countable intersection of cylinder sets.*

Proof. Observe that

$$F_D^{-1}(\infty) = \bigcap_{k\in\mathbb{Z}_{\geq 0}} \pi_k^{-1}\pi_k\big(F_D^{-1}(\infty)\big) \tag{2.5}$$

because a power series is identically zero if and only if its truncation to degree k is the zero polynomial, for all $k \in \mathbb{Z}_{\geq 0}$. It is easy to see that the sets $\pi_k(F_D^{-1}(\infty)) \subset J_k(Y)$ are constructible. $\square$

2.2.3 A measure μ on the space of formal arcs

In this section we define a measure μ on $J_\infty(Y)$ with respect to which the function F_D is measurable. The measure is not real-valued, so we begin this section by constructing the ring in which μ takes values.

Definition 2.8 Let $\mathcal{V}_\mathbb{C}$ denote the category of complex algebraic varieties. The Grothendieck group of $\mathcal{V}_\mathbb{C}$ is the free Abelian group on the isomorphism classes $[V]$ of complex algebraic varieties modulo the subgroup generated by elements of the form $[V] - [V'] - [V \setminus V']$ for a closed subset $V' \subseteq V$. The product of varieties induces a ring structure $[V] \cdot [V'] = [V \times V']$, and the resulting ring, denoted by $K_0(\mathcal{V}_\mathbb{C})$, is called the *Grothendieck ring of complex algebraic varieties.* Let

$$[\quad]\colon \mathrm{Ob}\mathcal{V}_\mathbb{C} \longrightarrow K_0(\mathcal{V}_\mathbb{C})$$

denote the natural map sending V to its class $[V]$ in the Grothendieck ring. This map is universal with respect to maps which are additive on disjoint unions of constructible subsets, and which respect products.

Write[2] $1 := [\text{point}]$ and $\mathbb{L} := [\mathbb{C}]$. Then

$$[\mathbb{C}^*] = [\mathbb{C} - \{0\}] = [\mathbb{C}] - [\{0\}] = \mathbb{L} - 1.$$

Also, if $f\colon Y \to X$ is a locally trivial fibration w.r.t. the Zariski topology and F is the fibre over a closed point then $[Y] = [F \times X]$.

Definition 2.9 Let $K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}] := S^{-1}K_0(\mathcal{V}_\mathbb{C})$ denote the ring of fractions of $K_0(\mathcal{V}_\mathbb{C})$ with respect to the multiplicative set $S := \{1, \mathbb{L}, \mathbb{L}^2, \dots\}$.

Definition 2.10 Recall that cylinder sets in $J_\infty(Y)$ are subsets $\pi_k^{-1}(B_k) \subset J_\infty(Y)$ for $k \in \mathbb{Z}_{\geq 0}$ and for $B_k \subseteq J_k(Y)$ a constructible subset. The function

$$\widetilde{\mu}\colon \Big\{\text{cylinder sets in } J_\infty(Y)\Big\} \longrightarrow K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}]$$

which assigns a 'measure' to each cylinder set is defined by

$$\pi_k^{-1}(B_k) \to [B_k] \cdot \mathbb{L}^{-n(k+1)}.$$

[2]See the Appendix A: the class of $\mathbb{C}$ in $K_0(\mathcal{V}_\mathbb{C})$ corresponds to the Tate motive $\mathbb{L}$.

Using the fact that the map [ ] introduced in Definition 2.8 is additive on disjoint unions of constructible sets, it is straightforward to show that

$$\widetilde{\mu}\left(\bigsqcup_{i=1}^{l} C_i\right) = \sum_{i=1}^{l} \widetilde{\mu}(C_i) \quad \text{for cylinder sets } C_1, \ldots, C_l.$$

For this reason we call $\widetilde{\mu}$ a finitely additive measure.

Proposition 2.6 states that for $s \in \mathbb{Z}_{\geq 0}$, the level set $F_D^{-1}(s)$ is a cylinder set, and is therefore $\widetilde{\mu}$-measurable. However, F_D is not $\widetilde{\mu}$-measurable because $F_D^{-1}(\infty)$ is not cylinder. To proceed, we extend $\widetilde{\mu}$ to a measure μ with respect to which $F_D^{-1}(\infty)$ is measurable.

The following discussion is intended to motivate the definition of μ (see Definition 2.12 to follow). The set $J_\infty(Y) \setminus F_D^{-1}(\infty)$ is a countable disjoint union of cylinder sets

$$J_\infty(Y) \setminus \pi_0^{-1}\pi_0(F_D^{-1}(\infty)) \sqcup \bigsqcup_{k \in \mathbb{Z}_{\geq 0}} \left(\pi_k^{-1}\pi_k(F_D^{-1}(\infty)) \setminus \pi_{k+1}^{-1}\pi_{k+1}(F_D^{-1}(\infty))\right); \tag{2.6}$$

to see this, take complements in equation (2.5) of Proposition 2.7. Our goal is to extend $\widetilde{\mu}$ to a measure μ defined on the collection of countable disjoint unions of cylinder sets so that the set $J_\infty(Y) \setminus F_D^{-1}(\infty)$, and hence its complement $F_D^{-1}(\infty)$, is μ-measurable. One would like to define

$$\mu\left(\bigsqcup_{i \in \mathbb{N}} C_i\right) := \sum_{i \in \mathbb{N}} \mu(C_i) = \sum_{i \in \mathbb{N}} \widetilde{\mu}(C_i) \quad \text{for cylinder sets } C_1, \ldots, C_l. \tag{2.7}$$

However, countable sums are not defined in $K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}]$. Furthermore, given a countable disjoint union $C = \bigsqcup_{i \in \mathbb{N}} C_i$, it is not clear a priori that $\mu(C)$ defined by formula (2.7) is independent of the choice of the C_i.

Kontsevich [Kon95] solved both of these problems at once by completing the ring $K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}]$, thereby allowing appropriate countable sums, in such a way that the measure of the set $C = \bigsqcup_{i \in \mathbb{N}} C_i$ is independent of the choice of the C_i, assuming that $\mu(C_i) \to 0$ as $i \to \infty$.

Definition 2.11 Let R denote the completion of the ring $K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}]$ with respect to the filtration

$$\cdots \supseteq F^{-1}K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}] \supseteq F^0 K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}] \supseteq F^1 K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}] \supseteq \cdots$$

where for each $m \in \mathbb{Z}$, $F^m K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}]$ is the subgroup of $K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}]$ generated by elements of the form $[V] \cdot \mathbb{L}^{-i}$ for $i - \dim V \geq m$. The natural completion map is denoted $\phi\colon K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}] \longrightarrow R$.

By composing $\widetilde{\mu}$ with the natural completion map ϕ, we produce a finitely additive measure with values in the ring R, namely

$$\widetilde{\mu} := \phi \circ \widetilde{\mu}\colon \pi_k^{-1}(B_k) \to \phi\left([B_k] \cdot \mathbb{L}^{-n(k+1)}\right)$$

which we also denote $\widetilde{\mu}$. Given a sequence of cylinder sets $\{C_i\}$ one may now ask whether or not $\widetilde{\mu}(C_i) \to 0$ as $i \to \infty$. We are finally in a position to define the measure μ on the space of formal arcs.

Definition 2.12 Let $\mathcal{C}$ denote the collection of countable disjoint unions of cylinder sets $\bigsqcup_{i \in \mathbb{N}} C_i$ for which $\widetilde{\mu}(C_i) \to 0$ as $i \to \infty$, together with the complements of such sets. Extend $\widetilde{\mu}$ to a measure μ defined on $\mathcal{C}$ which takes values in R given by

$$\bigsqcup_{i \in \mathbb{N}} C_i \longrightarrow \sum_{i \in \mathbb{N}} \widetilde{\mu}(C_i).$$

It is nontrivial to show (see [DL99b, §3.2] or [Bat98, §6.18]) that this definition is independent of the choice of the C_i.

Proposition 2.13 *F_D is μ-measurable, and $\mu(F_D^{-1}(\infty)) = 0$.*

Proof. We prove that $F_D^{-1}(\infty)$ (in fact its complement) lies in $\mathcal{C}$. It's clear from (2.6) that we need only prove that $\mu(\pi_k^{-1}\pi_k(F_D^{-1}(\infty))) \to 0$ as $k \to \infty$. Lemma 2.14 below reveals that $\mu(\pi_k^{-1}\pi_k(F_D^{-1}(\infty))) \in \phi(F^{k+1}K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}])$ which, by the nature of the topology on R, tends to zero as k tends to infinity. This proves the first statement. Using (2.6) we calculate

$$\begin{aligned} \mu\big(J_\infty(Y) \setminus F_D^{-1}(\infty)\big) &= \widetilde{\mu}\big(J_\infty(Y) \setminus \pi_0^{-1}\pi_0(F_D^{-1}(\infty))\big) \\ &\quad + \sum_{k \in \mathbb{Z}_{\geq 0}} \widetilde{\mu}\left(\pi_k^{-1}\pi_k(F_D^{-1}(\infty)) \setminus \pi_{k+1}^{-1}\pi_{k+1}(F_D^{-1}(\infty))\right). \end{aligned} \tag{2.8}$$

This equals $\mu(J_\infty(Y)) - \lim_{k\to\infty} \mu\left(\pi_k^{-1}\pi_k(F_D^{-1}(\infty))\right)$. By the above remark, this is simply $\mu(J_\infty(Y))$, so $\mu(F_D^{-1}(\infty)) = 0$ as required. $\square$

Lemma 2.14 $\widetilde{\mu}(\pi_k^{-1}\pi_k(F_D^{-1}(\infty))) \in F^{k+1}K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}]$

PROOF. It is enough to prove the result for a prime divisor D, since $F_D^{-1}(\infty)$ is the union of sets $F_{D_i}^{-1}(\infty)$. Choose coordinates on a chart U in which D is $(z_1 = 0)$. Each $\gamma_y \in F_D^{-1}(\infty) \cap \pi_0^{-1}(U)$ is an n-tuple $(p_1(z), \ldots, p_n(z))$ of power series over $y \in U \cap D$ such that $p_1(z)$ is identically zero. Truncating these power series to degree k leaves $n-1$ polynomials of degree k with zero constant term, and the zero polynomial $\pi_k(p_1(z))$. The space of all such polynomials is isomorphic to $\mathbb{C}^{(n-1)k}$, so that $\pi_k(F_D^{-1}(\infty) \cap \pi_0^{-1}(U)) \simeq (U \cap D) \times \mathbb{C}^{(n-1)k}$. Thus $[\pi_k(F_D^{-1}(\infty)] = [D] \cdot [\mathbb{C}^{(n-1)k}]$ and

$$\begin{aligned}
\widetilde{\mu}(\pi_k^{-1}\pi_k(F_D^{-1}(\infty)) &= [\pi_k(F_D^{-1}(\infty)] \cdot \mathbb{L}^{-n(k+1)} \\
&= [D] \cdot \mathbb{L}^{(n-1)k} \cdot \mathbb{L}^{-n(k+1)} \\
&= [D] \cdot \mathbb{L}^{-(n+k)}
\end{aligned}$$

This lies in $F^{k+1}K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]$ since D has dimension $n-1$. □

2.2.4 The motivic integral of a pair (Y, D)

Definition 2.15 Let Y be a complex manifold of dimension n, and choose an effective divisor $D = \sum_{i=1}^r a_i D_i$ on Y with only simple normal crossings. The *motivic integral of the pair* (Y, D) is

$$\int_{J_\infty(Y)} F_D \,\mathrm{d}\mu := \sum_{s \in \mathbb{Z}_{\geq 0} \cup \infty} \mu\left(F_D^{-1}(s)\right) \cdot \mathbb{L}^{-s}.$$

Since the set $F_D^{-1}(\infty) \subset J_\infty(Y)$ has measure zero (see Proposition 2.13), we need only integrate over $J_\infty(Y) \setminus F_D^{-1}(\infty)$, so we need only sum over $s \in \mathbb{Z}_{\geq 0}$.

We now show that the motivic integral converges in the ring R introduced in Definition 2.11. In doing so, we establish a user-friendly formula.

Theorem 2.16 *Let Y be a complex manifold of dimension n and $D = \sum_{i=1}^r a_i D_i$ an effective divisor on Y with only simple normal crossings. The motivic integral of the pair (Y, D) is*

$$\int_{J_\infty(Y)} F_D \,\mathrm{d}\mu = \sum_{J \subseteq \{1,\ldots,r\}} [D_J^\circ] \cdot \left(\prod_{j \in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_j+1} - 1} \right) \cdot \mathbb{L}^{-n}$$

where we sum over all subsets $J \subseteq \{1, \ldots, r\}$ including $J = \emptyset$.

PROOF. In the proof of Proposition 2.6 we cover Y by sets $\{U\}$ and prove that $\bigcap_{i=1,\ldots r} F_{D_i}^{-1}(m_i) \cap \pi_0^{-1}(U)$ is a cylinder set of the form

$$\pi_t^{-1}\left((U \cap D_J^\circ) \times \mathbb{C}^{tn-\sum_{j\in J} m_j} \times (\mathbb{C}^*)^{|J|}\right).$$

Since the map $[\quad]$ introduced in Definition 2.8 is additive on a disjoint union of constructible subsets, take the union over the cover $\{U\}$ of Y to see that $\bigcap_{i=1,\ldots r} F_{D_i}^{-1}(m_i) = \pi_t^{-1}(B_t)$ where

$$[B_t] = \left[D_J^\circ \times \mathbb{C}^{tn-\sum_{j\in J} m_j} \times (\mathbb{C}^*)^{|J|}\right] = [D_J^\circ] \cdot \mathbb{L}^{tn-\sum_{j\in J} m_j} \cdot (\mathbb{L}-1)^{|J|}.$$

Since $\mu\left(\pi_t^{-1}(B_t)\right) = [B_t] \cdot \mathbb{L}^{-(n+nt)}$, we have

$$\mu\left(\bigcap_{i=1,\ldots r} F_{D_i}^{-1}(m_i)\right) = [D_J^\circ] \cdot \mathbb{L}^{-\sum_{j\in J} m_j} \cdot (\mathbb{L}-1)^{|J|} \cdot \mathbb{L}^{-n}.$$

Now use the partition (2.2) of $F_D^{-1}(s)$ to compute the motivic integral:

$$\begin{aligned}
&\sum_{s\in\mathbb{Z}_{\geq 0}} \mu\left(F_D^{-1}(s)\right) \cdot \mathbb{L}^{-s} \\
&= \sum_{s\in\mathbb{Z}_{\geq 0}} \sum_{J\subset\{1,\ldots,r\}} \sum_{(m_1,\ldots,m_r)\in M_{J,s}} \mu\left(\bigcap_{i=1,\ldots r} F_{D_i}^{-1}(m_i)\right) \cdot \mathbb{L}^{-\sum_{j\in J} a_j m_j} \\
&= \sum_{s\in\mathbb{Z}_{\geq 0}} \sum_{J\subset\{1,\ldots,r\}} \sum_{(m_1,\ldots,m_r)\in M_{J,s}} [D_J^\circ] \cdot (\mathbb{L}-1)^{|J|} \cdot \mathbb{L}^{-n} \cdot \prod_{j\in J} \mathbb{L}^{-(a_j+1)m_j} \\
&= \sum_{J\subset\{1,\ldots,r\}} [D_J^\circ] \cdot \prod_{j\in J} \left((\mathbb{L}-1) \cdot \textstyle\sum_{m_j>0} \mathbb{L}^{-(a_j+1)m_j}\right) \cdot \mathbb{L}^{-n} \\
&= \sum_{J\subset\{1,\ldots,r\}} [D_J^\circ] \cdot \prod_{j\in J} \left((\mathbb{L}-1) \cdot \left(\frac{1}{1-\mathbb{L}^{-(a_j+1)}} - 1\right)\right) \cdot \mathbb{L}^{-n} \\
&= \sum_{J\subset\{1,\ldots,r\}} [D_J^\circ] \cdot \left(\prod_{j\in J} \frac{\mathbb{L}-1}{\mathbb{L}^{a_j+1}-1}\right) \cdot \mathbb{L}^{-n}.
\end{aligned}$$

This completes the proof of the theorem. □

Warning 2.17 There is a small error in the proof of the corresponding result in Batyrev [Bat98, §6.28] which leads to the omission of the $\mathbb{L}^{-n}$ term.

Corollary 2.18 *The motivic integral of the pair* (Y, D) *is an element of the subring*

$$\phi\big(K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]\big)\left[\left\{\frac{1}{\mathbb{L}^i - 1}\right\}_{i\in\mathbb{N}}\right]$$

of the ring R *introduced in Definition 2.11.*

2.2.5 The transformation rule for the integral

The discrepancy divisor $W := K_{Y'} - \alpha^*K_Y$ of a proper birational morphism $\alpha\colon Y' \to Y$ between smooth varieties is the divisor of the Jacobian determinant of α. The next result may therefore be viewed as the 'change of variables formula' for the motivic integral.

Theorem 2.19 *Let* $\alpha\colon Y' \longrightarrow Y$ *be a proper birational morphism of between smooth varieties and let* $W := K_{Y'} - \alpha^*K_Y$ *be the discrepancy divisor. Then*

$$\int_{J_\infty(Y)} F_D \,\mathrm{d}\mu = \int_{J_\infty(Y')} F_{\alpha^*D+W} \,\mathrm{d}\mu\,.$$

Proof. Composition defines maps $\alpha_t\colon J_t(Y') \to J_t(Y)$ for each $t \in \mathbb{Z}_{\geq 0}\cup\infty$. An arc in Y which is not contained in the locus of indeterminacy of α^{-1} has a birational transform as an arc in Y'. In light of (2.1) and Proposition 2.13, α_∞ is bijective off a subset of measure zero.

The sets $F_W^{-1}(k)$, for $k \in \mathbb{Z}_{\geq 0}$, partition $J_\infty(Y') \setminus F_W^{-1}(\infty)$. Thus, for any $s \in \mathbb{Z}_{\geq 0}$ we have, modulo the set $F_W^{-1}(\infty)$ of measure zero, a partition

$$F_D^{-1}(s) = \bigsqcup_{k\in\mathbb{Z}_{\geq 0}} \alpha_\infty(C_{k,s}) \quad \text{where} \quad C_{k,s} := F_W^{-1}(k) \cap F_{\alpha^*D}^{-1}(s). \tag{2.9}$$

The set $C_{k,s}$ is cylinder and, since the image of a constructible set is constructible ([Mum88, p. 72]), the set $\alpha_\infty(C_{k,s})$ is cylinder. Lemma 2.20 below states that $\mu\big(C_{k,s}\big) = \mu\big(\alpha_\infty(C_{k,s})\big)\cdot\mathbb{L}^k$. We use this identity and the partition (2.9) to calculate

$$\int_{J_\infty(Y)} F_D \,\mathrm{d}\mu = \sum_{k,s\in\mathbb{Z}_{\geq 0}} \mu\big(\alpha_\infty(C_{k,s})\big)\cdot\mathbb{L}^{-s} = \sum_{k,s\in\mathbb{Z}_{\geq 0}} \mu\big(C_{k,s}\big)\cdot\mathbb{L}^{-(s+k)}.$$

Set $s' := s + k$. Clearly $\bigsqcup_{0 \leq k \leq s'} C_{k,s'-k} = F^{-1}_{\alpha^* D+W}(s')$. Substituting this into the above leaves

$$\int_{J_\infty(Y)} F_D \, \mathrm{d}\mu = \sum_{s' \in \mathbb{Z}_{\geq 0}} \mu\big(F^{-1}_{\alpha^* D+W}(s')\big) \cdot \mathbb{L}^{-s'} = \int_{J_\infty(Y')} F_{\alpha^* D+W} \, \mathrm{d}\mu,$$

as required. □

Lemma 2.20 $\mu\big(C_{k,s}\big) = \mu\big(\alpha_\infty(C_{k,s})\big) \cdot \mathbb{L}^k$.

Discussion of proof. Both $C_{k,s}$ and $\alpha_\infty(C_{k,s})$ are cylinder sets so there exists $t \in \mathbb{Z}_{\geq 0}$ and constructible sets B'_t and B_t in $J_\infty(Y')$ and $J_\infty(Y)$ respectively such that the following diagram commutes:

$$\begin{array}{ccc} C_{k,s} \subset J_\infty(Y') & \xrightarrow{\alpha_\infty} & \alpha_\infty(C_{k,s}) \subset J_\infty(Y) \\ \pi_t \big\downarrow & & \big\downarrow \pi_t \\ B'_t \subset J_t(Y') & \xrightarrow{\alpha_t} & B_t \subset J_t(Y). \end{array}$$

We claim that the restriction of α_t to B'_t is a $\mathbb{C}^k$-bundle over B_t. It follows that $[B'_t] = [\mathbb{C}^k] \cdot [B_t]$ and we have

$$\mu\big(C_{k,s}\big) = [B'_t] \cdot \mathbb{L}^{-(n+nt)} = [B_t] \cdot \mathbb{L}^k \cdot \mathbb{L}^{-(n+nt)} = \mu\big(\alpha_\infty(C_{k,s})\big) \cdot \mathbb{L}^k$$

as required. The proof of the claim is a local calculation which is carried out in [DL99b, Lemma 3.4(b)]. The key observation is that the order of vanishing of the Jacobian determinant of α at $\gamma_y \in C_{k,s}$ is $F_W(\gamma_y) = k$. □

Definition 2.21 Let X denote a complex algebraic variety with at worst Gorenstein canonical singularities. The *motivic integral* of X is defined to be the motivic integral of the pair (Y, D), where $\varphi \colon Y \to X$ is any resolution of singularities for which the discrepancy divisor $D = K_Y - \varphi^* K_X$ has only simple normal crossings.

Note first that the discrepancy divisor D is effective because X has at worst Gorenstein canonical singularities. The crucial point however is that the motivic integral of (Y, D) is independent of the choice of resolution:

Proposition 2.22 *Let $\varphi_1 \colon Y_1 \longrightarrow X$ and $\varphi_2 \colon Y_2 \longrightarrow X$ be resolutions of X with discrepancy divisors D_1 and D_2 respectively. Then the motivic integrals of the pairs (Y_1, D_1) and (Y_2, D_2) are equal.*

Proof. Form a 'Hironaka hut'

$$\begin{array}{ccc} Y_0 & \xrightarrow{\psi_2} & Y_2 \\ \psi_1 \downarrow & \searrow & \downarrow \varphi_2 \\ Y_1 & \xrightarrow{\varphi_1} & X \end{array}$$

and let D_0 denote the discrepancy divisor of $\varphi_0 \colon Y_0 \longrightarrow X$. The discrepancy divisor of ψ_i is $D_0 - \psi_i^* D_i$. Indeed

$$\begin{aligned} K_{Y_0} &= \varphi_0^*(K_X) + D_0 = \psi_i^* \circ \varphi_i^*(K_X) + D_0 = \psi_i^*(K_{Y_i} - D_i) + D_0 \\ &= \psi_i^*(K_{Y_i}) + (D_0 - \psi_i^* D_i). \end{aligned}$$

The maps $\psi_i \colon Y_0 \longrightarrow Y_i$ are proper birational morphisms between smooth projective varieties so Theorem 2.19 applies:

$$\int_{J_\infty(Y_i)} F_{D_i}\, \mathrm{d}\mu = \int_{J_\infty(Y_0)} F_{\psi_i^* D_i + (D_0 - \psi_i^* D_i)}\, \mathrm{d}\mu = \int_{J_\infty(Y_0)} F_{D_0}\, \mathrm{d}\mu\,.$$

This proves the result. □

2.3 The stringy E-function

We now show that the motivic integral of X gives rise to a 'stringy Euler function' which encodes the Hodge–Deligne numbers of a resolution $Y \to X$. Kontsevich's proof of Conjecture 2.1 is then straightforward, given that the motivic integral of X, and hence the stringy Euler function of X, is independent of the choice of resolution.

Recall from Theorem 1.16 that the map $E \colon \mathcal{V}_{\mathbb{C}} \longrightarrow \mathbb{Z}[u,v]$ which associates to each complex variety X its E-polynomial is additive on a disjoint union of locally closed subvarieties, and satisfies $E(X \times Y) = E(X) \cdot E(Y)$. It follows from the universality of the map $[\ \ ]$ introduced in Definition 2.8 that E factors through the Grothendieck ring of algebraic varieties, inducing a function $E \colon K_0(\mathcal{V}_{\mathbb{C}}) \to \mathbb{Z}[u,v]$. By defining $E(\mathbb{L}^{-1}) := (uv)^{-1}$ we can extend this to a function[3].

$$E \colon K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}] \to \mathbb{Z}[u, v, (uv)^{-1}].$$

[3] One can use this function to define a finitely additive $\mathbb{Z}[u,v,(uv)^{-1}]$–valued measure $\widetilde{\mu}_E := E \circ \widetilde{\mu}$ on cylinder sets given by $\pi_k^{-1}(B_k) \to E(B_k) \cdot (uv)^{-n(k+1)}$. Then construct the stringy E-function directly; this is the approach adopted by Batyrev [Bat98, §6].

Proposition 2.23 *The map E can be extended uniquely to the subring*

$$\phi\big(K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]\big)\left[\left\{\frac{1}{\mathbb{L}^i-1}\right\}_{i\in\mathbb{Z}_{\geq 1}}\right]$$

of the ring R introduced in Definition 2.11.

Proof. The kernel of the completion map $\phi\colon K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}] \to R$ is

$$\bigcap_{m\in\mathbb{Z}} F^m K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]. \tag{2.10}$$

For $[V]\cdot\mathbb{L}^{-i} \in F^m K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]$, the degree of the E-polynomial $E\left([V]\cdot\mathbb{L}^{-i}\right)$ is $2\dim V - 2i \leq -2m$. The E-polynomial of an element Z in the intersection (2.10) must therefore be $-\infty$; that is, $E(Z)=0$. Thus E annihilates $\ker\phi$ and hence factors through $\phi\left(K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]\right)$. The result follows when we define $E(1/(\mathbb{L}^i-1)) := 1/((uv)^i-1)$ for $i\in\mathbb{N}$. $\square$

By Corollary 2.18 the motivic integral of the pair (Y,D) lies in the subring of Proposition 2.23. We now consider the image of the integral under E.

Warning 2.24 As we remarked in Warning 2.17, the derivation of the motivic integral in [Bat98] contains a small error which leads to the omission of an $\mathbb{L}^{-n}$ term. However, in practise it is extremely convenient to omit this term (!). As a result, in our definition of the stringy E-function to follow we consider the image under E of the motivic integral times $\mathbb{L}^n$. In short, our stringy E-function agrees with that in [Bat98], even though our calculation of the motivic integral differs.

Definition 2.25 Let X be a complex algebraic variety of dimension n with at worst Gorenstein canonical singularities. Let $\varphi\colon Y\to X$ be a resolution of singularities for which the discrepancy divisor $D=\sum_{i=1}^r a_iD_i$ has only simple normal crossings. The *stringy E-function* of X is

$$\begin{aligned} E_{\mathrm{st}}(X) &:= E\left(\int_{J_\infty(Y)} F_D\,\mathrm{d}\mu\cdot\mathbb{L}^n\right)\\ &= \sum_{J\subseteq\{1,\dots,r\}} E(D_J^\circ)\cdot\left(\prod_{j\in J}\frac{uv-1}{(uv)^{a_j+1}-1}\right), \end{aligned} \tag{2.11}$$

where we sum over all subsets $J\subseteq\{1,\dots,r\}$ including $J=\emptyset$.

Theorem 2.26 ([Kon95]) *Let X be a complex projective variety with at worst Gorenstein canonical singularities. If X admits a crepant resolution $\varphi\colon Y \to X$ then the Hodge numbers of Y are independent of the choice of crepant resolution, as predicted by Conjecture 2.1.*

PROOF. The discrepancy divisor $D = \sum_{i=1}^{r} a_i D_i$ of the crepant resolution $\varphi\colon Y \to X$ is by definition zero, so the motivic integral of X is the motivic integral of the pair $(Y, 0)$. Since each $a_i = 0$ it's clear that

$$E_{\mathrm{st}}(X) = \textstyle\sum_{J \subseteq \{1,\dots,r\}} E(D_J^\circ) = E(Y).$$

The stringy E-function is independent of the choice of the resolution φ. In particular, $E(Y) = E_{\mathrm{st}}(X) = E(Y_2)$ for $\varphi_2 : Y_2 \to X$ another crepant resolution. It remains to note that $E(Y)$ determines the Hodge–Deligne numbers of Y, and hence the Hodge numbers since Y is smooth and projective. □

2.4 Calculating the motivic integral

To get a better feeling for the motivic integral we now consider several examples. To perform nontrivial calculations of the stringy E-function we must choose varieties which admit no crepant resolution.

A nice family of examples is provided by Gorenstein terminal cyclic quotient singularities: for $G \subset \mathrm{SL}(n, \mathbb{C})$ the quotient $X = \mathbb{C}^n/G$ is Gorenstein by Proposition 1.6, and terminal when the criteria of Theorem 1.13(i) are satisfied. We know that 2- and 3-dimensional Gorenstein quotients admit crepant resolutions (see §1.1.1 and Theorem 1.24(i)) so instead we begin by considering 4-dimensional singularities of the form $\frac{1}{r}(1, r-1, a, r-a)$ with $\gcd(r, a) = 1$. Morrison and Stevens [MS84, Theorem 2.4(ii)] prove that these are the only terminal Gorenstein 4-fold cyclic quotient singularities.

Remark 2.27 In each example below we calculate both $E(Y)$ and $E_{\mathrm{st}}(X)$ after resolving the singularity $\varphi\colon Y \to X$. Note that $E(Y)$ is not equal to the E-polynomial of the exceptional fibre $D = \varphi^{-1}(\pi(0))$ because

$$E(Y) = E(Y \setminus D) + E(D) = (uv)^n - 1 + E(D),$$

for $n = \dim X$. The point is that the E-polynomial encodes the Hodge–Deligne numbers of compactly supported cohomology, yet

$$H_c^*(D, \mathbb{C}) = H^*(D, \mathbb{C}) = H^*(Y, \mathbb{C}) \neq H_c^*(Y, \mathbb{C}),$$

as explained in the proof of Theorem 1.31 in §1.2.5.

Example 2.28 Write $X = X_{L,\sigma}$ for the cyclic quotient singularity of type $\frac{1}{2}(1,1,1,1)$. Add the ray τ generated by the vector $v = \frac{1}{2}(1,1,1,1)$ to the cone σ, then take the simplicial subdivision of σ. This determines a toric resolution $\varphi\colon Y \to X$ with a single exceptional divisor $D = X_{L(\tau),\mathrm{Star}(\tau)} \cong \mathbb{P}^3$. The discrepancy of D is 1 by Theorem 1.13(i). Using Proposition 1.17 we calculate

$$\begin{aligned} E(Y) &= E(Y \setminus D) + E(\mathbb{P}^3) \\ &= \big((uv)^4 - 1\big) + \big((uv)^3 + (uv)^2 + uv + 1\big) \\ &= (uv)^4 + (uv)^3 + (uv)^2 + uv. \end{aligned}$$

Compare this with the stringy E-function:

$$E_{\mathrm{st}}(X) = E(Y \setminus D) + E(\mathbb{P}^3) \cdot \frac{uv - 1}{(uv)^2 - 1} = (uv)^4 + (uv)^2.$$

Example 2.29 Write $X = X_{L,\sigma}$ for the cyclic quotient singularity of type $\frac{1}{3}(1,2,1,2)$. Add rays τ_1 and τ_2 generated by the vectors $v_1 = \frac{1}{3}(1,2,1,2)$ and $v_2 = \frac{1}{3}(2,1,2,1)$ respectively to the cone σ, then take the simplicial subdivision of σ. The resulting fan Σ is determined by its cross-section Δ_2 (see Definition 1.26) illustrated in Figure 2.1.

There are eight 3-dimensional simplices in Δ_2 (four contain a face of the tetrahedron and four contain the edge joining v_1 to v_2). Each of these simplices determines a 4-dimensional cone in Σ which is generated by a basis of the lattice L, so $Y = X_{L,\Sigma} \to X_{L,\sigma}$ is a resolution. The union of all eight 3-dimensional simplices in Δ_2 contain eighteen faces, fifteen edges and six vertices. Write d_k for the number of cones of dimension k in Σ, so

$$d_4 = 8; \quad d_3 = 18; \quad d_2 = 15; \quad d_1 = 6; \quad d_0 = 1 \text{ (the origin in } L_{\mathbb{R}}).$$

Apply Proposition 1.17 to compute

$$E(Y) = (uv)^4 + 2(uv)^3 + 3(uv)^2 + 2uv.$$

To compute $E_{\mathrm{st}}(X)$ observe first that for $j = 1$ or 2 the exceptional divisor $D_j := X_{L(\tau_j),\mathrm{Star}(\tau_j)}$ has discrepancy 1 by Theorem 1.13(i). Write $d_k(\tau_j)$ for the number of cones of dimension k in $\mathrm{Star}(\tau_j)$, so

$$d_3(\tau_j) = 6; \quad d_2(\tau_j) = 9; \quad d_1(\tau_j) = 5; \quad d_0(\tau_j) = 1 \text{ (the origin in } L(\tau_j)).$$

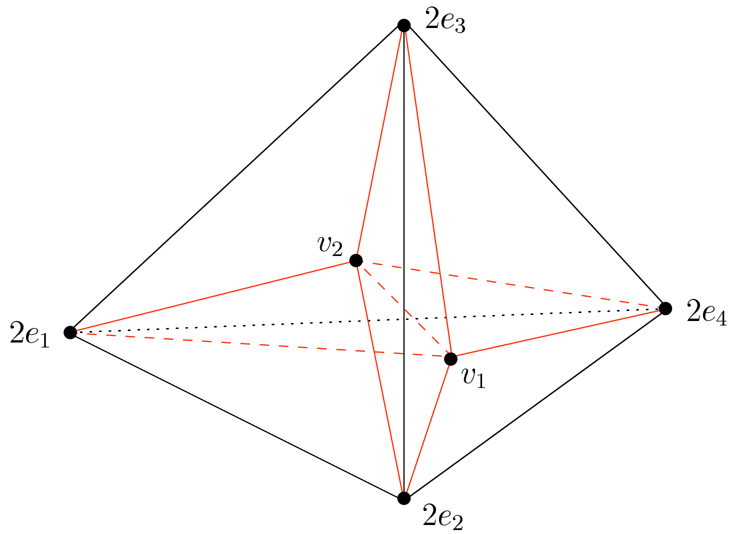


Figure 2.1: The simplex Δ_2 for $\frac{1}{3}(1, 2, 1, 2)$

Proposition 1.17 gives

$$E(D_j) = (uv)^3 + 2(uv)^2 + 2(uv) + 1 \quad \text{for } j = 1, 2.$$

Similarly, the fan $\mathrm{Star}(\langle \tau_1, \tau_2 \rangle)$ contains four faces, four edges and one vertex so Proposition 1.17 gives $E(D_1 \cap D_2) = (uv)^2 + 2(uv) + 1$. As a result

- $E(D^\circ_\emptyset) = E(Y \setminus (D_1 \cup D_2)) = (uv)^4 - 1$.
- $E(D^\circ_{\{1\}}) = E(D^\circ_{\{2\}}) = E(D_j) - E(D_1 \cap D_2) = (uv)^3 + (uv)^2$.
- $E(D^\circ_{\{1,2\}}) = E(D_1 \cap D_2) = (uv)^2 + 2(uv) + 1$.

Now compute the stringy E-function using formula (2.11):

$$\begin{aligned} E_{\mathrm{st}}(X) &= (uv)^4 - 1 + E(D^\circ_{\{1\}}) \cdot \left(\frac{uv - 1}{(uv)^2 - 1} \right) + E(D^\circ_{\{2\}}) \cdot \left(\frac{uv - 1}{(uv)^2 - 1} \right) \\ &\quad + E(D^\circ_{\{1,2\}}) \cdot \left(\frac{uv - 1}{(uv)^2 - 1} \right)^2 \\ &= (uv)^4 + 2(uv)^2. \end{aligned}$$

Example 2.30 Write $X = X_{L,\sigma}$ for the cyclic quotient singularity of type $\frac{1}{4}(1,3,1,3)$. Add rays τ_1, τ_2 and τ_3 generated by the vectors $v_1 = \frac{1}{4}(1,3,1,3)$ and $v_2 = \frac{1}{4}(2,2,2,2)$ and $v_3 = \frac{1}{4}(3,1,3,1)$ respectively to the cone σ, then take the simplicial subdivision of σ. The cross-section Δ_2 of the resulting fan Σ has three colinear points in the interior of the tetrahedron but is otherwise similar to that shown in Figure 2.1. There are twelve 3-dimensional simplices in Δ_2 containing 26 faces, 20 edges and 7 vertices. Proposition 1.17 calculates

$$E(Y) = (uv)^4 + 3(uv)^3 + 5(uv)^2 + 3uv.$$

For $j = 1,2,3$, the divisors $D_j := X_{L(\tau_j),\mathrm{Star}(\tau_j)}$ have discrepancy 1 by Theorem 1.13(i). Following the method of Example 2.29 we calculate

$$E(D_1) = E(D_3) = (uv)^3 + 2(uv)^2 + 2(uv) + 1$$

and $E(D_1 \cap D_2) = E(D_2 \cap D_3) = (uv)^2 + 2(uv) + 1$. To compute the E-polynomial of D_2 observe that

$$d_3(\tau_2) = 8; \quad d_2(\tau_2) = 12; \quad d_1(\tau_2) = 6; \quad d_0(\tau_2) = 1 \text{ (the origin in } L(\tau_2)),$$

where $d_k(\tau_2)$ denotes the number of cones of dimension k in $\mathrm{Star}(\tau_2)$. It follows from Proposition 1.17 that

$$E(D_2) = (uv)^3 + 3(uv)^2 + 3(uv) + 1.$$

Finally, since $D_1 \cap D_3 = \emptyset$ we have $E(D_1 \cap D_3) = E(D_1 \cap D_2 \cap D_3) = 0$. As a result

- $E(D^\circ_\emptyset) = E(Y \setminus (D_1 \cup D_2 \cup D_3)) = (uv)^4 - 1.$
- $E(D^\circ_{\{1\}}) = E(D^\circ_{\{3\}}) = (uv)^3 + (uv)^2.$
- $E(D^\circ_{\{2\}}) = (uv)^3 + (uv)^2 - (uv) - 1.$
- $E(D^\circ_{\{1,2\}}) = E(D^\circ_{\{2,3\}}) = (uv)^2 + 2(uv) + 1.$
- $E(D^\circ_{\{1,3\}}) = E(D^\circ_{\{1,2,3\}}) = 0.$

Apply formula (2.11) to compute

$$E_{\mathrm{st}}(X) = (uv)^4 + 3(uv)^2.$$

Remark 2.31 The above examples feature only exceptional divisors with discrepancy 1. To obtain examples of Gorenstein terminal cyclic quotient singularities which admit resolutions containing divisors having discrepancy larger than one we must work in dimension higher than four.

Example 2.32 Write $X = X_{L,\sigma}$ for the cyclic quotient singularity of type $\frac{1}{r}(1,1,1,\ldots,1)$ where $n := \dim X = kr$ for some $k \in \mathbb{Z}$ (by assuming that r divides n we ensure that X is Gorenstein). Add a single ray τ generated by the vector $v_1 = \frac{1}{r}(1,1,1,\ldots,1)$ to the cone σ, then take the simplicial subdivision of σ. This determines a toric resolution $\varphi\colon Y \to X$ with a single exceptional divisor $D = X_{L(\tau),\mathrm{Star}(\tau)} \cong \mathbb{P}^{n-1}$. The discrepancy of D is $k-1$ by Theorem 1.13(i). Using Proposition 1.17 we calculate

$$\begin{aligned} E(Y) &= E(Y \setminus D) + E(\mathbb{P}^{n-1}) \\ &= \big((uv)^n - 1\big) + \big((uv)^{n-1} + (uv)^{n-2} + \cdots + uv + 1\big) \\ &= (uv)^n + (uv)^{n-1} + \cdots + uv. \end{aligned}$$

Compare this with the stringy E-function:

$$\begin{aligned} E_{\mathrm{st}}(X) &= E(Y \setminus D) + E(\mathbb{P}^{n-1}) \cdot \frac{uv-1}{(uv)^k - 1} \\ &= (uv)^n + (uv)^{n-k} + \cdots + (uv)^{2k} + (uv)^k. \end{aligned}$$

Example 2.33 Let $X = X_{L,\sigma}$ denote the cyclic quotient singularity of type $\frac{1}{3}(1,2,1,2,1,2)$ (compare Example 2.29). Add rays τ_1 and τ_2 generated by the vectors $v_1 = \frac{1}{3}(1,2,1,2,1,2)$ and $v_2 = \frac{1}{3}(2,1,2,1,2,1)$ respectively to the cone σ, then take the simplicial subdivision of σ. Both v_1 and v_2 lie in the simplex Δ_3 of the resulting fan Σ so the corresponding exceptional divisors D_1 and D_2 each have discrepancy 2 by Theorem 1.13(i). The cross-section Δ_3 is difficult to draw (it is 5-dimensional!) but, using Figure 2.1 as a guide, one can show that

$$d_6 = 15; \quad d_5 = 48; \quad d_4 = 68; \quad d_3 = 56; \quad d_2 = 28; \quad d_1 = 8; \quad d_0 = 1,$$

where d_k denotes the number of cones of dimension k in Σ. Hence

$$E(Y) = (uv)^6 + 2(uv)^5 + 3(uv)^4 + 4(uv)^3 + 3(uv)^2 + 2(uv).$$

As with Example 2.29, for $j = 1,2$ write $d_k(\tau_j)$ for the number of cones of dimension k in $\mathrm{Star}(\tau_j)$, so

$$d_5(\tau_j) = 12;\ d_4(\tau_j) = 30;\ d_3(\tau_j) = 34;\ d_2(\tau_j) = 21;\ d_1(\tau_j) = 7;\ d_0(\tau_j) = 1.$$

Proposition 1.17 gives

$$E(D_j) = (uv)^5 + 2(uv)^4 + 3(uv)^3 + 3(uv)^2 + 2(uv) + 1 \quad \text{for } j = 1, 2.$$

Similarly, counting simplices in the fan $\mathrm{Star}(\langle \tau_1, \tau_2 \rangle)$ gives

$$E(D_1 \cap D_2) = (uv)^4 + 2(uv)^3 + 3(uv)^2 + 2(uv) + 1.$$

Now compute the stringy E-function using formula (2.11):

$$\begin{aligned} E_{\mathrm{st}}(X) &= (uv)^6 - 1 + E(D^\circ_{\{1\}}) \cdot \left(\frac{uv - 1}{(uv)^3 - 1} \right) + E(D^\circ_{\{2\}}) \cdot \left(\frac{uv - 1}{(uv)^3 - 1} \right) \\ &\qquad + E(D^\circ_{\{1,2\}}) \cdot \left(\frac{uv - 1}{(uv)^3 - 1} \right)^2 \\ &= (uv)^6 + 2(uv)^3. \end{aligned}$$

2.5 The orbifold E-function

Following the introduction of the orbifold Euler number for the action of a finite group G on a manifold M (see §1.2.3), Vafa [Vaf89] and Zaslow [Zas93] considered the *orbifold Hodge numbers*

$$h^{p,q}(M, G) := \sum_{[g] \in \mathrm{Conj}(G)} \sum_{i=1}^{m(g)} \dim_{\mathbb{C}} H_c^{p - \mathrm{age}[g], q - \mathrm{age}[g]}(M_i(g)/C(g)), \tag{2.12}$$

where $M^g = M_1(g) \cup \cdots \cup M_{m(g)}(g)$ are the smooth connected components of the fixed point set (compare formula (1.9) for the orbifold Euler number).

Definition 2.34 The *orbifold E-function* of the pair (M, G) is

$$\begin{aligned} E_{\mathrm{orb}}(M, G) &:= \sum_{p,q} (-1)^{p+q} \, h^{p,q}(M, G) \, u^p \, v^q \\ &= \sum_{[g] \in \mathrm{Conj}(G)} \sum_{i=1}^{m(g)} E(M_i(g)/C(g)) \cdot (uv)^{\mathrm{age}[g]}, \end{aligned}$$

where E is the standard E-polynomial (see §1.15) and $C(g)$ is the centraliser of $g \in G$. The second formula follows from a straightforward substitution. When $E_{\mathrm{orb}}(M, G)$ is evaluated at $u = v = 1$ we produce the orbifold Euler number $e(M, G)$ introduced in §1.2.3.

Theorem 2.35 ([Bat99b]) *For $G \subset \mathrm{SL}(n, \mathbb{C})$ a finite subgroup we have*

$$E_{\mathrm{orb}}(\mathbb{C}^n, G) = E_{\mathrm{st}}(\mathbb{C}^n/G). \tag{2.13}$$

Remark 2.36 In fact, Theorem 2.35 is a special case of the following more general result established by Batyrev: for any finite group G acting regularly on a manifold M we have

$$E_{\mathrm{orb}}(M, G) = E_{\mathrm{st}}(X, \Delta_X), \tag{2.14}$$

where $X = M/G$, $\Delta_X \subset X$ is the ramification divisor of the quotient map $\pi\colon M \to M/G$ and $E_{\mathrm{st}}(X, \Delta_X)$ is the stringy E-function of the pair (X, Δ_X). We choose not to formally introduce $E_{\mathrm{st}}(X, \Delta_X)$ because in the special case $M = \mathbb{C}^n$ and $G \subset \mathrm{SL}(n, \mathbb{C})$ we have $\Delta_X = 0$ and $E_{\mathrm{st}}(X, 0) = E_{\mathrm{st}}(X)$. Thus Theorem 2.35 follows from (2.14) above. See Batyrev [Bat99b, §3.7,§7.5] for the definition of $E_{\mathrm{st}}(X, \Delta_X)$ and a proof of (2.14).

We digress momentarily to discuss recent work of Chen and Ruan [CR00] on a new cohomology theory for orbifolds. Rather than go into the details for general orbifolds, we focus on the special case $X = \mathbb{C}^n/G$.

Definition 2.37 *Set $X_{(g)} := (\mathbb{C}^n)^g / C(g)$ and define compactly supported orbifold cohomology to be*

$$H^i_{\mathrm{orb,c}}(X, \mathbb{Q}) := \bigoplus_{[g] \in \mathrm{Conj}(G)} H_c^{i - 2\,\mathrm{age}[g]}(X_{(g)}, \mathbb{Q}),$$

together with the corresponding Dolbeault cohomology groups

$$H^{p,q}_{\mathrm{orb,c}}(X, \mathbb{C}) := \bigoplus_{[g] \in \mathrm{Conj}(G)} H_c^{p - \mathrm{age}[g], q - \mathrm{age}[g]}(X_{(g)}, \mathbb{C}).$$

The dimensions of these spaces coincide with the orbifold Hodge numbers of Vafa and Zaslow (2.12) by construction, so the orbifold E-function encodes the orbifold Hodge numbers in the same way that the standard E-polynomial encodes the classical Hodge numbers.

This is not in itself remarkable. After all, Vafa and Zaslow could have formulated the above definitions when discussing orbifold Hodge numbers. The remarkable aspect of the Chen and Ruan construction is an 'orbifold Gromov–Witten invariant' which determines a cup product

$$\cup_{\mathrm{orb}}\colon H^i_{\mathrm{orb,c}}(X, \mathbb{Q}) \times H^j_{\mathrm{orb,c}}(X, \mathbb{Q}) \longrightarrow H^{i+j}_{\mathrm{orb,c}}(X, \mathbb{Q}) \quad \text{for} \quad 0 \le i, j \le 2n,$$

making $H^*_{\mathrm{orb,c}}(X, \mathbb{Q}) = \bigoplus_{0 \le i \le 2n} H^i_{\mathrm{orb,c}}(X, \mathbb{Q})$ into a ring with a unit.

2.6 Strong McKay via motivic integration

We conclude this chapter with a look at how Batyrev [Bat99b, Bat00] used motivic integration to prove the strong McKay conjecture 1.30 (an alternative proof has subsequently been provided by Denef and Loeser [DL99a]).

Proposition 2.38 ([Bat99b]) *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite subgroup. Then*

$$E_{\mathrm{st}}(\mathbb{C}^n/G) = \sum_{[g]\in \mathrm{Conj}(G)} (uv)^{n-\mathrm{age}[g]}, \tag{2.15}$$

where we sum over the conjugacy classes of G.

Proof. From (2.13) above it is enough to compute the orbifold Euler number of the pair $(\mathbb{C}^n, G)$. Write $M := \mathbb{C}^n$ so that $M^g \cong \mathbb{C}^{\dim\ker(g-I)}$. It follows that $E(M^g/C(g)) = (uv)^{\dim\ker(g-I)}$ and, since

$$\mathrm{age}[g] + \mathrm{age}[g^{-1}] = \mathrm{rank}(g - I) = n - \dim\ker(g - I),$$

we have $E(M^g/C(g)) \cdot (uv)^{\mathrm{age}[g]} = (uv)^{n-\mathrm{age}[g^{-1}]}$. The result follows by summing over all conjugacy classes $[g^{-1}]$ of G. $\square$

Theorem 2.39 (strong McKay correspondence) *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite subgroup and suppose that the quotient $X = \mathbb{C}^n/G$ admits a crepant resolution $\varphi\colon Y \to X$. The nonzero Betti numbers of Y are*

$$\dim_{\mathbb{C}} H^{2k}(Y, \mathbb{C}) = \#\Big\{\textit{age } k \textit{ conjugacy classes of } G\Big\}.$$

for $k = 0, \dots, n-1$, so $e(Y) = \#\{$conjugacy classes of $G\}$.

Proof. The Hodge structure in $H^i_c(Y, \mathbb{Q})$ is pure for each i and Poincaré duality $H^{2n-i}_c(Y, \mathbb{C}) \otimes H^i(Y, \mathbb{C}) \to H^{2n}_c(Y, \mathbb{C})$ respects the Hodge structure, so it is enough to show that the only nonzero Hodge–Deligne numbers of the compactly supported cohomology of Y are

$$h^{n-k,n-k}\big(H^{2n-2k}_c(Y, \mathbb{C})\big) = \#\Big\{\text{age } k \text{ conjugacy classes of } G\Big\}.$$

Now $h^{n-k,n-k}\big(H^{2n-2k}_c(Y, \mathbb{C})\big)$ is the coefficient of $(uv)^{n-k}$ in the E-polynomial of Y. Moreover, the resolution $\varphi\colon Y \to X$ is crepant so $E(Y) = E_{\mathrm{st}}(X)$ and the result follows from (2.15). $\square$

Remark 2.40 Reid [Rei97, Rei99] has long held the philosophy that the McKay correspondence can be written as a tautology. We close this chapter with a short explanation of this point of view by generalising Theorem 2.39 using the orbifold cohomology of Chen and Ruan.

Observe that Proposition 2.38 holds even when $\mathbb{C}^n/G$ does not admit a crepant resolution. Indeed, for the Gorenstein terminal quotient singularity of Example 2.28, where the nontrivial element of the group $G = \mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{C}^4$ has age two, formula (2.15) gives

$$E_{\mathrm{st}}(\mathbb{C}^4/G) = (uv)^4 + (uv)^2,$$

as shown in §2.4 (check that the same holds for the other examples of §2.4). But if we remove the assumption that $X = \mathbb{C}^n/G$ admits a crepant resolution $\varphi\colon Y \to X$ what is the appropriate generalisation of Theorem 2.39?

One answer[4] can be phrased in terms of the new orbifold cohomology of Chen and Ruan discussed in §2.5. Recall the trivial calculation

$$E_{\mathrm{orb}}(\mathbb{C}^n, G) = \sum_{[g]\in \mathrm{Conj}(G)} (uv)^{n-\mathrm{age}[g]}$$

from the proof of Proposition 2.38. This calculation can be rephrased in the following terms:

Tautology 2.41 *Let $G \subset \mathrm{SL}(n,\mathbb{C})$ be a finite subgroup. Then*

$$\dim_{\mathbb{C}} H^{2n-2k}_{\mathrm{orb,c}}(\mathbb{C}^n/G, \mathbb{C}) = \#\Big\{\textit{age } k \textit{ conjugacy classes of } G\Big\}$$

for $k = 0, \dots, n-1$, so $e(\mathbb{C}^n, G) = \#\{$conjugacy classes of $G\}$.

This is nothing more than a refinement of the observation (1.10) made by Hirzebruch and Höfer, and should be regarded as a tautology. Nevertheless, it is a remarkable consequence of motivic integration, and specifically of formula (2.13), that Tautology 2.41 specialises to the highly nontrivial Theorem 2.39 when $\mathbb{C}^n/G$ admits a crepant resolution Y. Indeed, if $\varphi\colon Y \to X = \mathbb{C}^n/G$ is a crepant resolution then

$$E_{\mathrm{orb}}(\mathbb{C}^n, G) = E_{\mathrm{st}}(\mathbb{C}^n/G) = E(Y)$$

[4]The tautological McKay correspondence of Reid [Rei99, §4,7] is written in terms of the *stringy motive*. See §A for more on the stringy motive.

so $\dim_{\mathbb{C}} H^{2n-2k}_{\text{orb,c}}(X,\mathbb{C}) = \dim_{\mathbb{C}} H^{2n-2k}_{c}(Y,\mathbb{C})$. This equals $\dim_{\mathbb{C}} H^{2k}(Y,\mathbb{C})$ by Poincaré duality.

Remark 2.42 In fact Chen and Ruan's orbifold cohomology is well defined for non-Gorenstein varieties so one can further generalise Tautology 2.41 by considering quotients $\mathbb{C}^n/G$ for finite subgroups $G \subset \mathrm{GL}(n,\mathbb{C})$. We choose to say no more on this topic here.

Chapter 3

How to calculate A-Hilb $\mathbb{C}^3$

Nakamura [Nak00] introduced the G-Hilbert scheme G-Hilb $\mathbb{C}^3$ for a finite subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$, and conjectured that it is a crepant resolution of the quotient $\mathbb{C}^3/G$. He proved this for an Abelian subgroup $A \subset \mathrm{SL}(3,\mathbb{C})$ by introducing an explicit algorithm that calculates A-Hilb $\mathbb{C}^3$. In this chapter we calculate A-Hilb $\mathbb{C}^3$ much more simply, in terms of fun with continued fractions plus regular tesselations by equilateral triangles. We state our two main results in Section 3.1 and prove the first of these in Section 3.2. Section 3.3 contains a simple procedure to calculate A-Hilb $\mathbb{C}^3$, together with a pair of worked examples. Section 3.4 investigates a dichotomy in the construction which plays a key role in Chapter 4. The rest of this chapter consists of a proof of our second main result.

3.1 Statement of the results

Let $A \subset \mathrm{SL}(3,\mathbb{C})$ be a finite diagonal subgroup acting on $\mathbb{C}^3$ with coordinates x, y, z. The quotient singularity $\mathbb{C}^3/A$ is the toric variety $X_{L,\sigma}$ introduced in §1.1.4. The junior simplex Δ (see Definition 1.26) has 3 vertices

$$e_1 = (1,0,0), \quad e_2 = (0,1,0) \quad \text{and} \quad e_3 = (0,0,1).$$

Write $\mathbb{R}^2_\Delta$ for the affine plane spanned by Δ and $\mathbb{Z}^2_\Delta = L \cap \mathbb{R}^2_\Delta$ for the corresponding affine lattice. Taking each e_i in turn as origin, construct the Newton polygons obtained as the convex hull of the lattice points in $\Delta \setminus e_i$ (see Figure 3.1(a)):

$$f_{i,0}, f_{i,1}, f_{i,2}, \ldots, f_{i,m_i+1}, \tag{3.1}$$

where $f_{i,0}$ and f_{i,m_i+1} extend along the sides e_ie_{i-1} and e_ie_{i+1} respectively (the indices $i, i\pm 1$ are cyclic). Since e_i is the origin, the notation $f_{i,j}$ denotes both the lattice point and the vector from e_i to $f_{i,j}$. The vectors $f_{i,j}$ out of e_i are subject to the Jung–Hirzebruch continued fraction rule (see §1.1.4)

$$f_{i,j-1} + f_{i,j+1} = a_{i,j} \cdot f_{i,j} \quad \text{for } j = 1, \ldots, m_i, \tag{3.2}$$

where $a_{i,j} \geq 2$ is the *strength* of $f_{i,j}$. By writing

$$\mathbb{Z}^2_\Delta = \mathbb{Z} \cdot f_{i,0} + \mathbb{Z} \cdot f_{i,m_i+1} + \mathbb{Z} \cdot f_{i,1} = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{r_i}(\alpha_i, 1),$$

with $\alpha_i < r_i$ and coprime to r_i, the integers $a_{i,j}$ can be computed via the continued fraction $\frac{r_i}{\alpha_i} = [a_{i,1}, \ldots, a_{i,m_i}]$. For each $i = 1, 2, 3$, draw lines $L_{i,j}$ from e_i to the lattice points $f_{i,j}$. The resulting fan at e_i corresponds to the Jung–Hirzebruch resolution of the surface singularity $\mathbb{C}^2_{(x_i=0)}/A$. The picture so far is the simplex Δ with a number of lines $L_{i,j}$ growing out of each of the 3 vertices (see Figure 3.1(a)).

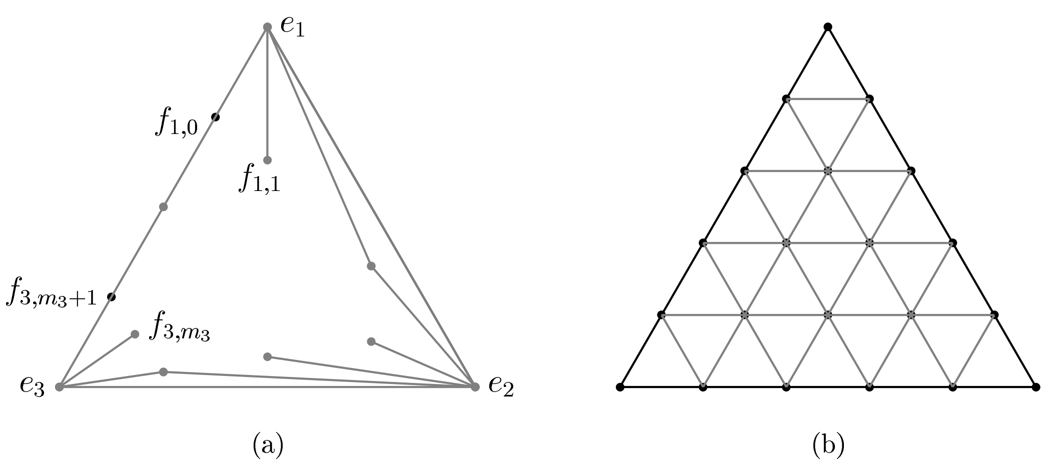


Figure 3.1: (a) Lines $L_{i,j}$ from e_i; (b) Regular tesselation of a regular triangle

Definition 3.1 A *regular triple* is a set of three vectors v_1, v_2, v_3 from among the $f_{i,j}$, any two of which form a basis of the group of translations of the affine lattice $\mathbb{Z}^2_\Delta$, such that $\pm v_1 \pm v_2 \pm v_3 = 0$. A triangle $T \subset \mathbb{R}^2_\Delta$ with vertices in $\mathbb{Z}^2_\Delta$ is a *regular triangle* if each of its sides is (part of) a line $L_{i,j}$ extending some $f_{i,j}$ and the 3 primitive vectors v_1, v_2, v_3 pointing along its sides form a regular triple. It is said to have *side* r if there are $r+1$ lattice points along each side. The *regular tesselation* of a regular triangle of side r is the subdivision into r^2 basic triangles with sides parallel to v_1, v_2, v_3 (see Figure 3.1(b) for the case $r = 5$; remove the grey tesselating lines to leave a regular triangle of side 5).

A regular triangle is what you get as the junior simplex for the group

$$A = \mathbb{Z}/r \oplus \mathbb{Z}/r = \left\langle \frac{1}{r}(1,-1,0), \frac{1}{r}(0,1,-1) \right\rangle \subset \mathrm{SL}(3,\mathbb{C}).$$

The tesselation consists of basic triangles with vertices in Δ, so corresponds to a crepant resolution of the quotient singularity $\mathbb{C}^3/A$. It is known (see [Rei97, §2.2]) that in this case A-Hilb $\mathbb{C}^3$ is the toric variety associated with its regular tesselation.

We are now in a position to state the main results of this chapter:

Theorem 3.2 *The regular triangles partition the junior simplex* Δ*.*

In §3.2 we introduce a combinatorial procedure involving continued fractions to determine the partition of Δ. Worked examples appear in §3.3.

Theorem 3.3 *Let* $\Sigma \subset L \otimes \mathbb{R}$ *denote the toric fan determined by the regular tesselation of the regular triangles of Theorem 3.2. The toric variety* $X_{L,\Sigma}$ *is Nakamura's A-Hilbert scheme* A-Hilb $\mathbb{C}^3$*.*

Corollary 3.4 ([Nak00]) A-Hilb $\mathbb{C}^3 \to \mathbb{C}^3/A$ *is a crepant resolution.*

The construction of Σ reveals that every internal vertex has valency $3, 4, 5$ or 6, so the compact exceptional surfaces in A-Hilb $\mathbb{C}^3$ are characterised as follows:

Corollary 3.5 *Every compact exceptional surface in* A-Hilb $\mathbb{C}^3$ *is either* $\mathbb{P}^2$*, a scroll* $\mathbb{F}_n$ *or a scroll blown up in one or two points (including* dP_6*, the del Pezzo surface of degree 6).*

3.2 Concatenating three continued fractions

In this section we prove Theorem 3.2. The key observation is that easy games with continued fractions provide all of the regular triples v_1, v_2, v_3 (see Definition 3.1) from among the vectors $f_{i,j}$. First, translate the three Newton polygons at e_1, e_2, e_3 to a common vertex to get the propellor shape of Figure 3.2, in which three hexants (the blades of the propellor) have convex basic subdivisions. The primitive vectors are read in cyclic order

$$f_{1,0}, f_{1,1}, \dots, f_{1,m_1}, f_{1,m_1+1} = -f_{2,0}, f_{2,1}, \quad \text{etc.}$$

Inverting any blade (that is, multiplying by -1) makes the three hexants into a basic subdivision of halfspace.

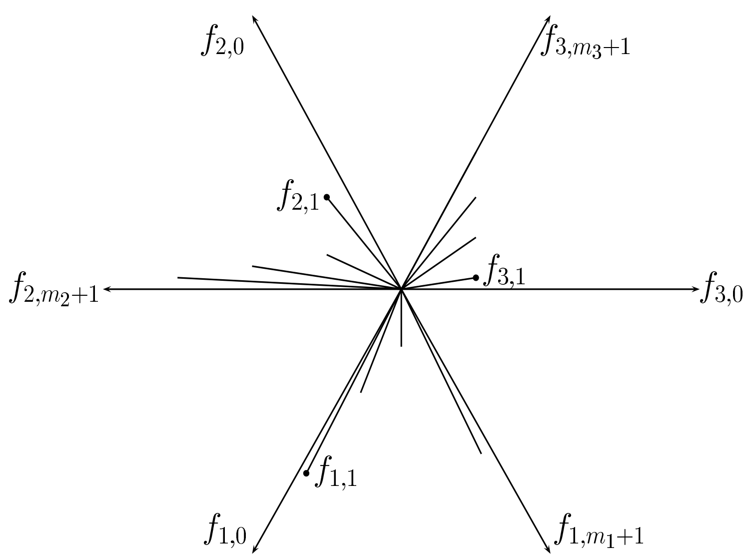


Figure 3.2: ‘Propellor’ with three ‘blades’

To concatenate the three continued fractions $[a_{i,1}, \dots, a_{i,m_i}]$ arising from the propellor of Figure 3.2 as a cyclic continued fraction we study the change of basis from the last basis f_{1,m_1}, f_{1,m_1+1} of the e_1 hexant to the first basis $f_{2,0}, f_{2,1}$ of the e_2 hexant. Clearly $f_{2,0} = -f_{1,m_1+1}$, and we claim that

$$f_{2,1} - f_{1,m_1} = a_{2,0} \cdot f_{2,0} \quad \text{for some} \quad a_{2,0} \geq 1. \tag{3.3}$$

Indeed, $-f_{1,m_1}, f_{2,0}$ and $f_{2,0}, f_{2,1}$ are two oriented bases (the usual argument).

Definition 3.6 The integer $a_{2,0}$ arising from relation (3.3) is the *strength* of the side $f_{2,0}$; we define the strength of $f_{1,0}$ and $f_{3,0}$ similarly. The side e_ie_j of the simplex Δ is a *long side* if the primitive vector $f_{i,0}$ along e_ie_j has strength strictly larger than 1.

We illustrate a long side in Figure 3.1(a). The lattice point $f_{1,0}$ lies between e_1 and e_3. Since $f_{1,1} - f_{3,m_3} = 2 \cdot f_{1,0}$ we see that e_1e_3 is a long side.

Lemma 3.7 *Δ has at most one long side.*

Proof. If e_1e_2 and e_1e_3 (say) both have coefficient larger than 1 then the basic subdivision of halfspace obtained by inverting the bottom blade of the propellor in Figure 3.2 would be convex at each ray, a contradiction. $\square$

Lemma 3.7 gives rise to a dichotomy in the construction (we investigate this further in §3.4): the basic subdivision of halfspace has either two or three convex sectors which lie between vectors with strength 1 arising from (3.3). For convenience we rotate halfspace so that one such vector is horizontal; we let v_0 denote this vector and write $c_0 = 1$ for its strength. Relabel the other vectors $f_{i,j}$ as $v_1, \dots v_n$ with strength $c_1, \dots, c_n$ respectively, where $v_n = -v_0$ and $c_n = c_0 = 1$:

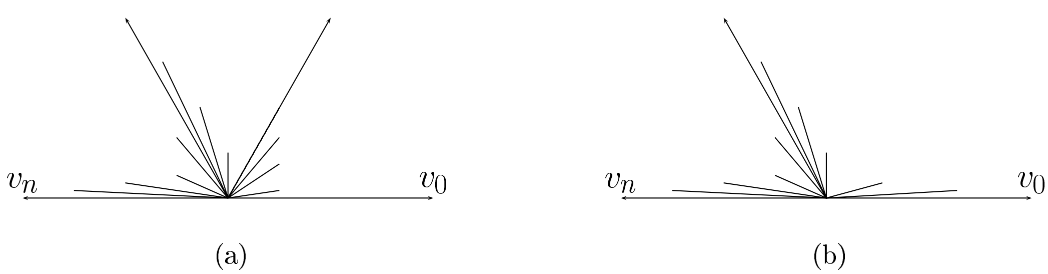


Figure 3.3: Subdivision of halfspace: (a) no long side; (b) one long side

Adjacent cones $\langle v_{i-1}, v_i \rangle$ and $\langle v_i, v_{i+1} \rangle$ in the subdivision of halfspace are related by the transformation

$$\begin{pmatrix} v_i \\ v_{i+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & c_i \end{pmatrix} \begin{pmatrix} v_{i-1} \\ v_i \end{pmatrix}.$$

Successive change of bases from $\langle v_0, v_1\rangle$ to $\langle -v_0, -v_1\rangle$ anti-clockwise around halfspace gives

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & c_{n-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & c_1 \end{pmatrix}. \tag{3.4}$$

As with the Jung–Hirzebruch continued fraction in the 2-dimensional case, the numbers c_i can be encoded in a continued fraction

$$[1, c_{n-1}, \ldots, c_1] := 1 - \cfrac{1}{c_{n-1} - \frac{1}{\cdots - \frac{1}{c_1}}} \tag{3.5}$$

which evaluates to $[1,1,1] = 1 - \frac{1}{0} = -\infty$. The continued fraction is cyclic: a different choice of side with strength 1 for v_0 gives rise to a continued fraction which is a cyclic permutation of the above.

Remark 3.8 When $c_i = 1$ the relation $v_i = \pm v_{i-1} \pm v_{i+1}$ holds so v_{i-1}, v_i, v_{i+1} is a regular triple. In this case the cone $\langle v_{i-1}, v_{i+1}\rangle$ is basic and removing the vector v_i leaves a new basic subdivision $v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ of halfspace. The new continued fraction is obtained by contracting the string

$$[\ldots, c_{i+1}, 1, c_{i-1}, \ldots] \quad \text{to} \quad [\ldots, c_{i+1} - 1, c_{i-1} - 1, \ldots];$$

this contraction corresponds to the matrix identity

$$\begin{pmatrix} 0 & 1 \\ -1 & c_{i+1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & c_{i-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & c_{i+1} - 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & c_{i-1} - 1 \end{pmatrix}.$$

The new continued fraction must have either two or three 1's for the reason outlined in Lemma 3.7. Thus we may repeat this process to establish a chain of contractions which converges to $[1,1,1]$, plus a final contraction which produces the empty continued fraction.

Definition 3.9 A chain of contractions of the cyclic continued fraction is called an *MMP*[1].

Lemma 3.10 Every regular triple appears in every MMP.

[1]The combinatorics are the same as for contracting -1-curves in a chain of rational curves on a surface with self-intersection the negatives of $1, c_{n-1}, \ldots, c_1$.

Proof. We prove the lemma by counting the number of regular triples and the number of contractions in an MMP. It's clear from the MMP algorithm that each vector v_i appears in precisely c_i regular triples. It follows that the disjoint union of all regular triangles has $\sum_{i=0}^{n-1} c_i$ edges, so there are $\frac{1}{3}\sum_{i=0}^{n-1} c_i$ distinct regular triples. On the other hand, in a given MMP each contraction reduces the total strength (i.e. the sum of the numbers in the continued fraction) by three so there are $\frac{1}{3}\sum_{i=0}^{n-1} c_i$ contractions. The result follows from the observation that a regular triple cannot correspond to more than one contraction in a given MMP. $\square$

Proof of Theorem 3.2. It follows from Lemma 3.10 that the simplex Δ has a unique subdivision into regular triangles, and any MMP computes it. This completes the proof of Theorem 3.2. $\square$

3.3 It's a knock-out!

The MMP in cyclic continued fractions has an entertaining interpretation as a contest between the lines $L_{i,j}$ which emanate from the 3 vertices e_i. As a result, the fan Σ of A-Hilb $\mathbb{C}^3$ can be calculated using the following 3-step procedure (see Example 3.11):

1. Draw lines $L_{i,0}, \dots, L_{i,m_i+1}$ emanating from the corners e_i of Δ as described in §3.1. The integer $a_{i,j} \geq 2$ determined by the Jung–Hirzebruch continued fraction rule (3.2) is the strength of $L_{i,j}$.

2. Extend the lines $L_{i,1}, \dots, L_{i,m_i}$ until they are 'defeated' by lines $L_{k,l}$ from e_k ($i \neq k$) according to the following rule: when two or more lines meet at a point, the line with greater strength extends but its strength decreases by 1 for every rival it defeats. Lines which meet with equal strength all die. As a consequence, strength 2 lines always die.

3. Step 2 produces the partition of Δ into regular triangles of Theorem 3.2. The regular tesselation of the regular triangles gives Σ.

Example 3.11 Consider the cyclic quotient singularity of type $\frac{1}{11}(1,2,8)$. The three continued fractions are

$$\begin{aligned} &\text{at } e_1: && \tfrac{11}{4} = [3,4] && (\text{because } \tfrac{1}{11}(2,8) \sim \tfrac{1}{11}(1,4)),\\ &\text{at } e_2: && \tfrac{11}{7} = [2,3,2,2] && (\text{because } \tfrac{1}{11}(8,1) \sim \tfrac{1}{11}(1,7)),\\ &\text{at } e_3: && \tfrac{11}{2} = [6,2]. \end{aligned}$$

In Figure 3.4(a) we illustrate the result of Step 1 of the procedure where, for example, the integers 6 and 2 marking the lines from e_3 come from the continued fraction $\frac{11}{2} = 6 - \frac{1}{2}$ of the surface singularity $\mathbb{C}^2_{(z=0)}/A = \frac{1}{11}(1,2)$.

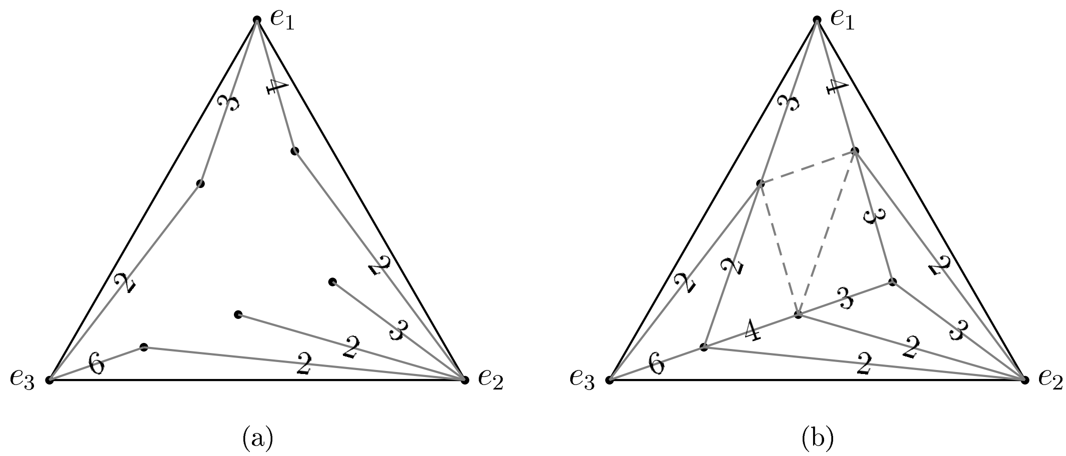


Figure 3.4: (a) Step 1; (b) Step 2 (solid lines) and Step 3 (dotted lines)

The solid lines in Figure 3.4(b) show the result of Step 2. For example, the line from e_1 with strength 3 intersects the line from e_3 with strength 2; the procedure says that the line from e_1 extends with strength 2 while the line from e_3 terminates. The resulting partition of Δ contains only one regular triangle of side $r > 1$. To perform Step 3 we tesselate this triangle (i.e. add the dotted lines to Figure 3.4(b)) which gives the fan Σ of A-Hilb $\mathbb{C}^3$.

Example 3.12 Consider the cyclic quotient singularity of type $\frac{1}{30}(25,2,3)$. Note that $\mathrm{hcf}(30,25) = 5$ and, because of the common factor, the three continued fractions are

$$\begin{array}{lll} \text{at } e_1: & [5] & (\text{because } \frac{1}{30}(2,3) \sim \frac{1}{5}(1,1)), \\ \text{at } e_2: & [2] & (\text{because } \frac{1}{30}(25,3) \sim \frac{1}{2}(1,1)), \\ \text{at } e_3: & [2,2] & (\text{because } \frac{1}{30}(25,2) \sim \frac{1}{3}(2,1)). \end{array}$$

The solid lines in Figure 3.5, each marked with the appropriate strength, show the partition of the junior simplex of $\frac{1}{30}(25,2,3)$ into regular triangles of side two and three. The dotted lines tesselate the regular triangles.

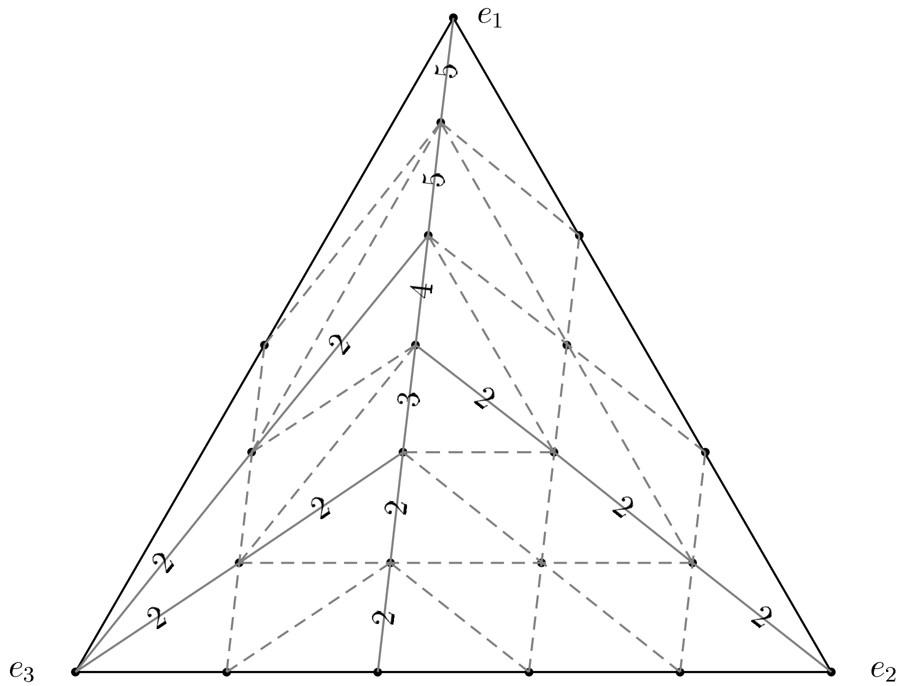


Figure 3.5: The partition of the junior simplex into regular triangles

The side e_2e_3 is a long side. Indeed, the primitive vector along e_3e_2 is $f_{3,0} = -f_{2,2} = (0, 6, -6)$ (we omit denominators $\frac{1}{30}$ throughout). Since $f_{2,1} = (5, -8, 3)$ and $f_{3,1} = (5, 4, -9)$ we see that $f_{3,1} - f_{2,1} = 2 \cdot f_{3,0}$, so e_3e_2 has strength $a_{3,0} = 2$.

3.4 A long side or a meeting of champions

Before proceeding to A-Hilb $\mathbb{C}^3$ and the proof of Theorem 3.3 we investigate the dichotomy in the construction of the fan Σ arising from the number of sectors in the subdivision of halfspace. Recall from Figure 3.3 that a long side exists if and only if the subdivision of halfspace has two convex sectors. Convexity ensures that any two vectors from a regular triple which lie in the same (closed) convex sector are adjacent. Thus a regular triple is in one of two possible orientations:

Type 1: two consecutive vectors in the same closed blade of the propellor, for example, $f_{1,2} = f_{1,1} + f_{3,1}$ of Figure 3.2; or

Type 2: an interior vector in each blade, for example $f_{1,2} + f_{2,2} + f_{3,1} = 0$.

If a triple of Type 2 exists then the lines $L_{i,j}$ which form the edges of the corresponding regular triangle are the champions of the knock-out competition that meet after eliminating all their less successful rivals. For instance, in Figure 3.4(b) the lines which meet at the interior vertex with valency 3 are the champions of Example 3.11.

Definition 3.13 A regular triangle corresponding to a triple of Type 2 is called the *meeting of champions*. This triangle may have side zero in which case three lines $L_{i,j}$, one from each vertex e_i, meet at a point.

Proposition 3.14 *There is either a unique long side or a unique meeting of champions.*

PROOF. If there is a long side e_2e_3, it is subdivided by a line from e_1, and Type 2 cannot occur. We claim that if there is no long side, there is a unique regular triple of Type 2. Uniqueness is almost obvious from the topology: if it exists, a meeting of champions divides Δ into 4 regions (one possibly empty), and any other line is confined to one region (it is knocked out by any champion it meets).

For the existence, the idea is that it is natural to deconstruct Δ by eating in from one side. The cyclic continued fraction (3.5) has three 1's, so that each side of Δ takes part in one regular triangle. Choose one side (say e_1e_3) and, preserving the other two, eat as many regular triangles as we can along e_1e_3 (that is, with sides through e_1 or e_3, as in Figure 3.6(a)). Every regular triple of Type 1 is associated with a well defined side of Δ, and is eaten in this way starting from that side. The union of regular triangles along each side forms its *catchment area*.

We now view an MMP as successively deleting dividing lines of the subdivision of Figure 3.2. Eating triangles in the catchment area of side e_1e_3 only deletes lines in the two hexants in the top right of Figure 3.2, between $f_{2,0}$ and $f_{3,0}$. Deleting a line joins two old cones to make a new cone, which is always basic; we conclude that the two vectors v, v' bounding the catchment area of e_1e_3 form a basis. After this, by assumption, no remaining line in these two hexants is marked with 1, so that the cone $\langle f_{2,0}, f_{3,0}\rangle$ now has its standard Newton polygon subdivision. If we now complete an MMP anyhow from this position, the same vectors v, v' must occur in some regular triple. By what we have said, the remaining vector must be in the interior of the third hexant. Thus a regular triple of Type 2 exists. $\square$

3.5 Regular triangles and invariant ratios

The regular triples v_1, v_2, v_3 of Definition 3.1 live in L. Passing to the dual lattice M of invariant monomials is a clever exercise in elementary coordinate geometry in an affine lattice that plays a key role in the proof of Theorem 3.3.

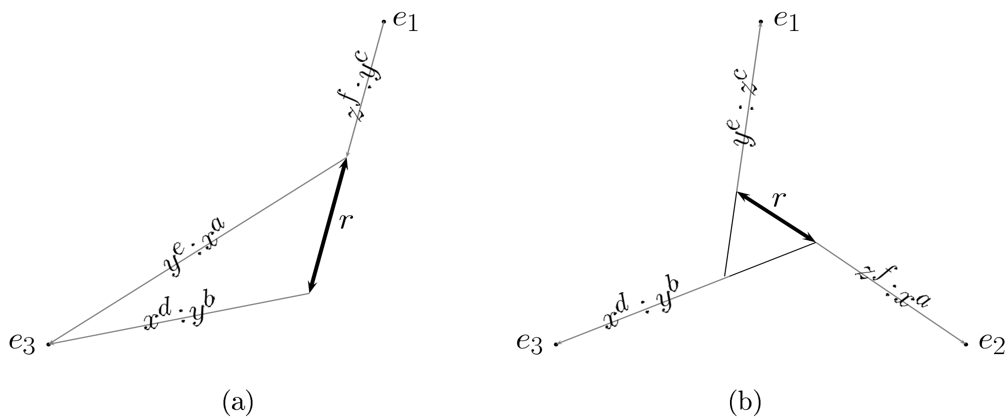


Figure 3.6: (a) corner triangle; (b) meeting of champions

Proposition 3.15 *Every regular triangle of side r gives rise to the invariant ratios of Figure 3.6 (we permute x, y, z if necessary). Moreover,*

$$d - a = e - b - c = f = r \quad \textit{in Case a,} \tag{3.6}$$
$$d - a = e - b = f - c = r \quad \textit{in Case b.} \tag{3.7}$$

Note: b, d (etc.) are not necessarily coprime; but x^d/y^b is primitive in M, that is, not a power of an *invariant* monomial.

Proposition 3.16 *Let l be any lattice line of $\mathbb{Z}^2_\Delta$, and $\mathbf{m} \in M$ an invariant monomial that bases its orthogonal $l^\perp \cap M$ (as explained at the start of the proof of Proposition 3.15). Then the lattice lines of $\mathbb{Z}^2_\Delta$ parallel to l are orthogonal to $\mathbf{m}(xyz)^i$ for $i \in \mathbb{Z}$. It follows that the regular tesselations of the regular triangles of Figure 3.6 are cut out by the ratios*

$$x^{d-i} : y^{b+i} z^i, \quad y^{e-j} : z^j x^{a+j}, \quad z^{f-k} : x^k y^{c+k} \quad \textit{in Case a,} \tag{3.8}$$
$$x^{d-i} : y^{b+i} z^i, \quad y^{e-j} : z^{c+j} x^j, \quad z^{f-k} : x^{a+k} y^k \quad \textit{in Case b,} \tag{3.9}$$

for $i, j, k = 0, \ldots, r-1$.

Proof. The overlattice L is based by e_i, v_1, v_2 for any $i = 1, 2$ or 3 and any regular triple v_1, v_2, v_3. In contrast, recall from §1.1.4 that e_1, e_2, e_3 base $\overline{L} = \mathbb{Z}^3 \subset L$, and x, y, z base the dual lattice $\overline{M} = \mathbb{Z}^3$ of monomials on $\mathbb{C}^3$. The invariant monomials form the sublattice $M \subset \overline{M}$ on which L is integral, so that $M = \operatorname{Hom}(L, \mathbb{Z})$.

Write T for one of the regular triangles of Figure 3.6. Each side of T defines a sublattice (say) $\{e_3, v_1\}^\perp \cap M \cong \mathbb{Z}$. The ratio $x^d : y^b$ in Figure 3.6, or the monomial $\xi = x^d/y^b$, is the basis of $\{e_3, v_1\}^\perp \cap M$ on which the triangle is positive, say $v_2(\xi) > 0$. So much for Figure 3.6.

For the equalities (3.6) in Case a, note that Figure 3.6(a) gives v_1, v_2, v_3 up to proportionality:

$$\begin{aligned} v_1 &\sim (b, d, -(b+d)), \\ v_2 &\sim (e, a, -(a+e)), \\ v_3 &\sim (c+f, -f, -c). \end{aligned} \tag{3.10}$$

We claim that *the constants of proportionality are all equal, and equal to*

$$\frac{1}{de - ab} = \frac{1}{ac + af + ef} = \frac{1}{bf + cd + df}.$$

(The denominators are the 2×2 minors in the array of (3.10).) For this, write

$$\xi = \frac{x^d}{y^b}, \quad \eta = \frac{y^e}{x^a}, \quad \zeta = \frac{z^f}{y^c}.$$

These 3 monomials are *not* a basis of M (unless $r = 1$, when our regular triangle is basic). But any two of them are *part of a basis*. Indeed, let e be any vertex of R and $\pm v_i, \pm v_j$ primitive vectors along its two sides; then $\{e, \pm v_i, \pm v_j\}$ is a basis of L, and the two monomials along the sides are part of the dual basis of M. Now there are lots of dual bases around, and the claim follows at once from

$$v_1(\eta) = v_2(\xi) = v_3(\xi) = 1, \quad v_1(\zeta) = v_2(\zeta) = v_3(\eta) = -1.$$

Equating components of $v_1 + v_3 = v_2$ gives $e = b + c + f$ and $a = d - f$, the first two equalities of (3.6). For the final equality, if we start from e_3 and take f steps along the vector v_1, we arrive at

$$e_3 + f v_1 = \frac{1}{de - ab}\Big(bf, df, de - ab - bf - df\Big)$$

The final entry $de - ab - bf - df$ evaluates to cd. Thus this point has last two entries df, cd proportional to f, c, so lies on the third side of R. Therefore $r = f$. The proof of (3.7) in Case b is similar.

For Proposition 3.16, write $m, u \in M_{\mathbb{R}}$ for the linear forms on L corresponding to the monomials $\mathbf{m}, xyz \in M$. The junior plane $\mathbb{R}^2_\Delta$ is defined by $u = 1$; therefore $\{(m + iu)^\perp\}_{i \in \mathbb{R}}$ is a pencil of parallel lines in $\mathbb{R}^2_\Delta$. For any lattice point $P \in \mathbb{Z}^2_\Delta$ we have $m(P) \in \mathbb{Z}$ and $u(P) = 1$, so $(m + iu)^\perp$ can only contain a lattice point for $i \in \mathbb{Z}$. $\square$

Example 3.17 Consider once again $\frac{1}{11}(1, 2, 8)$. The line (with strength 6 in Figure 3.4) from e_3 to the lattice point $\frac{1}{11}(1, 2, 8)$ represents a 2-dimensional cone τ in $\mathbb{R}^3$ (with origin behind the paper) with normal vector $\pm(2, -1, 0)$. The corresponding toric stratum $X_{L(\tau),\mathrm{Star}(\tau)} \subset X_{L,\Sigma}$ is $\mathbb{P}^1$ obtained by gluing Spec $\mathbb{C}[x^2y^{-1}]$ to Spec $\mathbb{C}[x^{-2}y]$, so it is parametrised by the A-invariant ratio $x^2 : y$. Repeating this calculation for all lines in Σ leads to Figure 3.7. Of course, the edges of Σ are not cut out by ratios; rather, the edges determine a single copy of $\mathbb{C} \subset X_{L,\Sigma}$ with coordinate an invariant monomial. That is, the image of the x, y or z-axis of $\mathbb{C}^3$ under the quotient map $\pi\colon \mathbb{C}^3 \to \mathbb{C}^3/A$.

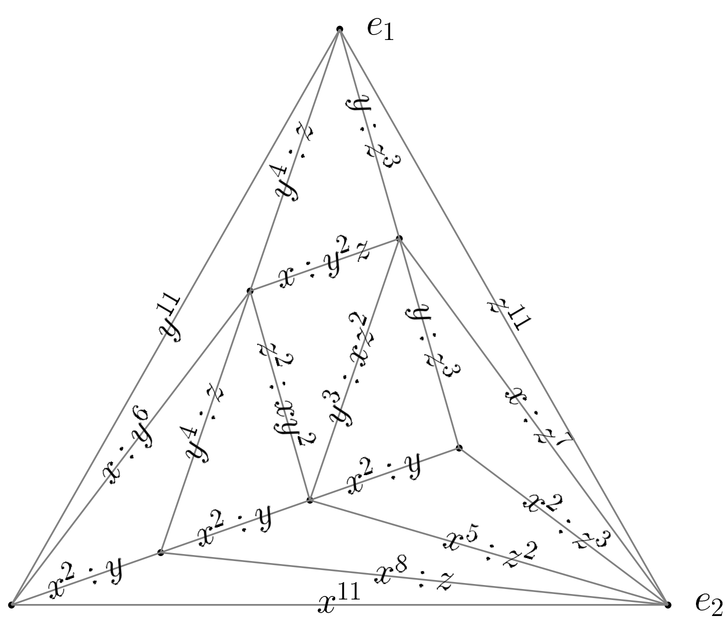


Figure 3.7: Ratios on the exceptional curves in A-Hilb $\mathbb{C}^3$ for $\frac{1}{11}(1, 2, 8)$

Remark 3.18 The coordinates of points of the tesselation can be calculated in many ways: for example, in Case a, we get

$$e_3 + iv_1 + jv_2 = \frac{1}{de-ab}\big(bj+ei, dj+ai, de-ab-(a+e)i-(b+d)j\big),$$

which could be used to prove Proposition 3.16; or from the 2×2 minors of

$$\begin{pmatrix} d-i & -(b+i) & -i \\ -(a+j) & e-j & j \end{pmatrix}.$$

It is curious that these explicit calculations in the general case shed almost no light on Propositions 3.15–3.16, even when you know the answers.

3.6 Basic triangles and dual monomial bases

The regular tesselation of a regular triangle R of side r is a simple and familiar object. A moment's thought shows that every basic triangle T is one of the following two types (see Figure 3.8 for the subgroup $\mathbb{Z}/r^2 \subset \mathrm{SL}(3,\mathbb{Z})$):

'up' For $i,j,k \geq 0$ with $i+j+k = r-1$, push the three sides of R inwards by i, j and k lattice steps respectively. (There are $\binom{r+1}{2}$ choices.) We visualise three shutters closing in until they leave a single basic triangle T. Note that T is a scaled down copy of R, parallel to R and in the same orientation; in other words, up to a translation, it is $\frac{1}{r}R$.

'down' For $i,j,k > 0$ with $i+j+k = r+1$, push the three sides of R inwards by i, j and k lattice steps (giving $\binom{r}{2}$ choices). Now the shutters close over completely, until they have a triple overlap consisting of a single basic triangle T, in the opposite orientation to R; up to translation, it is $-\frac{1}{r}R$.

Proposition 3.19 *Let R be one of the regular triangle of Figure 3.6. Its basic 'up' triangles have dual bases*

$$\xi = x^{d-i}/y^{b+i}z^i, \quad \eta = y^{e-j}/z^j x^{a+j}, \quad \zeta = z^{f-k}/x^k y^{c+k} \quad \textit{in Case a}$$
$$\xi = x^{d-i}/y^{b+i}z^i, \quad \eta = y^{e-j}/z^{c+j} x^{j}, \quad \zeta = z^{f-k}/x^{a+k} y^{k} \quad \textit{in Case b}$$

for $i,j,k \geq 0$ with $i+j+k = r-1$. Its basic 'down' triangles have dual bases

$$\lambda = y^{b+i}z^i/x^{d-i}, \quad \mu = z^j x^{a+j}/y^{e-j}, \quad \nu = x^k y^{c+k}/z^{f-k} \quad \textit{in Case a}$$
$$\lambda = y^{b+i}z^i/x^{d-i}, \quad \mu = z^{c+j} x^{j}/y^{e-j}, \quad \nu = x^{a+k} y^{k}/z^{f-k} \quad \textit{in Case b}$$

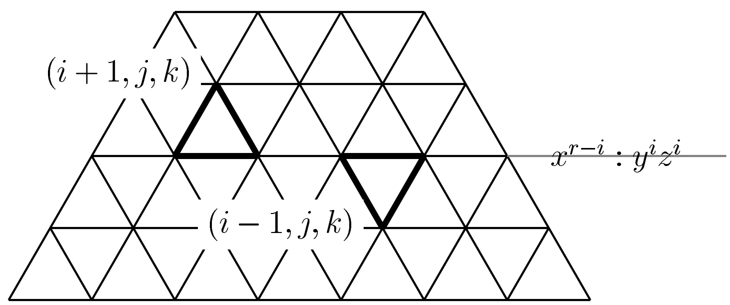
for $i,j,k > 0$ with $i+j+k = r+1$.

Proof. A basic triangle T has a basic dual cone in the lattice M, based by 3 monomials perpendicular to the 3 sides of T. These monomials are given by Proposition 3.16, or more explicitly as listed above. $\square$

Example 3.20 Up triangle for $A = \mathbb{Z}/r \oplus \mathbb{Z}/r$. The lattice is

$$\mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(1, -1, 0) + \mathbb{Z} \cdot \frac{1}{r}(0, 1, -1),$$

and Δ is spanned as usual by $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. We omit denominators, writing lattice points of Δ as (a, b, c) with $a + b + c = r$.



(a) (b)

Figure 3.8: (a) Up triangle; (b) down triangle (same i, nonspecific j, k)

An up triangle T has vertexes $(i+1, j, k)$, $(i, j+1, k)$ and $(i, j, k+1)$ for some $i, j, k \geq 0$ with $i + j + k = r - 1$ as in Figure 3.8(a). Since T is basic, so is its dual cone in the lattice of monomials, so the dual cone has the basis

$$\xi = x^{r-i}/y^i z^i, \quad \eta = y^{r-j}/x^j z^j, \quad \zeta = z^{r-k}/x^k y^k.$$

Thus the affine piece $Y_T = \mathbb{C}^3_{\xi,\eta,\zeta} \subset Y_\Sigma$ parametrises equations of the form

$$\begin{aligned} x^{r-i} &= \xi y^i z^i, & y^{i+1} z^{i+1} &= \eta\zeta x^{r-i-1}, \\ y^{r-j} &= \eta x^j z^j, & x^{j+1} z^{j+1} &= \xi\zeta y^{r-j-1}, & \quad xyz = \xi\eta\zeta. \\ z^{r-k} &= \zeta x^k y^k, & x^{k+1} y^{k+1} &= \xi\eta z^{r-k-1}, \end{aligned} \tag{3.11}$$

A down triangle T has vertices $(i-1,j,k)$, $(i,j-1,k)$ and $(i,j,k-1)$ for some $i,j,k \geq 0$ with $i+j+k = r+1$ as in Figure 3.8(b). The sides of T again correspond to the invariant ratios $x^{r-i} : y^i z^i$ etc., and its dual has basis

$$\lambda = y^i z^i / x^{r-i}, \quad \mu = x^j z^j / y^{r-j}, \quad \nu = x^k y^k / z^{r-k}.$$

The affine piece $Y_T = \mathbb{C}^3_{\lambda,\mu,\nu} \subset Y_\Sigma$ parametrises the equations

$$\begin{array}{lll} y^i z^i = \lambda x^{r-i}, & x^{r-i+1} = \mu\nu y^{i-1} z^{i-1}, & \\ x^j z^j = \mu y^{r-j}, & y^{r-j+1} = \lambda\nu x^{j-1} y^{j-1}, & xyz = \lambda\mu\nu. \\ x^k y^k = \nu z^{r-k}, & z^{r-k+1} = \lambda\mu x^{k-1} y^{k-1}, & \end{array} \tag{3.12}$$

Remark 3.21 The standard construction of toric geometry is that the toric variety $X_{L,\Sigma}$ is the union of the affine pieces $X_{L,\tau} = \operatorname{Spec} \mathbb{C}[\tau^\vee \cap M]$ taken over all 3-dimensional cones $\tau \in \Sigma$, or equivalently over every triangle T in the triangulation Σ. Proposition 3.19 says that $\mathbb{C}[\tau^\vee \cap M] = \mathbb{C}[\xi,\eta,\zeta]$ (respectively $\mathbb{C}[\lambda,\mu,\nu]$); that is, $X_{L,\tau} \cong \mathbb{C}^3 \subset X_{L,\Sigma}$ with affine coordinates ξ,η,ζ (respectively λ,μ,ν).

On the other hand Proposition 3.19 also causes $X_{L,\tau}$ to parametrise systems of equations such as

$$x^{d-i} = \xi y^{b+i} z^i, \quad y^{e-j} = \eta z^j x^{a+j}, \quad z^{f-k} = \zeta x^k y^{c+k}, \quad \text{etc.}$$

To prove Theorem 3.3 we show that these equations determine the equations of a certain A-cluster of $\mathbb{C}^3$, and conversely, every A-cluster occurs in this way; thus $X_{L,\tau}$ is naturally a parameter space for A-clusters. The details are given in §3.8 below.

3.7 Nakamura's theorem

The literature uses two a priori different notions of G-Hilb $\mathbb{C}^3$. The original construction goes as follows: set $n := |G|$, take the Hilbert scheme $\operatorname{Hilb}^n(\mathbb{C}^3)$ of all *clusters* (i.e., zero-dimensional subschemes) $Z \subset \mathbb{C}^3$ of length n, then take the fixed locus $(\operatorname{Hilb}^n(\mathbb{C}^3))^G$ and, finally, define G-Hilb $\mathbb{C}^3$ to be the irreducible component containing the general G-orbit. This is a 'dynamic' definition: a cluster $Z \subset \mathbb{C}^3$ corresponds to a point in G-Hilb $\mathbb{C}^3$ if it is a flat deformation of a genuine G-orbit of n distinct points. Thus the dynamic G-Hilb $\mathbb{C}^3$ is irreducible by definition, but we don't know which functor it represents. Also, the definition involves the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^3)$ of n points in $\mathbb{C}^3$ which is almost always very badly singular.

Here we use the following algebraic definition:

Definition 3.22 For a finite subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$, a *$G$-cluster* Z is a G-invariant subscheme $Z \subset \mathbb{C}^3$ for which $H^0(Z, \mathcal{O}_Z)$ is the regular representation of G. The *G-Hilbert scheme* G-Hilb $\mathbb{C}^3$ is the moduli space of G-clusters.

Ito and Nakajima [IN99, §2.1] proved that the algebraic and the dynamic definitions of G-Hilb $\mathbb{C}^3$ coincide for a finite Abelian subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$. More recently, Bridgeland, King and Reid [BKR99] proved that the definitions coincide for a finite (not necessarily Abelian) subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$.

Theorem 3.23 ([Nak00]) *(I) For every subgroup $A \subset \mathrm{SL}(3,\mathbb{C})$ which is finite and diagonal, and every A-cluster Z, generators of the ideal $\mathcal{I}_Z$ can be chosen as the system of 7 equations*

$$\begin{array}{lll} x^{l+1} = \xi y^b z^f, & y^{b+1} z^{f+1} = \lambda x^l, & \\ y^{m+1} = \eta z^c x^d, & z^{c+1} x^{d+1} = \mu y^m, & xyz = \pi. \\ z^{n+1} = \zeta x^a y^e, & x^{a+1} y^{e+1} = \nu z^n, & \end{array} \tag{3.13}$$

Here $a,b,c,d,e,f,l,m,n \geq 0$ are integers, and $\xi,\eta,\zeta,\lambda,\mu,\nu,\pi \in \mathbb{C}$ are constants satisfying

$$\lambda\xi = \mu\eta = \nu\zeta = \pi. \tag{3.14}$$

(II) Moreover, exactly one of the following cases holds:

$$\text{‘up’} \quad \begin{cases} \lambda = \eta\zeta, \quad \mu = \zeta\xi, \quad \nu = \xi\eta, \quad \pi = \xi\eta\zeta \\ l = a+d, \quad m = b+e, \quad n = c+f; \quad \textit{or} \end{cases} \tag{3.15}$$

$$\text{‘down’} \quad \begin{cases} \xi = \mu\nu, \quad \eta = \nu\lambda, \quad \zeta = \lambda\mu, \quad \pi = \lambda\mu\nu \\ l = a+d+1, \quad m = b+e+1, \quad n = c+f+1. \end{cases} \tag{3.16}$$

Remark 3.24 The group A doesn't really come into our arguments, which deal with *all* diagonal groups at one and the same time. For example, $A = 0$ makes perfectly good sense. The particular group for which Z is an A-cluster is determined from the exponents in (3.13) as follows: its character group A^* is generated by its eigenvalues χ_x, χ_y, χ_z on x, y, z, and related by

$$\chi_x + \chi_y + \chi_z = 0 \quad \text{and} \quad \begin{aligned} (l+1)\chi_x &= b\chi_y + f\chi_z \\ (m+1)\chi_y &= c\chi_z + d\chi_x \\ (n+1)\chi_z &= a\chi_x + e\chi_y. \end{aligned} \tag{3.17}$$

This is a presentation of A^* as a $\mathbb{Z}$-module, as a little 4×3 matrix; all our stuff about regular triples, regular tesselations and so on, can be viewed as a classification of different presentations of A^* of type (3.17).

PROOF OF THEOREM 3.23(I). By Definition 3.22 the ring $H^0(Z, \mathcal{O}_Z) = k[x,y,z]/I_Z = \mathcal{O}_{\mathbb{C}^3}/\mathcal{I}_Z$ of Z is the regular representation so each character of A has exactly a 1-dimensional eigenspace in $H^0(Z, \mathcal{O}_Z)$ (written as $\mathcal{O}_Z$ hereafter). Arguing on the identity character and using the assumption that $A \subset \mathrm{SL}(3,\mathbb{C})$ provides an equation $xyz = \pi$ for some $\pi \in \mathbb{C}$.

Since $k[x,y,z]$ is based by monomials, their images span $\mathcal{O}_Z$; monomials are eigenfunctions of the A action. Obviously, each eigenspace in $\mathcal{O}_Z$ contains a nonzero image of a monomial $\mathbf{m}$, and is based by any such. Moreover, if $\mathbf{m}$ is a multiple of an invariant monomial, say $\mathbf{m} = \mathbf{m}_0\mathbf{m}_1$ with $\mathbf{m}_0$ invariant under A, and is nonzero in $\mathcal{O}_Z$, then the other factor $\mathbf{m}_1$ is also a basis of the same eigenspace. From now on, we say *basic monomial* in $\mathcal{O}_Z$ to mean the nonzero image in $\mathcal{O}_Z$ of a monomial that is not a multiple of an invariant monomial; in particular, it is not a multiple of xyz, so involves at most two of x, y, z.

Lemma 3.25 below shows how to choose the equations in (3.13). Indeed $x^{l+1}, y^b z^f$ belong to a common eigenspace, and therefore, because xyz is invariant, also x^l and $y^{b+1}z^{f+1}$ belong to a common eigenspace. This is based by x^l by choice of l, hence we get the relation $y^{b+1}z^{f+1} = \lambda x^l$. Finally, since $y^b z^f$ is a basic monomial, $\lambda\xi = \pi$ corresponds to the syzygy $\lambda(\mathrm{i}) + x(\mathrm{ii}) - y^b z^f(\mathrm{iii})$ between the three relations

$$(\mathrm{i}) \quad x^{l+1} = \xi y^b z^f, \quad (\mathrm{ii}) \quad y^{b+1}z^{f+1} = \lambda x^l, \quad (\mathrm{iii}) \quad xyz = \pi.$$

The relations involving y^{m+1} and z^{n+1} arise similarly.

Now (I) says that, for any A and any A-cluster Z, once the relations (3.13) are derived as above, $\mathcal{O}_Z$ is based by the monomials in the tripod of Figure 3.9, and the relations reduce any monomial $\mathbf{m}$ to one of these. We

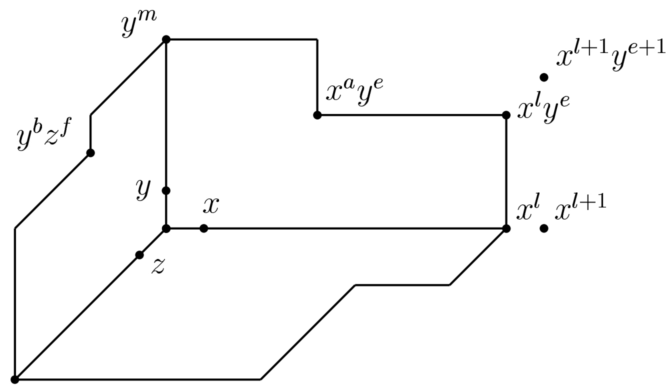


Figure 3.9: Tripod of monomials basing $\mathcal{O}_Z$

derived the relations in pairs $x^{l+1} \mapsto y^b z^f$ and $y^{b+1}z^{f+1} \mapsto x^l$. The first type reduces pure powers of x higher than x^l. Suppose we have a further relation in the first quadrant, (say) $x^\alpha y^\varepsilon \mapsto \mathbf{m}$: if $\mathbf{m}$ involves x or y the new relation would be a multiple of a simpler relation. On the other hand, if $\mathbf{m} = z^\gamma$ is a pure power of z, the above argument shows the new relation is paired with a relation $z^{\gamma+1} \mapsto x^{\alpha-1}y^{\varepsilon-1}$, which contradicts our choice of n (in the exponent of z^{n+1}). This concludes the proof of (I), assuming Lemma 3.25. $\square$

Lemma 3.25 *Let x^r be the first power of x that is A-invariant. Then there is (at least) one $l \in [0, r-1]$ such that $1, x, x^2, \ldots, x^l \in \mathcal{O}_Z$ are basic monomials, and x^{l+1} is a multiple of some basic monomial $y^b z^f$ in the same eigenspace, say $x^{l+1} = \xi y^b z^f$ for some $\xi \in \mathbb{C}$.*

Proof. If $x^{r-1} \neq 0 \in \mathcal{O}_Z$ it is a basic monomial, and one choice is to take $l = r-1$ and $b = f = 0$, and to take the relation $x^{l+1} = x^r = \xi \cdot 1$. (Other choices arise if the eigenspace of some $x^{l'+1}$ with $l' < l$ also contain a basic monomial $y^{b'}z^{f'}$).

If not, there is some l with $0 \leq l \leq r-1$ such that $1, x, x^2, \ldots, x^l$ are basic monomials and $x^{l+1} = 0 \in \mathcal{O}_Z$. Now the eigenspace of x^{l+1} must contain a basic monomial $\mathbf{m}$; under the current assumptions, we assert that $\mathbf{m}$ is of the form $y^b z^f$, which proves the lemma. We need only prove that $\mathbf{m}$ is not a multiple of x. If $\mathbf{m} = x\mathbf{m}'$ then $\mathbf{m}'$ must in turn be a basic monomial in the same eigenspace as x^l. But then $x^l = (\text{unit}) \cdot \mathbf{m}'$ contradicts $x^{l+1} = 0$ and $x\mathbf{m}' \neq 0$. $\square$

PROOF OF THEOREM 3.23(II) The point is that a monomial just outside one of the shoulders of the tripod of Figure 3.9 such as $x^{l+1}y^{e+1}$ or $y^{m+1}z^{f+1}$, etc., reduces to a basic monomial in two steps involving two of the ξ, η, ζ relations, or two of the λ, μ, ν relations. (Compare [Rei97, Remark 7.3] for a discussion.)

The first reduction applies if $b + e \geq m$:

$$x^{l+1}y^{e+1} \mapsto \xi y^{b+e+1}z^f \mapsto \xi\eta y^{b+e-m}x^d z^{c+f}$$

This implies that the monomials $x^{l-d+1}y^{m-b+1}$ and z^{c+f} are in the same eigenspace, and the existence of the relation

$$x^{l-d+1}y^{m-b+1} = \xi\eta z^{c+f}$$

between them. But from the argument in (I), there is only one relation in this quadrant, namely $x^{a+1}y^{e+1} = \nu z^n$. Therefore $l - d = a$, $m - b = e$, $c + f = n$ and $\nu = \xi\eta$. Now $a + d \geq l$ and $c + f \geq n$, so that we can run the same two-step reduction to other monomials to get $\lambda = \eta\zeta$ and $\mu = \xi\zeta$.

The second type of reduction applies if $m \geq b + e + 1$

$$y^{m+1}z^{f+1} \mapsto \lambda y^{m-b}x^l \mapsto \lambda\nu x^{l-a-1}y^{m-b-e-1}z^n$$

Therefore the two monomials y^{b+e+2} and $x^{l-a-1}z^{n-f-1}$ are in the same eigenspace, and $y^{b+e+2} = \lambda\nu x^{l-a-1}z^{n-f-1}$. As before, this must be identical to the η relation, so that $m+1 = b+e+2$, $l-a-1 = d$, $n-f-1 = c$ and $\eta = \lambda\nu$. This proves the theorem. □

3.8 Proof of Theorem 3.3

The point is to identify the objects in the conclusion of Proposition 3.19 and of Theorem 3.23; this is really just a mechanical translation. To distinguish between the two sets of symbols in the monomial bases of Proposition 3.19, we first substitute for d, e, f from (3.6–3.7) of Proposition 3.15, and then replace

$$a \mapsto A, \quad b \mapsto B, \quad c \mapsto C.$$

Each of the monomial bases of Proposition 3.19 gives rise to a triple of equations, either up:

$$x^{A+r-i} = \xi y^{B+i}z^i, \quad y^{B+C+r-j} = \eta z^j x^{A+j}, \quad z^{r-k} = \zeta x^k y^{C+k} \quad \text{in Case a}$$
$$x^{A+r-i} = \xi y^{B+i}z^i, \quad y^{B+r-j} = \eta z^{C+j}x^j, \quad z^{C+r-k} = \zeta x^{A+k}y^k \quad \text{in Case b}$$

with $i,j,k \geq 0$ and $i+j+k = r-1$; or down:

$$\begin{array}{llll} y^{B+i}z^i = \lambda x^{A+r-i}, & z^j x^{A+j} = \mu y^{B+C+r-j}, & x^k y^{C+k} = \nu z^{r-k} & \text{in Case a} \\ y^{B+i}z^i = \lambda x^{A+r-i}, & z^{C+j} x^j = \mu y^{B+r-j}, & x^{A+k} y^k = \nu z^{C+r-k} & \text{in Case b} \end{array}$$

with $i,j,k > 0$ and $i+j+k = r+1$.

Each triple can be completed to the equations of an A-cluster; for example, the first triple gives:

$$\begin{aligned} x^{A+r-i} &= \xi y^{B+i} z^i & y^{B+r-j-k} z^{r-j-k} &= \eta\zeta x^{A+j+k} \\ y^{B+C+r-j} &= \eta z^j x^{A+j} & z^{r-i-k} x^{A+r-i-k} &= \zeta\xi y^{B+C+k+i} \qquad xyz = \xi\eta\zeta. \\ z^{r-k} &= \zeta x^k y^{C+k} & x^{r-i-j} y^{C+r-i-j} &= \xi\eta z^{i+j} \end{aligned}$$

(The method is to multiply together any two of the equations and cancel common factors.) Since $i+j+k = r-1$, these are of the form of Theorem 3.23, with $l = A+j+k$, $b = B+i$, $f = i$, etc.. The other cases are similar. Therefore each affine piece $X_{L,\tau} \cong \mathbb{C}^3 \subset Y_\Sigma$ determined by the triangle $T = \tau \cap \Delta$ parametrises A-clusters.

Conversely, we prove that for $A \subset \mathrm{SL}(3,\mathbb{C})$ a finite diagonal subgroup and Z an A-cluster with equations as in Theorem 3.23, Z belongs to one of the families parametrised by $X_{L,\tau}$. If Z is 'up' its equations are determined by the first three:

$$x^{a+d+1} = \xi y^b z^f, \quad y^{b+e+1} = \eta z^c x^d, \quad z^{c+f+1} = \zeta x^a y^e. \tag{3.18}$$

Consider first just two of the possibilities for the signs of $f-b$, $d-c$, $e-a$.

1. Suppose $b \geq f$, $d \geq c$ and $e \geq a$. We define A,B,C,i,j,k by

   $$A = d-c, \quad B = b-f, \quad C = e-a, \quad i = f, \quad j = c, \quad k = a$$

   and set $r = i+j+k+1$. Then, obviously,

   $$a = k, \quad b = B+i, \quad c = j, \quad d = A+j, \quad e = C+k, \quad f = i.$$

   Substituting these values in the exponents of (3.18), puts the equations of Z in the form up, Case a.

2. Similarly, if $b \geq f$, $c \geq d$ and $a \geq e$, we fix up A,B,C,i,j,k so that

   $$a = A+k, \quad b = B+i, \quad c = C+j, \quad d = j, \quad e = k, \quad f = i.$$

   Substituting in (3.18), shows that Z is up, Case b.

One sees that the permutation $y \leftrightarrow z$ leads to $b \leftrightarrow f$, $a \leftrightarrow d$ and $c \leftrightarrow e$, and the other possibilities for the signs of $e - a$, $f - b$, $d - c$ all reduce to these two cases on permuting x, y, z. In fact, Figure 3.6(a) has 6 different images on permuting x, y, z (corresponding to the choices of e_1 and e_3), and Figure 3.6(b) has 2 different images (corresponding to the cyclic order).

If Z is 'down' its equations can be deduced from the second three:

$$y^{b+1}z^{f+1} = \lambda x^{a+d+1}, \quad z^{c+1}x^{d+1} = \mu x^{b+e+1}, \quad x^{a+1}y^{e+1} = \nu z^{c+f+1} \tag{3.19}$$

Exactly as before, if $b \geq f$, $d \geq c$ and $e \geq a$ then we can fix up $A, B, C \geq 0$ and $i, j, k > 0$ so that

$$\begin{gathered} a+1 = k, \quad b+1 = B+i, \quad c+1 = j, \\ d+1 = A+j, \quad e+1 = C+k, \quad f+1 = i, \end{gathered}$$

which puts (3.19) in the form down, Case a. The rest of the proof is a routine repetition. This proves Theorem 3.3. □

Chapter 4

The McKay correspondence for A-Hilb $\mathbb{C}^3$

For a finite Abelian subgroup $A \subset \mathrm{SL}(3, \mathbb{C})$, Ito and Nakajima [IN00] established an isomorphism between the K-theory of Nakamura's A-Hilbert scheme A-Hilb $\mathbb{C}^3$ and the representation ring of A. This leads to a basis of the rational cohomology of A-Hilb $\mathbb{C}^3$ in one-to-one correspondence with the irreducible representations of A. In this chapter we construct an explicit basis of the integral cohomology of A-Hilb $\mathbb{C}^3$ in one-to-one correspondence with the irreducible representations of A, as conjectured by Reid [Rei97].

4.1 Statement of the results

Gonzalez-Sprinberg and Verdier provided a geometric explanation for the classical McKay correspondence by constructing tautological bundles on the minimal resolution $Y \to \mathbb{C}^2/G$ of a Kleinian singularity as described in §1.2.2. Reid [Rei97] generalised the Gonzalez-Sprinberg and Verdier construction to higher dimensions and formulated the following conjecture:

Conjecture 4.1 (Reid's second McKay conjecture) *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite subgroup and suppose that $Y := G$-Hilb $\mathbb{C}^n$ is a crepant resolution of the quotient $X := \mathbb{C}^n/G$. Then*

(i) *the Gonzalez-Sprinberg and Verdier sheaves $\mathcal{F}_i$ on Y are locally free and form a $\mathbb{Z}$-basis of the K-theory of Y.*

(ii) *a certain cookery with the Chern classes of the sheaves $\mathcal{F}_i$ leads to a $\mathbb{Z}$-basis of $H^*(Y,\mathbb{Z})$ for which the following bijection holds:*

$$\Big\{\textit{irreducible representations of } G\Big\} \longleftrightarrow \textit{ basis of } H^*(Y,\mathbb{Z}). \qquad (4.1)$$

To support this conjecture Reid calculated $Y = A\text{-Hilb}\,\mathbb{C}^3$ for several examples of finite Abelian subgroups $A \subset \mathrm{SL}(3,\mathbb{C})$. In each example, the first Chern classes of the tautological bundles $c_1(\mathcal{F}_i)$ span $H^2(Y,\mathbb{Z})$. By using the relations in $\mathrm{Pic}(Y)$ between tautological bundles, Reid introduced a recipe to cook up virtual bundles $\mathcal{V}_m$ on Y whose second Chern classes $c_2(\mathcal{V}_m)$ base $H^4(Y,\mathbb{Z})$. Moreover, the virtual bundles $\mathcal{V}_m$ are indexed by certain characters χ_m of the group A and, by removing the corresponding classes $c_1(\mathcal{F}_m)$ from the set spanning $H^2(Y,\mathbb{Z})$, he produced a basis for $H^2(Y,\mathbb{Z})$. Thus, the McKay correspondence bijection (4.1) holds for each of Reid's examples.

Ito and Nakajima [IN00] subsequently proved part (i) of Conjecture 4.1 for a finite Abelian subgroup $A \subset \mathrm{SL}(3,\mathbb{C})$. By applying the Chern character they established a basis of $H^*(Y,\mathbb{Q})$ in one-to-one correspondence with the irreducible representations of A, a rational version of bijection (4.1).

The main result of this chapter establishes part (ii) of Conjecture 4.1 for a finite Abelian subgroup $A \subset \mathrm{SL}(3,\mathbb{C})$. We begin by constructing a basis of $H^4(Y,\mathbb{Z})$:

Theorem 4.2 *For any finite Abelian subgroup $A \subset \mathrm{SL}(3,\mathbb{C})$ we construct virtual bundles $\mathcal{V}_m$ on $Y = A\text{-Hilb}\,\mathbb{C}^3$ indexed by certain characters χ_m of the group A. Moreover, the classes $c_2(\mathcal{V}_m)$ form a basis of $H^4(Y,\mathbb{Z})$ dual to the basis $[S] \in H_4(Y,\mathbb{Z})$ of the compact exceptional surfaces S of the resolution $\varphi\colon Y \to X$.*

Remark 4.3 The proof of Theorem 4.2 uncovers certain relations between tautological bundles of the form, say, $\mathcal{F}_m = \mathcal{F}_k \otimes \mathcal{F}_l$ for characters $\chi_m = \chi_k \otimes \chi_l$. In fact, one such relation arises for each compact exceptional surface S of the map φ and determines the virtual bundle $\mathcal{V}_m$ on Y according to "Reid's recipe" described in §4.3. However it is not true in general that the map $\chi_i \to \mathcal{F}_i$ is multiplicative. See Remark 4.38 for more on this point.

The first Chern classes $c_1(\mathcal{F}_i)$ of the tautological bundles span $H^2(Y,\mathbb{Z})$, but they do not form a $\mathbb{Z}$-basis in general. However, we determine a subset which does base $H^2(Y,\mathbb{Z})$:

Theorem 4.4 *Given the first Chern classes of all nontrivial tautological bundles, discard those classes* $c_1(\mathcal{F}_m)$ *determined by characters* χ_m *which form the indexing set of the basis* $c_2(\mathcal{V}_m)$ *of* $H^4(Y,\mathbb{Z})$. *The remaining classes form a* $\mathbb{Z}$*-basis of* $H^2(Y,\mathbb{Z})$.

The trivial character determines the trivial tautological bundle $\mathcal{F}_0 = \mathcal{O}_Y$ which generates $H^0(Y,\mathbb{Z})$. This leads immediately to our main result:

Corollary 4.5 *The McKay correspondence bijection (4.1) holds (replace* G *by* A*) for all finite Abelian subgroups* $A \subset \mathrm{SL}(3,\mathbb{C})$.

Remark 4.6 Bridgeland, King and Reid [BKR99] proved Conjecture 4.1(i) for a finite subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$ by identifying the K-theory of Y with the G-equivariant K-theory of $\mathbb{C}^3$. However, Conjecture 4.1(ii) is still an open problem for non-Abelian subgroups $G \subset \mathrm{SL}(3,\mathbb{C})$.

4.2 Tautological line bundles on A-Hilb $\mathbb{C}^3$

Let $A \subset \mathrm{SL}(3,\mathbb{C})$ be a finite Abelian subgroup. Write $\pi\colon \mathbb{C}^3 \to X := \mathbb{C}^3/A$ for the quotient, $Y := A$-Hilb $\mathbb{C}^3$ for the A-Hilbert scheme and $\varphi\colon Y \to X$ for the crepant resolution.

Definition 4.7 For an irreducible representation $\rho_i\colon A \to \mathrm{GL}(V_i)$, let

$$M_i := \mathrm{Hom}_{\mathbb{C}[A]}\big(V_i, \mathbb{C}[x,y,z]\big)$$

denote the $\mathcal{O}_X$-module generated by monomials $x^\alpha y^\beta z^\gamma$ in the χ_i-character space. Define $\mathcal{F}_i := \varphi^* M_i/\mathrm{Tors}_{\mathcal{O}_Y}$, where $\mathrm{Tors}_{\mathcal{O}_Y}$ denotes the $\mathcal{O}_Y$-torsion of $\varphi^* M_i$. The sheaf $\mathcal{F}_i$ is invertible [Rei97, §5.5] and is called the *tautological line bundle* on Y associated to ρ_i.

Reid [Rei97, §6] constructs A-Hilb $\mathbb{C}^3$ as the 'simultaneous dual Newton polyhedron' of the modules M_i and proves that A-Hilb $\mathbb{C}^3$ is the smallest resolution of $\mathbb{C}^3/A$ on which every sheaf $\mathcal{F}_i$ is locally free. A worked example can be found in Reid [Rei97, p. 20], but for convenience we provide our own:

Example 4.8 We return to the singularity $\frac{1}{11}(1,2,8)$ of Example 3.11. A conventional representation of the Newton polyhedron of M_1 is shown in Figure 4.1(a), where we draw only the spanning monomials as a plane figure.

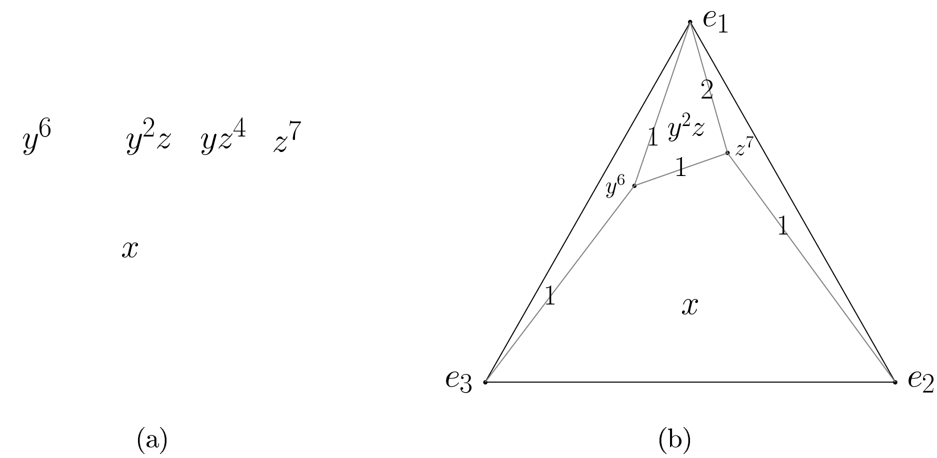


Figure 4.1: (a) Newton polyhedron of M_1; (b) the minimal partial resolution on which $\mathcal{F}_1$ is locally free and the monomial generators on each open set

The weights $(11,0,0)$ and $(6,1,4)$ have common minima on y^6 and y^2z, so the line parametrised by their ratio $y^4 : z$ joins e_1 to $\frac{1}{11}(6,1,4)$ in the partial resolution. We draw the resulting fan of the partial resolution and record the monomial generator of $\mathcal{F}_1$ on each open set (i.e. each triangle) in Figure 4.1(b).

The degree of $\mathcal{F}_1$ is nonzero on the curves which appear on the partial resolution (we write the degree on the corresponding line in Figure 4.1(b)), and is zero on every other curve in Y. In fact the degree can be read directly from the Newton polyhedron of M_1. For instance, the monomials y^6 and y^2z are adjacent in Figure 4.1(a) and a calculation similar to that from Example 1.21 shows that the degree of $\mathcal{F}_1$ on the curve cut out by their common ratio $y^4 : z$ is one. However, the monomial yz^4 lies on the line joining y^2z and z^7 in Figure 4.1(a) and it follows that the degree of $\mathcal{F}_1$ on the curve cut out by $y : z^3$ is two. Finally, $\mathcal{F}_1$ has degree one on the curves cut out by the ratios $x : y^6$, $x : y^2z$ and $x : z^7$ whose monomials lie in M_1.

Definition 4.9 A line l in Σ, or equivalently an exceptional curve $\mathbb{P}^1 \subset Y$, is said to be *marked* with the character χ if the monomials in the ratio parametrising the $\mathbb{P}^1$ lie in the χ-character space (see also Proposition 3.16).

Lemma 4.10 *The bundle $\mathcal{F}_i$ has degree one on each curve marked with χ_i.*

Proof. Consider a line in Σ marked with the character χ_i, and suppose by permuting x, y, z if necessary that the corresponding curve $\mathbb{P}^1 \subset Y$ is parametrised by the ratio $z^{f-k} : x^k y^{c+k}$. Then $z^{f-k}, x^k y^{c+k} \in M_i$. Moreover, z^{f-k} generates $\mathcal{F}_i$ on the open sets defined by the triangles to one side of the line, while $x^k y^{c+k}$ generates $\mathcal{F}_i$ on the other side. The result follows from a calculation similar to that of Example 1.21 □

The converse to Lemma 4.10 is false because the bundle $\mathcal{F}_1$ of Example 4.8 has degree one on the curve cut out by $y^4 : z$.

4.3 Reid's recipe for a dual basis of $H^4(Y, \mathbb{Z})$

In this section we prove Theorem 4.2. First we establish that Reid's recipe for constructing virtual bundles $\mathcal{V}_m$ on Y indexed by certain characters χ_m of the group A applies for any finite Abelian subgroup $A \subset \mathrm{SL}(3, \mathbb{C})$. Worked examples are provided in §4.3.2 to illustrate the construction. We conclude by proving the classes $c_2(\mathcal{V}_m)$ define a basis of $H^4(Y, \mathbb{Z})$ dual to the basis in homology $[S] \in H_4(Y, \mathbb{Z})$ determined by the compact exceptional surfaces S of the resolution $\varphi \colon Y \to X$.

4.3.1 Virtual bundles on A-Hilb $\mathbb{C}^3$

Every compact exceptional surface of the crepant resolution $\varphi \colon Y \to X$ corresponds to an internal vertex of the triangulation Σ. We now demonstrate that for every such vertex there is a relation between the tautological bundles $\mathcal{F}_i$ on Y. Following Reid's examples [Rei97], we use these relations to cook up virtual bundles $\mathcal{V}_m$ on Y having trivial rank and trivial first Chern class.

Definition 4.11 For a 3-dimensional cone τ in the fan of A-Hilb $\mathbb{C}^3$ (or equivalently, for a triangle in the triangulation Σ), write $f_{i,\tau}$ for the generator of $\mathcal{F}_i$ on the open affine subset $U_\tau = \mathrm{Spec}\ \mathbb{C}[\tau^\vee \cap M]$.

For each vertex v in Σ we uncover a relation between the generators $f_{i,\tau}$ on every open set U_τ in the open cover of Y. The vertices in Σ have valency 3, 4, 5 or 6 (see Corollary 3.5) so we perform a case by case analysis:

Case 1: A vertex of valency 3.

A vertex v of valency 3 in Σ determines an exceptional $\mathbb{P}^2$. This happens only when three champion lines cut out by $x^a : y^b$, $y^b : z^c$ and $z^c : x^a$ meet at v, so each line is marked with the same character, say χ_l.

Proposition 4.12 *For $\chi_m := \chi_l \otimes \chi_l$, we have $\mathcal{F}_m = \mathcal{F}_l \otimes \mathcal{F}_l$.*

PROOF. Each of the monomials x^a, y^b and z^c generates $\mathcal{F}_l$ on certain sets U_τ of the open cover of Y, as illustrated by Figure 4.2(a). The

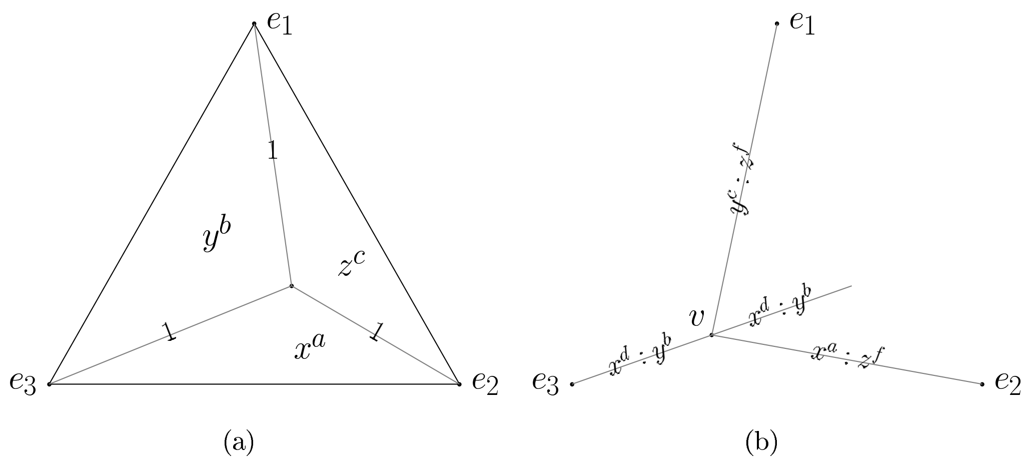


Figure 4.2: (a)The generators of $\mathcal{F}_l$; (b) Ratios on the lines meeting at $\mathbb{F}_r$

monomials x^{2a}, $x^a y^b$, y^{2b}, $y^b z^c$, z^{2c} and $x^a z^c$ lie in M_m, but only x^{2a}, y^{2b} and z^{2c} already generate $\mathcal{F}_m|_{U_\tau}$. Clearly $f_{m,\tau} = f_{l,\tau} \cdot f_{l,\tau}$ on every open set U_τ in the cover of Y, proving the result. □

Definition 4.13 (Reid's recipe, Case 1) The virtual bundle

$$\mathcal{V}_m := \big(\mathcal{F}_l \oplus \mathcal{F}_l\big) \ominus \big(\mathcal{F}_m \oplus \mathcal{O}_Y\big)$$

has trivial rank and, by Proposition 4.12, trivial first Chern class. We mark χ_m on the vertex of valency 3 in Σ to indicate that $\mathcal{V}_m$ is associated to the corresponding surface $S_m := \mathbb{P}^2$.

CASE 2: A VERTEX OF VALENCY 4.

A vertex v of valency 4 in Σ determines an exceptional scroll $\mathbb{F}_r$.

Lemma 4.14 *There are distinct characters χ_k and χ_l which each mark a pair of lines meeting at a vertex v of valency 4.*

PROOF. A vertex v of valency 4 occurs only when a line $L_{\alpha,\beta}$ from e_α defeats lines emanating from both of the other corners of Δ. By permuting x, y, z if necessary we assume that $\alpha = 3$ (see Figure 4.2(b)). Let χ_k denote the common character space of the monomials x^d and y^b in the ratio marking $L_{3,\beta}$. If there are no vertices on $L_{3,\beta}$ between e_3 and v then z is one of the monomials in the ratios marking the defeated lines from e_1 and e_2. More generally, it follows from the calculation of A-Hilb $\mathbb{C}^3$ that if $(f-1)$ vertices lie between e_3 and v then z^f occurs in both A-invariant ratios on the defeated lines, as shown in Figure 4.2(b). In particular, if z^f lies in the χ_l-character space then the lines from e_1 and e_2 are marked with the character χ_l. Finally, $c > b$ as $L_{3,\beta}$ defeats the line from e_1, so $\chi_k \neq \chi_l$. $\square$

Proposition 4.15 *Set $\chi_m := \chi_k \otimes \chi_l$, with χ_k and χ_l from Lemma 4.14. The relation $\mathcal{F}_m = \mathcal{F}_k \otimes \mathcal{F}_l$ holds.*

PROOF. It is enough to show that $f_{m,\tau} = f_{k,\tau} \cdot f_{l,\tau}$ on every open set U_τ in the cover of Y. We adopt the notation of Lemma 4.14 and Figure 4.2(b). Each of the monomials x^a, y^c and z^f generates $\mathcal{F}_l$ on certain sets U_τ of the open cover of Y, as illustrated by Figure 4.3(a). One can similarly compute that $f_{k,\tau} = x^d$ and $f_{m,\tau} = x^{a+d}$ whenever $f_{l,\tau} = x^a$, and that $f_{k,\tau} = y^b$ and $f_{m,\tau} = y^{b+c}$ whenever $f_{l,\tau} = y^c$.

It remains to prove that $f_{m,\tau} = z^f \cdot f_{k,\tau}$ whenever $f_{l,\tau} = z^f$. This clearly holds on two of the open sets $U_\tau \subset \mathbb{F}_r$ defined by triangles adjacent to the vertex v, as shown in Figure 4.3(b). By considering the positions of $x^d z^f$ and $y^b z^f$ in the Newton polyhedron of M_m it is clear that, for every open set U_τ for which $f_{l,\tau} = z^f$, the monomial $f_{m,\tau}$ must be divisible by z^f. Thus, for every such τ we have $f_{m,\tau}/z^f \in M_k$, hence $f_{m,\tau} = z^f \cdot f_{k,\tau}$ as required. $\square$

Definition 4.16 (Reid's recipe, Case 2) The virtual bundle

$$\mathcal{V}_m := \big(\mathcal{F}_k \oplus \mathcal{F}_l\big) \ominus \big(\mathcal{F}_m \oplus \mathcal{O}_Y\big)$$

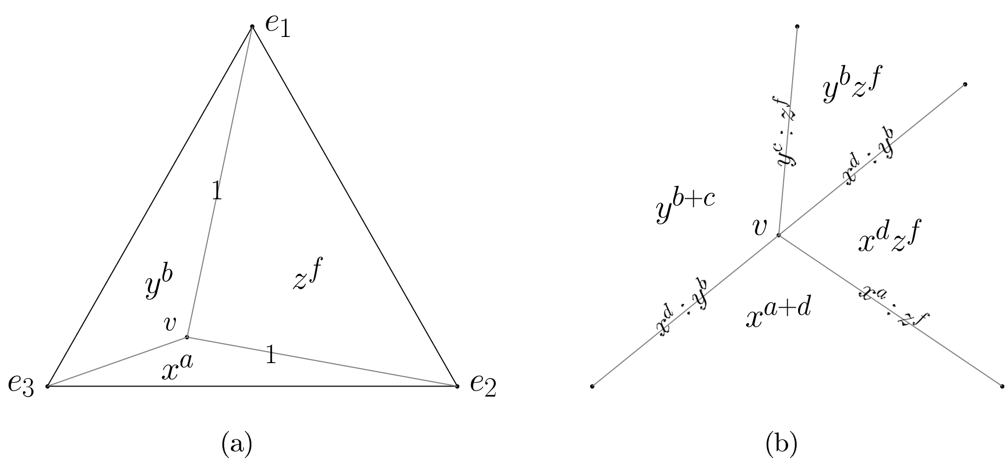


Figure 4.3: Generators of (a) $\mathcal{F}_l$; (b) $\mathcal{F}_m|_{U_\tau}$ for $U_\tau \subset \mathbb{F}_r$

has trivial rank and, by Proposition 4.15, has trivial first Chern class. We mark χ_m on the vertex of valency 4 in Σ to indicate that $\mathcal{V}_m$ is associated to the corresponding scroll $S_m := \mathbb{F}_r$.

Case 3: A vertex of valency 5 or 6, but not 3 straight lines. This is similar to the preceding case. A vertex v of valency 5 or 6 (but not 3 straight lines) corresponds to a scroll blown up in one or two torus-invariant points (but not the del Pezzo surface of degree six).

Lemma 4.17 *There are uniquely determined characters χ_k and χ_l which each mark a pair of lines meeting at a vertex v of valency 5 or 6 (not three straight lines); we say that these lines 'pass through' the vertex. The remaining line or pair of lines are marked with distinct characters.*

Proof. A vertex of valency 5 occurs only at the intersection point of a line $L_{\alpha,\beta}$ from e_α with a line $L_{\gamma,\delta}$ from e_γ. We may assume that $\alpha = 3, \gamma = 1$, and that $L_{3,\beta}$ is defeated so $L_{1,\delta}$ extends. This accounts for three lines meeting at v; the fourth and fifth lines are tesselating lines of a regular triangle T which is either a corner triangle from e_1, the meeting of champions triangle or a corner triangle from e_2. We illustrate the first case in Figure 4.4(a): the lines $L_{3,\beta}$ and $L_{1,\delta}$ are cut out by $x^a : y^e$ and $y^c : z^f$ respectively, while the tesselating lines which extend from v into T are cut out by $x^{d-r} : y^{b+r}z^r$, for r satisfying the relations (3.6), and by $z^{h-i} : x^i y^{g+i}$, for some g, h, i ($i \neq 0$).

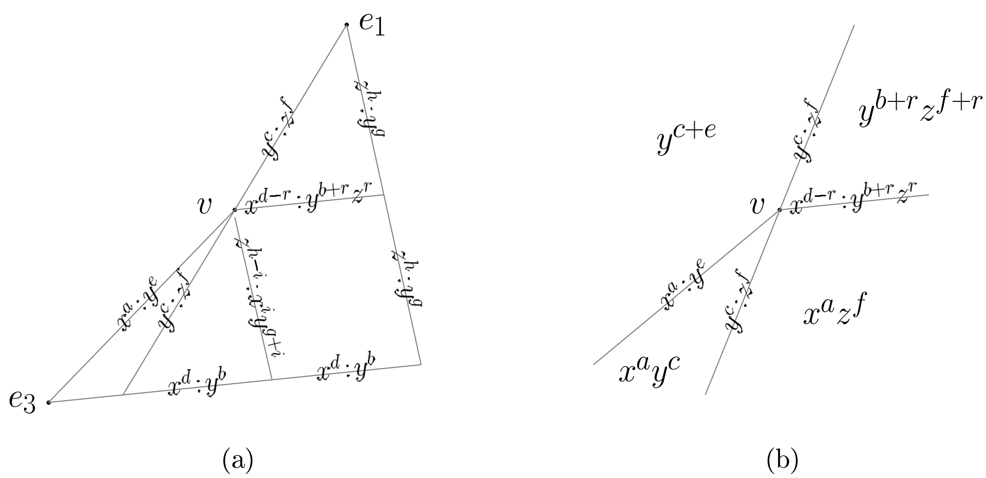


Figure 4.4: (a) Ratios on lines meeting at v; (b) Local generators of $\mathcal{F}_m|_{U_\tau}$

The character χ_k marking $L_{1,\delta}$ 'passes through' v. From (3.6) we have $x^{d-r} = x^a$, therefore a single character χ_l marks the lines cut out by $x^a : y^e$ and $x^{d-r} : y^{b+r}z^r$, so it also 'passes through' v. Finally, the character χ_j marking the fifth line is neither χ_k nor χ_l. Indeed, the relations (3.6) for T ensure that $h - i > f$, so $\chi_j \neq \chi_k$. Also, if $\chi_j = \chi_l$ then $\mathcal{F}_l$ has degree one on the line cut out by $z^{h-i} : x^i y^{g+i}$. But $\mathcal{F}_l$ is generated by x^a on both sides of the line, a contradiction. This proves the lemma when T is a corner triangle from e_1. When T is the meeting of champions or a corner triangle from e_2, the ratios cutting out the fourth and fifth lines are either $x^{d-i} : y^{b+i}z^i$ together with $z^{h-k} : x^{g+k}y^k$, or $x^{d-i} : y^i z^{b+i}$ and $z^{h-k} : x^{g+k}y^k$. In each case the same argument holds. This proves the lemma for a vertex of valency 5. The case where the vertex has valency 6 is similar and is left as an exercise. □

Proposition 4.18 *The relation $\mathcal{F}_m = \mathcal{F}_k \otimes \mathcal{F}_l$ holds for $\chi_m := \chi_k \otimes \chi_l$.*

Proof. As with Proposition 4.15 it is easy to show that $f_{m,\tau} = y^c \cdot f_{l,\tau}$ on the open sets U_τ where $\mathcal{F}_k$ is generated by y^c. On every other subset U_τ in the cover of Y we have either $f_{k,\tau} = z^f$ or $f_{l,\tau} = x^a$. In each case, making reference to Figure 4.4(b), repeat the argument at the end of Proposition 4.15 to establish the relation $f_{m,\tau} = f_{k,\tau} \cdot f_{l,\tau}$ as required. $\square$

Definition 4.19 (Reid's recipe, Case 3) The virtual bundle

$$\mathcal{V}_m := \big(\mathcal{F}_k \oplus \mathcal{F}_l\big) \ominus \big(\mathcal{F}_m \oplus \mathcal{O}_Y\big)$$

has trivial rank and, by Proposition 4.18, trivial first Chern class. We mark χ_m on the vertex of valency 5 or 6 in Σ to indicate that $\mathcal{V}_m$ is associated to the corresponding once or twice blown up scroll S_m.

Case 4: 3 straight lines through a vertex of valency 6. A vertex v of valency 6 defines a del Pezzo surface dP_6 of degree six.

Lemma 4.20 *The monomials defining the pair of morphisms* $\mathrm{dP}_6 \to \mathbb{P}^2$ *lie in uniquely determined character spaces* χ_l *and* χ_m *satisfying*

$$\chi_l \otimes \chi_m = \chi_i \otimes \chi_j \otimes \chi_k, \tag{4.2}$$

where χ_i, χ_j *and* χ_k *mark the straight lines through the vertex* v *defining the del Pezzo surface* dP_6.

Proof. A vertex v defines a del Pezzo surface only when three lines tesselating a regular triangle intersect. If v lies in a corner triangle then the three ratios listed in (3.8) which cut out the lines satisfy $i + j + k = r$ (the case where the lines are cut out by the ratios (3.9) is similar and is left as an exercise). These ratios determine a Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^7$ given by

$$\begin{aligned}&(x^{d-i}y^{e-j}z^{f-k} : x^{d-i+k}y^{e-j+c+k} : x^{d-i+a+j}z^{f-k+j} : x^{d-i+a+j+k}y^{c+k}z^j :\\ &y^{e-j+b+i}z^{f-k+i} : x^k y^{e-j+c+k+b+i}z^i : x^{a+j}y^{b+i}z^{f-k+i+j} : x^{a+j+k}y^{b+i+c+k}z^{i+j}).\end{aligned}$$

The del Pezzo $\mathrm{dP}_6 \subset \mathbb{P}^6$ is the intersection of the image of this map with the hyperplane $x_0 = x_7$, assuming that $x_0, \ldots, x_7$ are the coordinates on

$\mathbb{P}^7$. Moreover, the maps $\mathrm{dP}_6 \to \mathbb{P}^2$ are the restriction of the projections $(x_0 \colon x_2 \colon x_3)$ and $(x_0 \colon x_4 \colon x_5)$ to dP_6. After removing the common factors $x^{d-i}z^j$ and $y^{e-j}z^i$, and after simplifying the exponents using (3.6), these morphisms are

$$\left(y^{e-j}z^i \colon x^{a+j}z^{f-k} \colon x^{d-i}y^{c+k}\right) \quad \text{and} \quad \left(x^{d-i}z^j \colon y^{b+i}z^{f-k} \colon x^k y^{e-j}\right). \tag{4.3}$$

The required characters χ_l and χ_m are the common character spaces of the monomials defining these maps; i.e. the character spaces of the monomials $x^{a+j}z^{f-k}$ and $x^{d-i}z^j$. The product of these two monomials is equal to the product of x^{d-i}, z^{f-k} and $x^{a+j}z^j$, so the relation (4.2) holds as required. □

Proposition 4.21 *The relation* $\mathcal{F}_l \otimes \mathcal{F}_m = \mathcal{F}_i \otimes \mathcal{F}_j \otimes \mathcal{F}_k$ *holds for the characters* χ_i, χ_j, χ_k, χ_l *and* χ_m *of the previous lemma.*

Proof. The monomials listed in (4.3) generate $\mathcal{F}_l$ and $\mathcal{F}_m$ on the open sets U_τ defined by triangles adjacent to the vertex v, as shown in Figure 4.5.

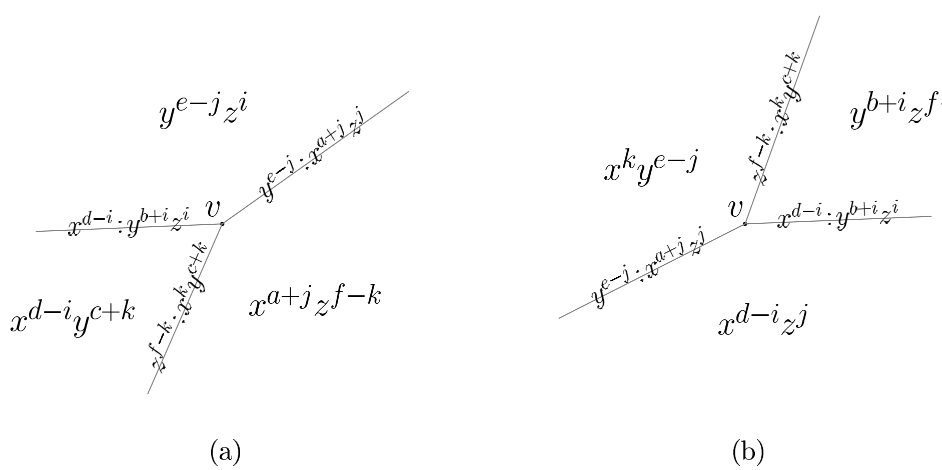


Figure 4.5: Local generators of (a) $\mathcal{F}_l$; (b) $\mathcal{F}_m$

Of course, the lines in Figure 4.5 should pass straight through v, leaving six triangles with v as a vertex. Three of these triangles lie to one side of the line cut out by $z^{f-k}:x^k y^{c+k}$ where $\mathcal{F}_k$ is generated by the monomial z^{f-k}. By extending the lines in Figure 4.5(a) through v, we see that $\mathcal{F}_l$ is generated by $y^{e-j}z^i$ on one of these triangles, and by $x^{a+j}z^{f-k}$ on the other two. If $U_\tau \subset \mathrm{dP}_6$ is an open set determined by one of these three triangles then

$$f_{l,\tau} = \left\{ \begin{array}{l} z^i \cdot y^{e-j} \\ z^i \cdot (x^{a+j}z^j) \end{array} \right\} = z^i \cdot f_{j,\tau}, \tag{4.4}$$

because $z^{f-k} = z^i \cdot z^j$ follows from (3.6). Also, $\mathcal{F}_m$ is generated by $y^{b+i}z^{f-k}$ on two of these triangles, and by $x^{d-i}z^j$ on the third. Hence

$$f_{m,\tau} = \left\{ \begin{array}{l} z^j \cdot (y^{b+i}z^i) \\ z^j \cdot x^{d-i} \end{array} \right\} = z^j \cdot f_{i,\tau}. \tag{4.5}$$

In particular, on these three subsets $U_\tau \subset \mathrm{dP}_6$ we have

$$f_{l,\tau} \cdot f_{m,\tau} = f_{i,\tau} \cdot f_{j,\tau} \cdot f_{k,\tau}. \tag{4.6}$$

We claim that (4.6) holds whenever $f_{k,\tau} = z^{f-k}$. The argument goes as before: by considering the positions of $y^{e-j}z^i$ and $x^{a+j}z^{f-k}$ in the Newton polyhedron of M_l it's clear that, on every open set U_τ for which $f_{k,\tau} = z^{f-k}$, the monomial $f_{l,\tau}$ must be divisible by z^i. Thus, for every such τ we have $f_{l,\tau}/z^i \in M_j$, hence (4.4) holds. The relation (4.5) holds similarly whenever $f_{k,\tau} = z^{f-k}$. This proves the claim. One can similarly show that

$$f_{l,\tau} = x^{a+j}f_{k,\tau} \quad \text{and} \quad f_{m,\tau} = x^k \cdot f_{j,\tau} \quad \text{whenever} \quad f_{i,\tau} = x^{d-i},$$

and that

$$f_{l,\tau} = y^{c+k} \cdot f_{i,\tau} \quad \text{and} \quad f_{m,\tau} = y^{b+i}f_{k,\tau} \quad \text{whenever} \quad f_{j,\tau} = y^{e-j}.$$

Finally, since the lines cut out by the ratios (3.8) intersect at v, it follows that for every open set U_τ in the cover of Y we have either $f_{i,\tau} = x^{d-i}$, $f_{j,\tau} = y^{e-j}$ or $f_{k,\tau} = z^{f-k}$. As a result, (4.6) holds for every U_τ. This proves the proposition, under the assumption that the vertex v lies inside a corner triangle whose tesselating lines are cut out by the ratios (3.8). We leave as an exercise the case where the vertex v lies inside the meeting of champions triangle whose tesselating lines are cut out by the ratios (3.9). □

Definition 4.22 (Reid's recipe, Case 4) The virtual bundle

$$\mathcal{V}_m := \big(\mathcal{F}_i \oplus \mathcal{F}_j \oplus \mathcal{F}_k\big) \ominus \big(\mathcal{F}_l \oplus \mathcal{F}_m \oplus \mathcal{O}_Y\big), \tag{4.7}$$

has trivial rank and, by Proposition 4.21, trivial first Chern class. We mark both χ_l and χ_m on the vertex of valency 6 in Σ to indicate that $\mathcal{V}_m$ and $\mathcal{F}_l$ are associated to the del Pezzo surface.

Remark 4.23 There are three maps $\mathrm{dP}_6 \to \mathbb{P}^1$ given by restriction of the bundles $\mathcal{F}_i$, $\mathcal{F}_j$, $\mathcal{F}_k$, and two maps $\mathrm{dP}_6 \to \mathbb{P}^2$ given by restriction of $\mathcal{F}_l$ and $\mathcal{F}_m$. The maps to $\mathbb{P}^1$ determine the embedding $\mathrm{dP}_6 \hookrightarrow \mathbb{P}^7$ of Lemma 4.20, but they do not generate the Picard group of dP_6. All five maps span $\mathrm{Pic}(\mathrm{dP}_6)$, and the relation of Proposition 4.21 holds. We break the symmetry in this relation by choosing the restriction of the bundles $\mathcal{F}_i$, $\mathcal{F}_j$, $\mathcal{F}_k$, $\mathcal{F}_l$ as a basis for $\mathrm{Pic}(\mathrm{dP}_6)$. We discard $\mathcal{F}_m$ and instead label the virtual bundle of Definition 4.22 with the character χ_m. However, we could equally well choose the restriction of $\mathcal{F}_m$ as the fourth basis element in which case the virtual bundle would be denoted $\mathcal{V}_l$. See Example 4.25 below.

4.3.2 Illustrating Reid's recipe

Example 4.24 The triangulation Σ for $\frac{1}{11}(1,2,8)$ is shown in Figure 4.6.

The lines meeting at the vertex of valency 3 are marked with $\chi_2 = \varepsilon^2$ (see Definition 4.9 and Figure 3.7). According to Definition 4.13 we mark the vertex of valency 3 with χ_4. The characters χ_2 and χ_8 mark lines passing through the vertex of valency 4 so, by Definition 4.16, we mark the vertex with χ_{10}. The remaining vertices have valency 5, so Definition 4.19 applies.

Example 4.25 Let $\mathbb{C}^3/A$ be the singularity of type $\frac{1}{30}(25,2,3)$. The fan Σ of A-Hilb $\mathbb{C}^3$ for this A-action is shown in Figure 4.7. There are three regular triangles of side 2 to the left of the line from e_1 to p, and two regular triangles of side 3 to the right.

Every internal vertex has valency 5 or 6. Most of the vertices are marked with a single character determined by Definition 4.19. However, inside each regular triangle of side 3 is a vertex of valency 6 which defines a del Pezzo surface dP_6. The characters χ_4, χ_5 and χ_{12} mark the lines passing through one of the vertices. The proof of Lemma 4.20 reveals that the monomials

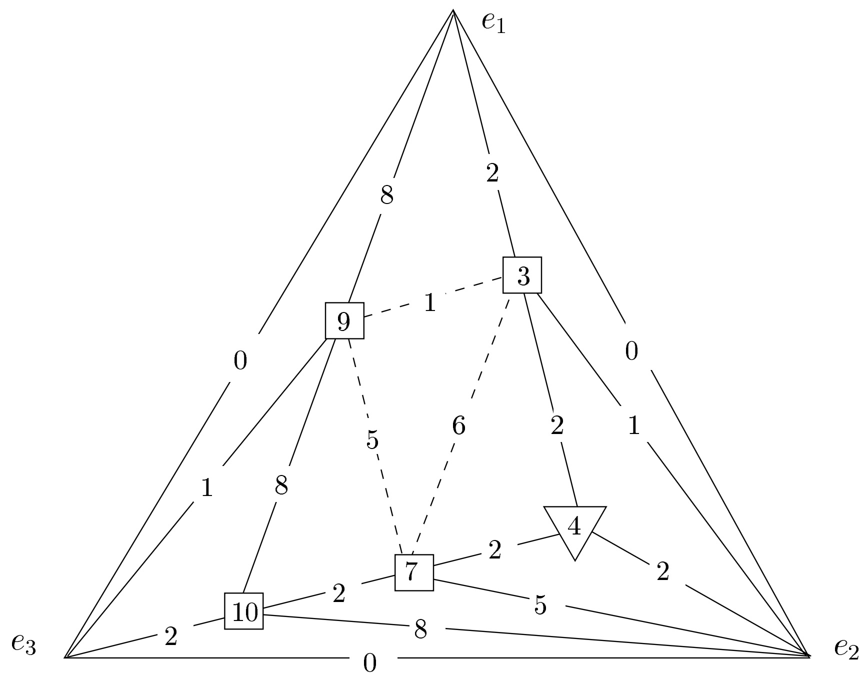


Figure 4.6: Reid's recipe for $\frac{1}{11}(1,2,8)$

defining the morphisms $\mathrm{dP}_6 \to \mathbb{P}^2$ are global sections of the bundles $\mathcal{F}_7$ and $\mathcal{F}_{14}$. According to Definition 4.22 we mark χ_7 and χ_{14} on the vertex of valency 6 to indicate that the bundles $\mathcal{V}_7$ and $\mathcal{F}_{14}$ (or $\mathcal{V}_{14}$ and $\mathcal{F}_7$, see Remark 4.23) are associated to the del Pezzo surface.

4.3.3 The basis of $H^4(Y,\mathbb{Z})$

We now prove the second part of Theorem 4.2, namely that the classes $c_2(\mathcal{V}_m)$ form a basis of $H^4(Y,\mathbb{Z})$ dual to the basis $[S] \in H_4(Y,\mathbb{Z})$ of compact exceptional surfaces S of the resolution $\varphi\colon Y \to X$.

Proof of Theorem 4.2 The $\mathbb{C}^*$-action $(x,y,z) \to (\lambda x, \lambda y, \lambda z)$ defines a retraction of Y onto the compactly supported exceptional locus of φ, so the homology classes of the compact exceptional surfaces form an integral basis of $H_4(Y,\mathbb{Z})$. To prove Theorem 4.2 we must show that

$$\int_{S_n} c_2(\mathcal{V}_m) = \delta_{mn}, \tag{4.8}$$

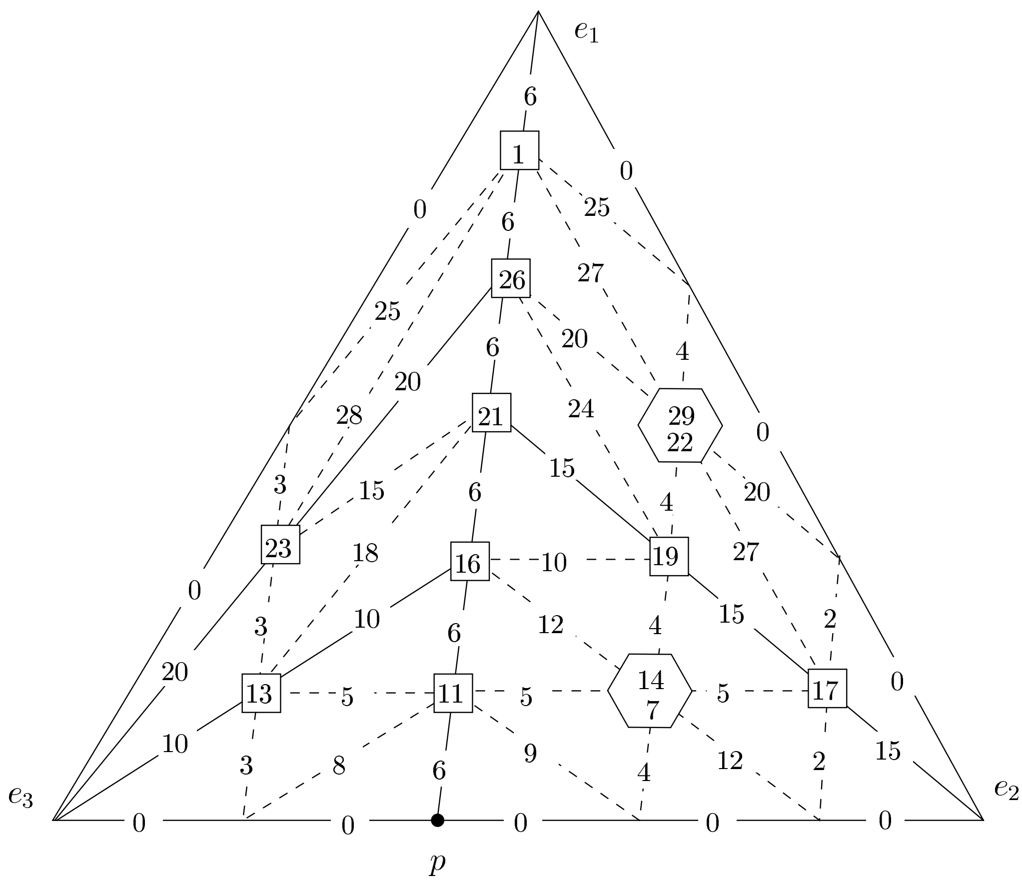


Figure 4.7: Reid's recipe for $\frac{1}{30}(25,2,3)$

where S_n is the exceptional surface corresponding to the vertex $v := v_n$ in Σ marked with the character χ_n, as described in §4.3.1. There are four cases:

CASE 1: A VERTEX OF VALENCY 3.

In the notation of §4.3.1 CASE 1, the bundle $\mathcal{F}_l$ has degree one on the curves in Y determined by lines in Figure 4.2(a), and degree zero on every other exceptional curve. Now, χ_m marks the vertex v_m of valency 3 corresponding to $S_m = \mathbb{P}^2$ and $\mathcal{F}_l|_{S_m} = \mathcal{O}_{\mathbb{P}^2}(1)$, so

$$\int_{S_m} c_2(\mathcal{V}_m) = \int_{S_m} c_1\big(\mathcal{F}_l|_{S_m}\big)^2 = \mathcal{O}_{\mathbb{P}^2}(1)^2 = 1,$$

as required. Next, consider a vertex $v_n \neq v_m$ in Σ. If v_n lies on a line from v_m to some e_j in Figure 4.2(a) then S_n is a (possibly once or twice blown up) scroll $\mathbb{F}_r$. The bundle $\mathcal{F}_l$ has degree one on the classes[1] M and D, and degree zero on F (and on each -1-curve E_i). It follows that $\mathcal{F}_l|_{S_n} = \mathcal{O}_{S_n}(F)$, so $c_1(\mathcal{F}_l|_{S_n})^2 = F^2 = 0$. Otherwise v_n lies inside one of the triangles pictured in Figure 4.2(a), in which case $\mathcal{F}_l|_{S_n} = \mathcal{O}_{S_n}$, so $c_1(\mathcal{F}_l|_{S_n})^2 = 0$. Hence

$$\int_{S_n} c_2(\mathcal{V}_m) = \int_{S_n} c_1\big(\mathcal{F}_l|_{S_n}\big)^2 = 0$$

for any $v_n \neq v_m$. This establishes relation (4.8) for Case 1.

Case 2: A vertex of valency 4.

In the notation of §4.3.1 Case 2, recall that χ_k marks the straight line through the vertex v_m of valency 4 so $\mathcal{F}_k$ has degree one on the classes M and D on the surface $S_m = \mathbb{F}_r$ corresponding to v_m. It follows that $\mathcal{F}_k|_{S_m} = \mathcal{O}_{S_m}(F)$. Also, $\mathcal{F}_l$ has degree one on F because χ_l marks the other two lines meeting at v_m, so $\mathcal{F}_l|_{S_m} = \mathcal{O}_{S_m}(M + c\cdot F)$, for some $c \in \mathbb{Z}$. Thus

$$\int_{S_m} c_2(\mathcal{V}_m) = \int_{S_m} c_1\big(\mathcal{F}_k|_{S_m}\big)\cdot c_1\big(\mathcal{F}_l|_{S_m}\big) = F\cdot(M + cF) = 1.$$

We now prove that the integral over every other surface is zero. From Figure 4.2(b) (and Figure 4.3(a)) we see that $\mathcal{F}_k$ (and $\mathcal{F}_l$) has degree one (respectively degree $d \geq 1$) on the line v_m to e_3, and degree zero (respectively degree 1) on the lines v_m to e_1 and v_m to e_2. If $v_n \neq v_m$ lies on the line v_m to e_3 then S_n is a scroll $\mathbb{F}_r$ (possibly blown up in one or two points) and, as above, it follows that $\mathcal{F}_k|_{S_n} = \mathcal{O}_{S_n}(F)$ and, similarly, $\mathcal{F}_l|_{S_n} = \mathcal{O}_{S_n}(dF)$, for some $d \in \mathbb{Z}$. Thus

$$\int_{S_n} c_2(\mathcal{V}_m) = \int_{S_n} c_1\big(\mathcal{F}_k|_{S_n}\big)\cdot c_1\big(\mathcal{F}_l|_{S_n}\big) = F\cdot(dF) = 0.$$

If $v_n \neq v_m$ lies on the line v_m to e_1 or v_m to e_2 then $\mathcal{F}_k|_{S_n} = \mathcal{O}_{S_n}$ so the Chern class calculation is trivially zero. Similarly, if $v_n \neq v_m$ does not lie on a line from v_m to some e_j then $\mathcal{F}_l|_{S_n} = \mathcal{O}_{S_n}$ and the calculation is trivially zero. Hence the relation (4.8) holds for Case 2.

[1] We adopt the following notation: let F, M and D denote the classes on a surface scroll $\mathbb{F}_r$ with selfintersection 0, r and $-r$ respectively; we use the same notation for the strict transforms of these classes in a once or twice blown up scroll.

CASE 3: A VERTEX OF VALENCY 5 OR 6, BUT NOT 3 STRAIGHT LINES. Similar to CASE 2, so we leave it as an exercise.

CASE 4: 3 STRAIGHT LINES THROUGH A VERTEX OF VALENCY 6. In the notation of §4.3.1 CASE 4, write v_m for the vertex marked with χ_l and χ_m defining a surface $S_m := \mathrm{dP}_6$. The divisor class group is

$$\mathrm{Div}(S_m) = \left\langle d_1, d_2, d_3, c_1, c_2 \;\middle|\; d_1 + d_2 + d_3 = c_1 + c_2 \right\rangle,$$

where $\mathcal{O}_{S_m}(c_\alpha)$ and $\mathcal{O}_{S_m}(d_\beta)$ define morphisms from S_m to $\mathbb{P}^2$ and $\mathbb{P}^1$ respectively. The characters χ_i, χ_j and χ_k mark the straight lines passing through v cut out by the ratios (3.8) or (3.9), and it follows that $\mathcal{F}_i|_{S_m} = \mathcal{O}_{S_m}(d_1)$, $\mathcal{F}_j|_{S_m} = \mathcal{O}_{S_m}(d_2)$ and $\mathcal{F}_k|_{S_m} = \mathcal{O}_{S_m}(d_3)$. Thus

$$\int_{S_m} c_2(\mathcal{F}_i \oplus \mathcal{F}_j \oplus \mathcal{F}_k) = \sum_{\alpha<\beta} d_\alpha \cdot d_\beta = 3,$$

Also, from the construction of the characters χ_l and χ_m in Lemma 4.20, we have $\mathcal{F}_l|_{S_m} = \mathcal{O}_{S_m}(c_1)$ and $\mathcal{F}_m|_{S_m} = \mathcal{O}_{S_m}(c_2)$, so

$$\int_{S_m} c_2\big(\mathcal{F}_l \oplus \mathcal{F}_m\big) = \int_{S_m} c_1(\mathcal{F}_l) \cdot c_1(\mathcal{F}_m) = c_1 \cdot c_2 = 2.$$

By computing the difference of these two integrals we see that, for the bundle $\mathcal{V}_m$ constructed in Definition 4.22, the relation (4.8) holds when $m = n$.

Next, fix $v_n \neq v_m$. Every basic triangle in Σ with v_n as a vertex defines an open subset $U_\tau \subset S_n$. The lines cut out by the ratios (3.8) intersect at $v_m \neq v_n$, so every $U_\tau \subset S_n$ satisfies either $f_{i,\tau} = x^{d-i}$, $f_{j,\tau} = y^{e-j}$ or $f_{k,\tau} = z^{f-k}$. Assume without loss of generality that $f_{k,\tau} = z^{f-k}$ on every open set $U_\tau \subset S_n$. Then $\mathcal{F}_k|_{S_n} = \mathcal{O}_{S_n}$, so $c_1(\mathcal{F}_k|_{S_n}) = 0$. It follows from (4.4) and (4.5) that $c_1(\mathcal{F}_l|_{S_n}) = c_1(\mathcal{F}_j|_{S_n})$ and $c_1(\mathcal{F}_m|_{S_n}) = c_1(\mathcal{F}_i|_{S_n})$. As a result

$$\begin{aligned}\int_{S_n} c_2(\mathcal{F}_i \oplus \mathcal{F}_j \oplus \mathcal{F}_k) &= c_1(\mathcal{F}_i|_{S_n}) \cdot c_1(\mathcal{F}_j|_{S_n}) \\ &= c_1(\mathcal{F}_l|_{S_n}) \cdot c_1(\mathcal{F}_m|_{S_n}) = \int_{S_n} c_2(\mathcal{F}_l \oplus \mathcal{F}_m),\end{aligned}$$

so (4.8) holds for $n \neq m$. This proves CASE 4.

This completes the proof of Theorem 4.2. □

4.4 Every character appears once on Σ

Reid's recipe calculates the character of A marking each vertex in Σ in terms of the characters marking the lines meeting at the vertex. It is not clear a priori from this construction that different vertices are marked with different characters. Nevertheless, this is the case in the worked examples of §4.3.2. In fact a much stronger statement holds: every character of A marks either lines in Σ according to Definition 4.9, or a vertex according to Definitions 4.13, 4.16, 4.19 or 4.22. In this section we prove that this statement is true for every finite Abelian subgroup $A \subset \mathrm{SL}(3, \mathbb{C})$.

4.4.1 A coarse subdivision of the fan Σ

There is a significant dichotomy in the calculation of A-Hilb $\mathbb{C}^3$: the fan Σ has either a meeting of champions or a unique 'long side' (see Proposition 3.14). If a meeting of champions exists we use the champion lines to subdivide Σ into four regions (see Figure 4.8(a)), three if the champion has side zero and one if the meeting of champions is the whole of Δ. Otherwise, permuting x, y, z if necessary, we subdivide Σ along a line from e_1 which cuts the long side e_2e_3 (see Figure 4.8(b)). There may be more than one line from e_1 which cuts the long side so this subdivision is not canonical.

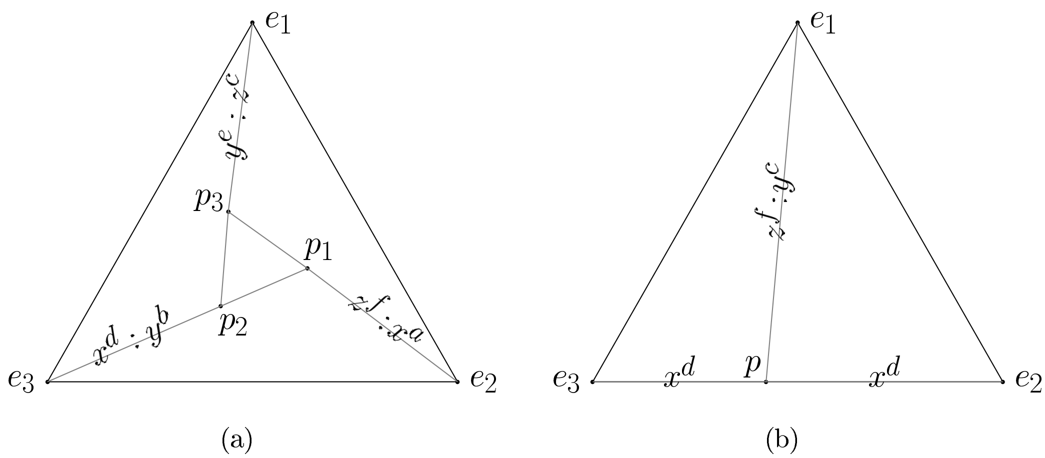


Figure 4.8: Coarse subdivision: (a) meeting of champions; (b) long side

In each case we produce a coarse subdivision of Σ into at most four regions which are themselves unions of regular triangles. Each region, apart from the interior triangle in Figure 4.8(a), is a triangle with vertices e_i, p_j, e_k; in Figure 4.8(b), the point $p = p_j$ lies on the edge e_2e_3 cut out by x^d.

4.4.2 Representing characters by monomials

In Examples 4.24 and 4.25 the integer i denotes the character $\chi_i = \varepsilon^i$, for some $\varepsilon^r = 1$. When tackling the general case we identify characters of A with monomials in the eigenspace of that character. There is of course no canonical monomial for each character. However we now show that the characters which mark the points and lines in any outer region (i.e. not the interior triangle in Figure 4.8(a)) prefer a single monomial above all other choices.

Proposition 4.26 *The characters which mark the points and lines which lie in the region $e_1p_2e_3$ of Figure 4.8(a) can be represented by the monomials*

$$x^i z^j \quad \textit{for} \quad i = 0, \dots, d\,; j = 0, \dots, f. \tag{4.9}$$

By permuting x, y, z if necessary, this proposition computes the characters marking any of the outer regions $e_ip_je_k$ in Figure 4.8(a) or 4.8(b). The proof of the proposition follows from the next lemma:

Lemma 4.27 *The characters which mark the regular triangles of side r in Figure 3.6 can be represented by the monomials*

$$\begin{array}{lllll} z^{f-k} & \textit{and} \quad x^{d-i}z^{f-k} & \textit{for} \quad i,k = 0,\dots,r & \textit{in Case a}, & (4.10) \\ x^i z^{f-k} & \textit{and} \quad x^{d-i}z^{c+k} & \textit{for} \quad 0 \le i+k \le r & \textit{in Case b}. & (4.11) \end{array}$$

Proof of Proposition 4.26, assuming the lemma. Starting from the edge e_1e_3, run an MMP (see Definition 3.9) which eats all regular triangles inside the region $e_1p_2e_3$ of Figure 4.8(a). We prove the proposition by induction on the number of contractions in the MMP. If the MMP consists of a single contraction then the region can be viewed as a regular corner triangle from e_3 (see Figure 3.6(a)). The ratio $x^a : y^e$ which cuts out the edge e_1e_3 is simply y^e, hence $a = 0$. Since $d - a = f = r$ holds by (3.6), substitute $d = f = r$ into the list (4.10) of characters which mark a corner triangle to see that the proposition holds in this case.

Suppose now that we have performed an MMP which has eaten all regular triangles in a region with vertices e_1qe_3 where the lines e_1q and e_3q are cut

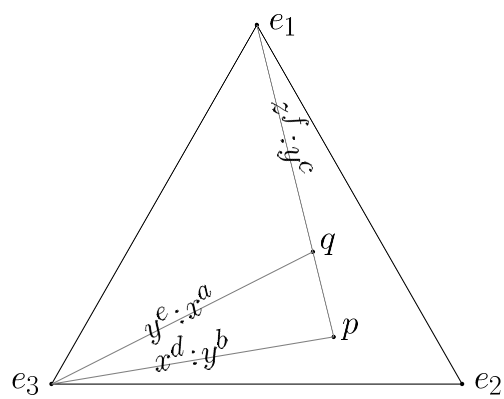


Figure 4.9: Inductive step in the proof of Proposition 4.26

out by the ratios $z^f : y^c$ and $x^a : y^e$ respectively (see Figure 4.9). We assume by induction that the characters which mark the union of regular triangles inside this region are

$$x^i z^j \quad \text{for} \quad i = 0, \dots, a; j = 0, \dots, f. \tag{4.12}$$

If the next contraction of the MMP eats a corner triangle from e_3, then the line $e_1 q$ extends to a lattice point p, and the line $e_3 p$ has ratio $x^d : y^b$ say, as shown in Figure 4.9. The characters which mark the new corner triangle are listed in (4.10). The region $e_1 p_2 e_3$ of Figure 4.8(a) is therefore marked with the union of characters (4.10) and (4.12); namely $x^i z^j$ for $i = 0, \dots d$; $j = 0, \dots f$ as required. The case where the final triangle is from e_1 is similar. $\Box$

Proof of Lemma 4.27. For Case a, the triangle is eaten by an MMP from the side $e_1 e_3$ so we choose to represent the characters marking this triangle by monomials in x, z. From (3.8), the characters which mark the tesselating lines of the triangle are

$$x^{d-i}; \quad x^{a+j} z^j; \quad z^{f-k} \quad \text{for} \quad i, j, k = 0, \dots r-1. \tag{4.13}$$

The vertices along the edges of the triangle are marked with the characters

$$x^a z^{f-k}; \quad x^d z^{f-k}; \quad x^{d-i} z^f \quad \text{for} \quad i, k = 0, \dots r-1. \tag{4.14}$$

Indeed, the edges emanating from e_3 are marked with x^a and x^d. The tesselating lines z^{f-k} cross them so, as explained in Definitions 4.13, 4.16 and

4.19, the product of these characters marks the vertices along these edges. Similarly, the tesselating lines x^{d-i} cross the edge z^f from e_1, so the characters $x^{d-i}z^f$ mark the vertices along this edge. Finally, from the proof of Lemma 4.20, we know that the characters

$$x^{a+j}z^{f-k}; \quad x^{d-i}z^j \quad \text{for} \quad i,j,k = 1,\dots r-1 \text{ such that } i+j+k=r \tag{4.15}$$

mark the internal vertices in the tesselation of the regular triangle of Figure 3.6. As a result, the union of the characters listed in (4.13), (4.14) and (4.15) mark the regular triangle of Figure 3.6(a). It is an easy combinatorial exercise to see that the union of these characters is equal to the list (4.10) as claimed. This proves Case a of Lemma 4.27.

To prove Case b, one proves similarly that the characters x^{d-i}, x^jz^{c+j} and z^{f-k} for $i,j,k = 0,\dots,r-1$ mark the lines of the regular tesselation; the characters x^dz^{f-k}, $x^{a+j}z^{c+j}$ and $x^{d-i}z^c$ for $i,j,k = 0,\dots,r-1$ mark the vertices along the edges of the triangle; and the characters x^jz^{f-k} and $x^{d-i}z^{c+j}$ for $i,j,k = 1,\dots r-1$ such that $i+j+k=r$ mark the internal vertices of the triangle. As with Case a, the union of these characters is equal to list (4.11) as claimed. □

Remark 4.28 There is symmetry in Lemma 4.27 Case b. We list the characters in terms of x,z, but equally we can write them in x,y or y,z using the relations (3.9). This doesn't alter the character, because the ratios in (3.9) are A-invariant. In short, the characters marking strata in the interior triangle in Figure 4.8(a) do not prefer a single monomial over all others.

4.4.3 Plotting characters on the McKay quiver

The condition $A \subset \mathrm{SL}(3,\mathbb{C})$ ensures that the monomial xyz is A-invariant so corresponds to the trivial character. Thus, characters of A correspond to Laurent monomials modulo xyz. We represent this as a tesselation of the plane by regular hexagons, part of which is illustrated in Figure 4.10.

Definition 4.29 The (universal cover of the) *McKay quiver* is the tesselation of the plane by regular hexagons, where the arrows are 'multiply by x,y or z'. Some power of each monomial x,y and z is A-invariant so the tesselation is periodic, and we say that any connected region in the quiver in one-to-one correspondence with the characters of A is a *fundamental domain*.

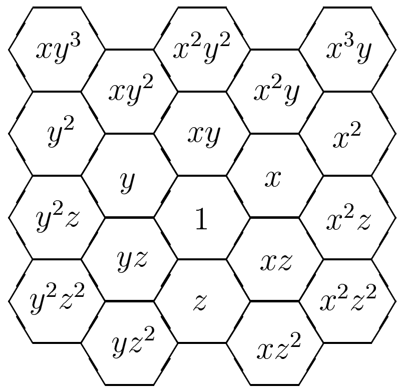


Figure 4.10: The McKay quiver as the lattice of monomials modulo xyz

Proposition 4.30 *The characters marking the points and lines in Σ form a fundamental region in the McKay quiver (assuming that a character marking two lines meeting at a vertex is recorded only once on the quiver).*

PROOF. The coarse subdivision of Σ is one of the two types shown in Figure 4.8. Beginning with Case a, we plot the characters which mark each region on the McKay quiver. The characters marking the three outer regions in the subdivision form parallelograms, by Proposition 4.26, and the characters marking the meeting of champions form a pair of triangles, by Lemma 4.27, Case b. The parallelograms and triangles intersect along characters $x^iy^e = x^iz^c$, $y^jz^f = y^jx^a$ and $x^dz^k = y^bz^k$ which mark the vertices on the champion lines as shown in Figure 4.11

The union of these regions is slightly larger than a fundamental domain in the quiver. However, the characters x^i, y^j, z^k around the edge of the shape in Figure 4.11 mark tesselating lines in different regions and have been plotted more than once. In each case, the lines marked with the same monomial meet at a scroll or a blown up scroll and pass from one region to another. From Lemmas 4.14 and 4.17 the monomials lie in the character space χ_i marking the pair of lines cut out by the divisor class F on the underlying scroll. Thus, the monomials x^i, y^j, z^k around the edge of the shape determine pairwise the same tautological bundle $\mathcal{F}_i$ whose character χ_i marks a pair of lines meeting at a (possibly blown up) surface scroll. As a result, we identify in pairs (the trivial character 1 appears three times) the monomials around the outside of the shape of Figure 4.11, leaving a fundamental domain.

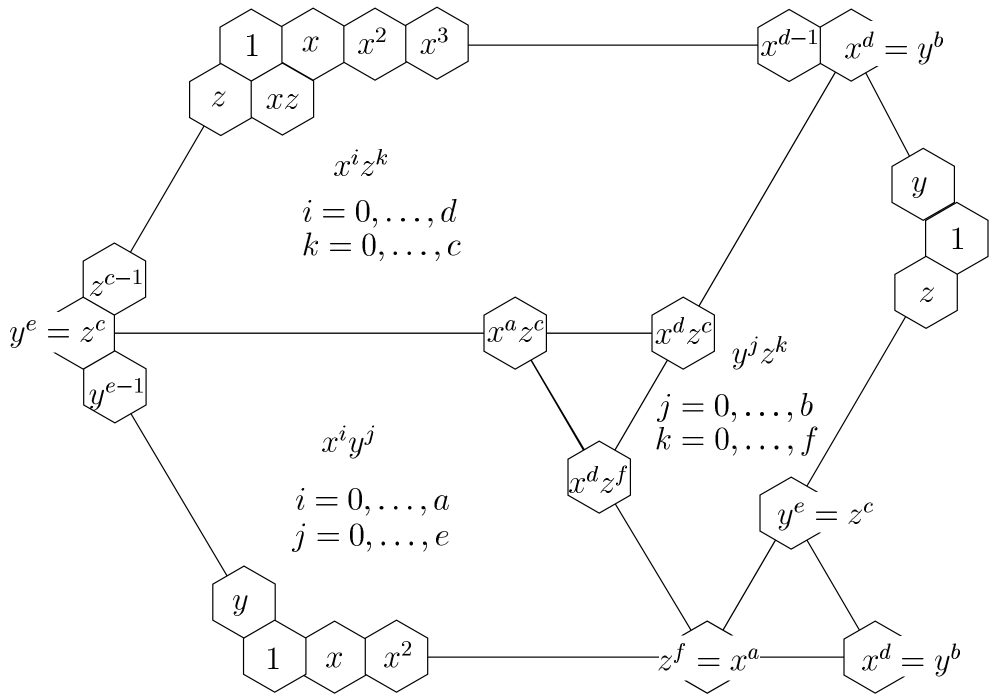


Figure 4.11: Three parallelograms and two triangles in the McKay quiver

Otherwise, the subdivision is Case b, as shown in Figure 4.8(b). The characters which mark the two regions are

$$x^i z^k \quad \text{and} \quad x^i y^j \quad \text{for} \quad i = 0, \dots d\,; j = 0, \dots c\,; k = 0, \dots, f. \tag{4.16}$$

These parallelograms intersect along the characters $x^i y^c = x^i z^f$ marking the vertices on the line of intersection of the regions. Since x^d is A-invariant, we identify the characters y^j and $x^d y^j$ pairwise, and similarly z^k and $x^d z^k$. Finally, as with Case a, we identify the two collections $1, x, \dots, x^d$ around the edge of the figure. □

Remark 4.31 We observed in §4.4.1 that the coarse subdivision of Σ is not canonical in Case b. Different subdivisions vary by a corner triangle T of side r from e_1 whose sides extend from e_1 to the long side. If the long side is cut out by the monomial x^f and the other sides of T are cut out by $y^a : z^e$ and $y^d : z^b$ say, where the relations (3.8) hold, then the characters which mark T

are $y^{d-i}x^{f-k}$ for $i, k = 0, \ldots, r$ by Lemma 4.27. This is a single parallelogram with sides of length r, and the pair of parallelograms which we plot on the McKay quiver would be translated. In particular, the overall result would be unchanged by choosing an alternative coarse subdivision.

Corollary 4.32 *Every character of A appears once on Σ and is either*

(i) *the trivial character χ_0 marking the edges of Δ; or*

(ii) *a character χ_i marking a line (possibly passing through several vertices) inside Σ; or*

(iii) *a character χ_m marking a vertex in Σ; or*

(iv) *the second character χ_l marking a vertex defining a del Pezzo surface.*

Example 4.33 (Subdivision with 3 regions) The coarse subdivision of Σ in Example 4.24 has 3 regions. Let p denote the vertex of valency 3 in Σ, i.e., the vertex marked with χ_4 in Figure 4.6.

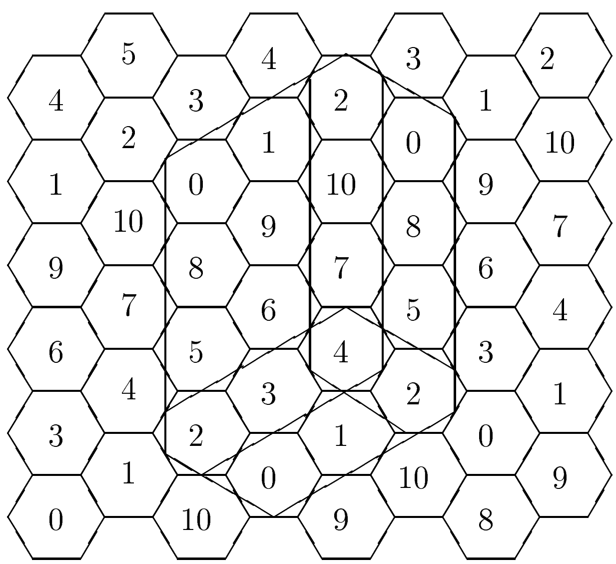


Figure 4.12: Part of the McKay quiver of $\frac{1}{11}(1, 2, 8)$

Each of the 3 regions in Σ is marked with characters which form a parallelogram when plotted on the McKay quiver. Indeed, the sides of the region

e_1pe_3 are cut out by $x^2 : y$ and $y : z^3$, so the characters

$$\left.\begin{matrix} 0 & 1 & 2 \\ 8 & 9 & 10 \\ 5 & 6 & 7 \\ 2 & 3 & 4 \end{matrix}\right\} \quad \text{corresponding to} \quad \left\{\begin{matrix} 1 & x & x^2 \\ z & xz & x^2z \\ z^2 & xz^2 & x^2z^2 \\ z^3 & xz^3 & x^2z^3 \end{matrix}\right.$$

mark this region. These characters form a parallelogram in the quiver, as shown in Figure 4.12. Similarly, the sides of the region e_1pe_2 are cut out by $y : z^3$ and $x^2 : z^3$, so this region is marked with the characters

$$\left.\begin{matrix} 2 & 3 & 4 \\ 0 & 1 & 2 \end{matrix}\right\} \quad \text{corresponding to} \quad \left\{\begin{matrix} y & xy & x^2y \\ 1 & x & x^2 \end{matrix}\right. .$$

Finally, the sides of the region e_2pe_3 are cut out by $x^2 : y$ and $x^2 : z^3$, so the characters which mark this region are

$$\left.\begin{matrix} 2 & 10 & 7 & 4 \\ 0 & 8 & 5 & 2 \end{matrix}\right\} \quad \text{corresponding to} \quad \left\{\begin{matrix} y & yz & yz^2 & yz^3 \\ 1 & z & z^2 & z^3 \end{matrix}\right. .$$

These three parallelograms are drawn on the McKay quiver of $\frac{1}{11}(1,2,8)$ in Figure 4.12, glued along the characters $2,3,4,7,10$ which mark the champion lines and the vertices on the champion lines. As in the proof of the Proposition 4.30, the characters around the edge of the shape in Figure 4.12 are identified, leaving exactly eleven hexagons marked with the characters $0,1,\dots,10$; that is, a fundamental domain for $A = \mathbb{Z}/11$.

Example 4.34 (Subdivision with 2 regions) The coarse subdivision of Σ in Example 4.25 has 2 regions. The line from e_1 to the lattice point p in Figure 4.7 which subdivides Σ is cut out by $y^3 : z^2$ and the edge e_2e_3 is cut out by $x^6 : 1$. The region e_1pe_2 is marked with the characters x^iy^j for $i = 0,\dots 6; j = 0,\dots,3$ as shown in Figure 4.13.

The region e_1pe_3 is marked with characters x^iz^k for $i = 0,\dots 6\,; k = 0,\dots 2$. The pair of parallelograms in the McKay quiver are shown in Figure 4.13, glued along the characters $6,11,16,21,26,1$ which mark the dividing line from e_1 and the vertices on this line. As before, the characters around the edge of the shape in Figure 4.13 are identified, leaving hexagons marked with the characters $0,1,\dots,29$; a fundamental domain for $A = \mathbb{Z}/30$.

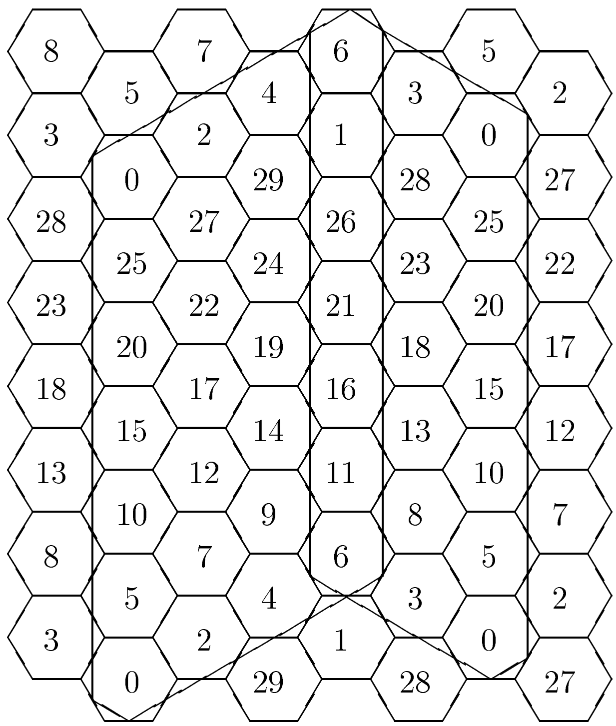


Figure 4.13: Part of the McKay quiver for $\frac{1}{30}(2,3,25)$

4.5 A basis for the cohomology of A-Hilb $\mathbb{C}^3$

The basis of $H^4(Y,\mathbb{Z})$ introduced in §4.3 is determined by the characters χ_m marking the internal vertices in Σ, denoted type (iii) in Corollary 4.32. We now prove Theorem 4.4 by showing that the remaining nontrivial characters, namely those denoted type (ii) and (iv) in Corollary 4.32, determine classes $c_1(\mathcal{F}_i)$ and $c_1(\mathcal{F}_l)$ which form an integral basis of $H^2(Y,\mathbb{Z})$.

Proof of Theorem 4.4 For a finite Abelian subgroup $A \subset \mathrm{SL}(3,\mathbb{C})$, Ito and Nakajima [IN00] established that the tautological bundles $\mathcal{F}_i$ span the K-theory of Y. In particular, they span $\mathrm{Pic}(Y)$ so the first Chern classes of the nontrivial tautological bundles $\mathcal{F}_i$ span $H^2(Y,\mathbb{Z})$, though in general they do not form a $\mathbb{Z}$-basis.

For every bundle $\mathcal{V}_m$ constructed in §4.3.1, remove the corresponding class $c_1(\mathcal{F}_m)$ from the set spanning $H^2(Y,\mathbb{Z})$. It follows from Propositions 4.12, 4.15, 4.18 and 4.21 that the first Chern classes of the remaining bundles still form a spanning set for $H^2(Y,\mathbb{Z})$. To prove that these classes form an integral basis, recall from the generalised McKay correspondence that the Euler number $e(Y)$ equals the order of A (see Theorem 1.24(ii)). Write $b_i(Y)$ for the i^{th} Betti number of Y, and write # as shorthand for the number of bundles of the appropriate type. Then

$$\begin{aligned}
\#\{\mathcal{F}_i\} + \#\{\mathcal{F}_l\} &= |A| - \#\{\mathcal{V}_m\} - 1 && \text{by Corollary 4.32}\\
&= e(Y) - \#\{\mathcal{V}_m\} - 1 \\
&= e(Y) - b_4(Y) - b_0(Y) && \text{by Theorem 4.2}\\
&= b_2(Y).
\end{aligned}$$

Thus the first Chern classes of the bundles $\mathcal{F}_i$ and $\mathcal{F}_l$, corresponding to the characters denoted type (ii) and (iv) in Corollary 4.32, form a basis of $H^2(Y,\mathbb{Z})$. This completes the proof of Theorem 4.4. □

Remark 4.35 An alternative approach to proving Theorem 4.4 would be to prove that

$$\int_{C_j} c_1(\mathcal{F}_i) = \delta_{ij}, \tag{4.17}$$

for the bundles denoted type (ii) and (iv) in Corollary 4.32, and for curves $C \subset Y$ indexed by the corresponding characters χ_i and χ_l whose classes $[C] \in H_2(Y,\mathbb{Z})$ form a basis. Such a basis of $H_2(Y,\mathbb{Z})$ exists, but in general the intersection matrix relating $c_1(\mathcal{F}_i)$ and C_j is not the identity, so the relation (4.17) does not hold; to see this, compute the degree of $\mathcal{F}_5$ and $\mathcal{F}_{18}$ on the curves marked with χ_3 in the example $\frac{1}{30}(25,2,3)$ illustrated in Figure 4.7.

The trivial character determines the trivial tautological bundle $\mathcal{F}_0 = \mathcal{O}_Y$ which generates $H^0(Y,\mathbb{Z})$. This leads immediately to our main result:

Corollary 4.36 *The McKay correspondence bijection*

$$\Big\{\textit{irreducible representations of } A\Big\} \longleftrightarrow \textit{basis of } H^*(Y,\mathbb{Z})$$

holds for all finite Abelian subgroups $A \subset \mathrm{SL}(3,\mathbb{C})$.

Warning 4.37 It follows from Remark 4.23 that if χ_l and χ_m mark the same vertex then there is a choice as to whether $c_1(\mathcal{F}_l)$ or $c_1(\mathcal{F}_m)$ is a basis element of $H^2(Y, \mathbb{Z})$, and as to whether we label the virtual bundle (4.7) as $\mathcal{V}_m$ or $\mathcal{V}_l$. In particular, when there is a del Pezzo surface $\mathrm{dP}_6 \subset Y$, there is no canonical answer to the question 'Which characters of A correspond to elements of $H^4(Y, \mathbb{Z})$ and which to $H^2(Y, \mathbb{Z})$?'.

Remark 4.38 Reid's recipe calculates a single relation between tautological bundles for each compact exceptional surface of the resolution $\varphi\colon Y \to X$. However the map $\chi_i \to \mathcal{F}_i$ is not multiplicative in general. Consider for example Reid's recipe applied to $Y = A$-Hilb $\mathbb{C}^3$ for the A-action $\frac{1}{11}(1,2,8)$ illustrated in Figure 4.6. The relation $\chi_6 = \chi_1 \otimes \chi_5$ holds, yet on the open set $U = \mathrm{Spec}\ \mathbb{C}[x/y^6, y^{11}, z/y^4] \subset A$-Hilb $\mathbb{C}^3$ we have

$$\mathcal{F}_1|_U = \langle y^6 \rangle, \quad \mathcal{F}_5|_U = \langle y^8 \rangle \quad \text{and} \quad \mathcal{F}_6|_U = \langle y^3 \rangle,$$

so $\mathcal{F}_6 \neq \mathcal{F}_1 \otimes \mathcal{F}_5$. In this example, the characters χ_1, χ_5 and χ_6 mark lines in the triangulation Σ of Figure 4.6. Theorem 4.4 reveals that the tautological bundles $\mathcal{F}_i$ associated to characters χ_i marking lines in Σ must be independent in $\mathrm{Pic}(Y)$ so that their first Chern classes are independent in $H^2(Y, \mathbb{Z})$.

Chapter 5

Moduli of representations of the McKay quiver

For a finite subgroup $G \subset \mathrm{SL}(3, \mathbb{C})$, G-Hilb $\mathbb{C}^3$ can be constructed as a moduli space $\mathcal{M}_\theta$ of representations of the McKay quiver. Given that G-Hilb $\mathbb{C}^3$ is a minimal model of $\mathbb{C}^3/G$ (see Chapter 3 for the Abelian case), it is natural to ask whether every minimal model of $\mathbb{C}^3/G$ is of the form $\mathcal{M}_\theta$ for some θ. For a finite Abelian subgroup $G \subset \mathrm{SL}(3, \mathbb{C})$, we propose a simple procedure to realise toric minimal models of $\mathbb{C}^3/G$ as moduli spaces of representations of the McKay quiver. Inspired by Nakamura's 'G-igsaw transformation'[1] which calculates G-Hilb $\mathbb{C}^3$, we introduce the more general notion of 'θ-stable G-igsaw transformation'. Worked examples are provided in Section 5.8 where every toric minimal model of the cyclic quotient singularities $\frac{1}{6}(1, 2, 3)$ and $\frac{1}{11}(1, 2, 8)$ is constructed as a moduli space $\mathcal{M}_\theta$ for some θ by performing a sequence of θ-stable G-igsaw transformations.

5.1 Nakamura's G-igsaw transformations

For a finite Abelian subgroup $G \subset \mathrm{SL}(3, \mathbb{C})$, Nakamura [Nak00] calculates G-Hilb $\mathbb{C}^3$ by performing a sequence of 'G-igsaw transformations'. In this section we describe the calculation using the notation from §1.1.4.

Given an ideal $I \subset \mathbb{C}[x, y, z]$ generated by monomials, write $\Gamma(I)$ for the set of monomials lying in the complement $\mathbb{C}[x, y, z] \setminus I$. Every monomial in

[1] In this chapter G (rather than A) denotes the finite Abelian subgroup of $\mathrm{SL}(3, \mathbb{C})$, otherwise the '$G$-igsaw' joke falls rather flat.

$\mathbb{C}[x,y,z]$ lies in a well defined character space of the G-action, giving rise to a map $\mathrm{wt}\colon \Gamma(I) \to G^\vee$ from $\Gamma(I)$ to the character group of G. The set $\Gamma(I)$ is called a *G-graph* if the map wt is one-to-one.

For each monomial ideal I_Z defining a G-cluster $Z \in G\text{-Hilb}\,\mathbb{C}^3$, the set $\Gamma(I_Z)$ is a G-graph (see Figure 3.9 for an illustration of $\Gamma(I_Z)$). Conversely, every G-graph Γ gives rise to a monomial ideal $I(\Gamma) \subset \mathbb{C}[x,y,z]$ such that the subscheme $Z(\Gamma) \subset \mathbb{C}^3$ is a G-cluster. According to Theorem 3.23, the ideal $I(\Gamma)$ is either 'up' or 'down', and the set $\mathrm{Def}(\Gamma)$ of toric parameters for deforming $Z(\Gamma)$ consists of

$$\begin{array}{llll} \xi = x^{l+1}/y^b z^f, & \eta = y^{m+1}/z^c x^d, & \zeta = z^{n+1}/x^a y^e & \text{if 'up'}, \\ \lambda = y^{b+1}z^{f+1}/x^l, & \mu = z^{c+1}x^{d+1}/y^m, & \nu = x^{a+1}y^{e+1}/z^n & \text{if 'down'}, \end{array}$$

for integers $a,b,c,d,e,f,l,m,n \geq 0$. Every such parameter lies in the lattice $M = \mathrm{Hom}(L,\mathbb{Z})$ of G-invariant Laurent monomials and defines a hyperplane in $L \otimes \mathbb{R}$, say $\xi^\perp = \{\alpha \in L \otimes \mathbb{R} \,|\, \xi(\alpha) = 0\}$. Taken together, all three hyperplanes support a 3-dimensional cone $\sigma(\Gamma) \subset L \otimes \mathbb{R}$ such that $Z(\Gamma)$ is the origin in the toric variety $X_{L,\sigma(\Gamma)} \cong \mathbb{C}^3$. For instance, in Case 'up'

$$\sigma(\Gamma) := \big\{\alpha \in L \otimes \mathbb{R} \;\big|\; \xi(\alpha) \geq 0,\ \eta(\alpha) \geq 0,\ \zeta(\alpha) \geq 0\big\} \tag{5.1}$$

and $X_{L,\sigma(\Gamma)} \cong \mathrm{Spec}\ \mathbb{C}[\xi,\eta,\zeta] \cong \mathbb{C}^3$. Case 'down' is similar.

Given a G-graph Γ, every other G-graph is determined by performing a sequence of *G-igsaw transformations*. The codimension 1 faces of $\sigma(\Gamma)$ are of the form $\tau = \sigma(\Gamma) \cap v^\perp$ for some $v \in \mathrm{Def}(\Gamma)$. Write $v_{\mathrm{num}}, v_{\mathrm{den}} \in \mathbb{C}[x,y,z]$ for the numerator and denomenator of v (e.g., if $v = \xi$ then $v_{\mathrm{num}} = x^{l+1}$ and $v_{\mathrm{den}} = y^b z^f$). Deform $I = I(\Gamma)$ in the v-direction to produce an ideal $I(v)$ for each $v \in \mathbb{A}^1$.

Key Lemma 5.1 Assume $v \notin \mathbb{C}[x,y,z]^G$. The ideal $I' := \lim_{v\to\infty} I(v)$ is a monomial ideal for which the set $\Gamma' := \Gamma(I')$ is a G-graph given by

$$\Gamma' = \big\{v^{c_{\max}(m)} \cdot m \,\big|\, m \in \Gamma\big\}, \tag{5.2}$$

for $c_{\max}(m) := \max\big\{c \in \mathbb{Z} \,\big|\, v^c \cdot m \in \mathbb{C}[x,y,z]\big\}$. That is, Γ' is obtained from Γ by replacing every occurence of $v_{\mathrm{den}} \in \Gamma$ by v_{num}.

The G-graph Γ' is called the *G-igsaw transform of Γ in the v-direction.* The cones $\sigma(\Gamma)$ and $\sigma(\Gamma')$ share the codimension 1 face $\tau = \sigma(\Gamma) \cap \sigma(\Gamma')$ so lie adjacent in $L \otimes \mathbb{R}$. Having performing one G-igsaw transformation, repeat the process using the G-graph Γ' and the ideal I'. Nakamura [Nak00] proves that this procedure sweeps out the whole of the fan Σ of $G\text{-Hilb}\,\mathbb{C}^3$:

Theorem 5.2 *Let $\Sigma \subset L \otimes \mathbb{R}$ be the fan consisting of cones $\sigma(\Gamma)$ defined by G-graphs Γ. Then Σ is obtained from a given G-graph by a finite sequence of G-igsaw transformations. Moreover, the toric variety $X_{L,\Sigma}$ is G-*Hilb $\mathbb{C}^3$.

Example 5.3 Consider the action on $\mathbb{C}^3$ by the group

$$G = \mathbb{Z}/2 \oplus \mathbb{Z}/2 = \left\langle \frac{1}{2}(1,-1,0), \frac{1}{2}(0,1,-1) \right\rangle \subset \mathrm{SL}(3,\mathbb{C}).$$

The ideal $I_1 = \langle x^2, y^2, z \rangle$ defines the G-graph $\Gamma_1 := \Gamma(I_1)$ shown in Figure 5.1. According to Theorem 3.23, $Z(\Gamma_1)$ has deformation parameters $\xi = x^2, \eta = y^2$ and $\zeta = z/xy$ which determine the three supporting hyperplanes of the cone $\sigma_1 := \sigma(\Gamma_1) = \{\alpha \in L \otimes \mathbb{R} \,|\, \xi(\alpha), \eta(\alpha), \zeta(\alpha) \geq 0\}$ in Figure 5.2.

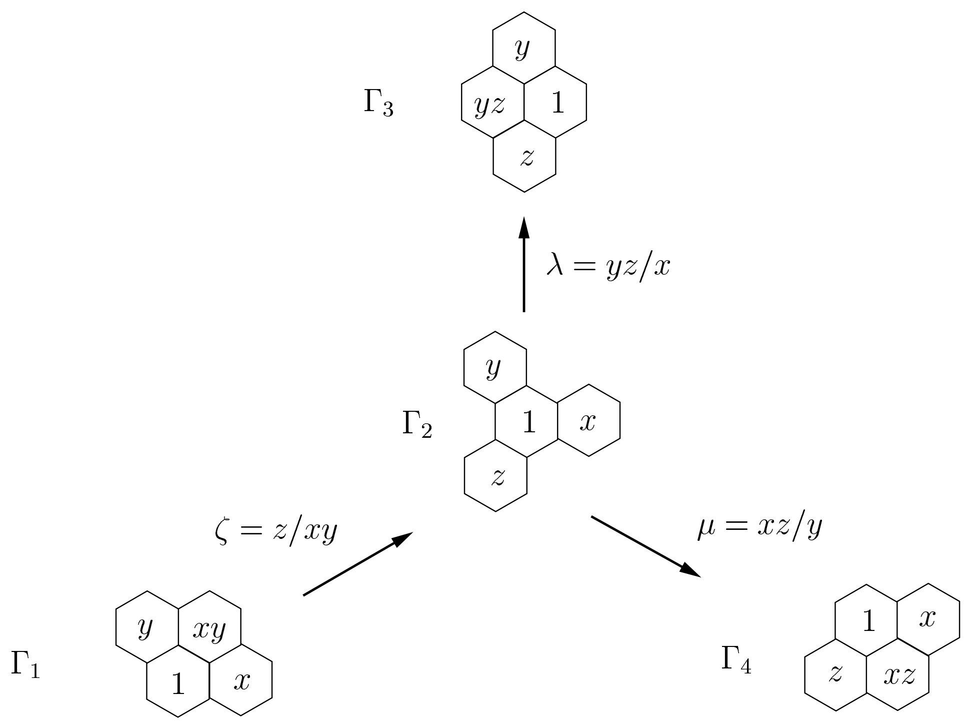


Figure 5.1: The G-igsaw puzzle for $\mathbb{Z}/2 \times \mathbb{Z}/2$

Neither $\xi = x^2$ nor $\eta = y^2$ give rise to G-igsaw transformations because $x^2, y^2 \in \mathbb{C}[x,y,z]^G$. However, deforming in the ζ-direction defines the ideal $I_1(\zeta) = \langle x^2, y^2, z - \zeta xy \rangle$ for any $\zeta \in \mathbb{A}^1$. The ideal $I_1' = \lim_{\zeta \to \infty} I_1(\zeta)$ is a monomial ideal defining the G-igsaw transform of Γ_1 in the ζ-direction. The corresponding G-graph $\Gamma_2 := \Gamma(I_1')$ is obtained by replacing xy in Γ_1 by z, as shown in Figure 5.1.

The ideal $I_1' = I(\Gamma_2) = \langle x^2, y^2, z^2, xy, xz, yz \rangle$ has deformation parameters $\lambda = yz/x, \mu = xz/y$ and $\nu = xy/z$ defining the cone $\sigma_2 := \sigma(\Gamma_2)$ of Figure 5.2. All three of these parameters determine G-igsaw transformations: the λ- and μ-directions define the graphs Γ_3 and Γ_4 of Figure 5.1 respectively, while the ν-direction leads straight back to Γ_1. This completes the 'G-igsaw puzzle' of Figure 5.1, where each G-graph Γ_i is drawn as a subset of the McKay quiver (see Definition 4.29). The resulting fan Σ shown in Figure 5.2 defines G-Hilb $\mathbb{C}^3$.

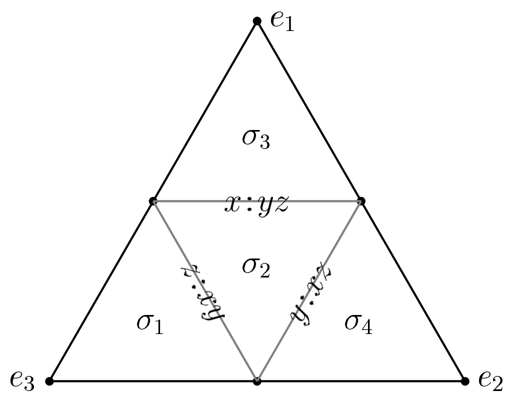


Figure 5.2: The fan Σ of G-Hilb $\mathbb{C}^3$ for the $\mathbb{Z}/2 \times \mathbb{Z}/2$-action

Remark 5.4 Nakamura [Nak00] calculates the 'dynamic' G-Hilbert scheme G-Hilb $\mathbb{C}^n$ for a finite Abelian subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ rather than the 'algebraic' version considered in this thesis (see §3.7 for the rival definitions). However, we are currently interested in the case $G \subset \mathrm{SL}(3, \mathbb{C})$ when these definitions agree.

5.2 Flops of G-Hilb $\mathbb{C}^3$ for $G = \mathbb{Z}/2 \times \mathbb{Z}/2$

The suggestion that it may be possible to adapt Nakamura's approach for calculating G-Hilb $\mathbb{C}^3$ in order to calculate flops of G-Hilb $\mathbb{C}^3$ was made independently by Akira Ishii and Alastair King. The basic idea is to deform a free orbit not as a G-equivariant subscheme of $\mathbb{C}^3$ but, more generally, as a G-equivariant $\mathbb{C}[x,y,z]$-module. We illustrate this idea by considering the simplest finite Abelian subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$ for which G-Hilb $\mathbb{C}^3$ admits a flop, namely the action of $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ introduced in Example 5.3.

Example 5.5 The G-graph Γ_1 from Example 5.3 is generated by 1 as a $\mathbb{C}[x,y,z]$-module. Replacing 1 by the G-invariant monomial x^2 gives rise to the G-equivariant $\mathbb{C}[x,y,z]$-module[2] $M_5 := \{x^2, x, y, xy\}$ whose monomials correspond one-to-one with the characters of $G = \mathbb{Z}/2 \times \mathbb{Z}/2$. To calculate the deformation parameters of M_5 as a G-equivariant $\mathbb{C}[x,y,z]$-module, write three 4×4 matrices corresponding to multiplication of $(x^2, x, y, xy)^t$ by x, y and z. Deforming these matrices G-equivariantly leads to

$$x = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \beta & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \\ \delta & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & 0 & \zeta \\ 0 & 0 & \eta & 0 \\ 0 & \theta & 0 & 0 \\ \iota & 0 & 0 & 0 \end{pmatrix}$$

(check: the module structure on M_5 is recovered when $\alpha = \cdots = \iota = 0$). These matrices commute pairwise after a flat deformation giving rise to relations between the parameters $\alpha, \ldots, \iota$, leaving only three free parameters α, δ, ι satisfying $x \cdot x^2 = \alpha x$, $y \cdot y = \delta x^2$ and $z \cdot x^2 = \zeta xy$. Thus, the deformation parameters of M_5 are

$$\alpha = x^2, \quad \delta = y^2/x^2 \quad \text{and} \quad \zeta = xz/y.$$

These Laurent monomials cut out the supporting hyperplanes $\alpha^\perp$, $\delta^\perp$ and $\zeta^\perp$ of the cone $\sigma_5 := \sigma(M_5) \subset L \otimes \mathbb{R}$ shown in Figure 5.3. Put another way, M_5 lies at the origin in the toric variety $X_{L,\sigma_5} \cong \mathbb{C}^3$.

It is natural to ask whether the method introduced in §5.1 can be applied in this more general setting to calculate the entire fan shown in Figure 5.3. With this goal in mind, mechanically replace every occurence of $y \in M_5$ by the monomial xz to simulate a G-igsaw transformation of M_5 in the $\zeta = xz/y$ direction. The resulting module $M_4 := \{x^2, x, xz, x^2z\}$ has deformation parameters $x^2, y/xz$ and z^2 which cut out the cone σ_4 of Figure 5.3, so M_4 lies at the origin of $X_{L,\sigma_4} \cong \mathbb{C}^3$. It appears then that our simulated G-igsaw transformation actually corresponds to a deformation across the common face $\tau = \sigma_5 \cap \sigma_4$ cut out by the ratio $y : xz$.

[2]Hereafter we represent $\mathbb{C}[x,y,z]$-modules by writing down a $\mathbb{C}$-basis only.

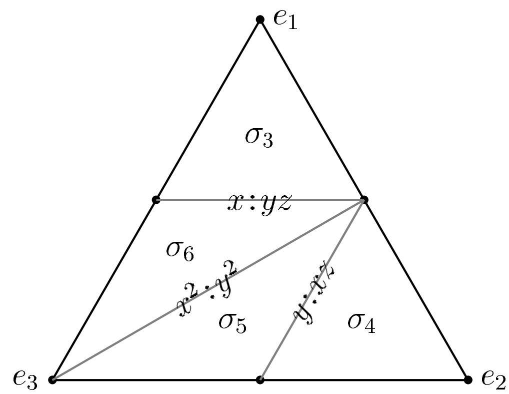


Figure 5.3: The fan Σ' of a flop of G-Hilb $\mathbb{C}^3$ for $G = \mathbb{Z}/2 \times \mathbb{Z}/2$

Similarly, simulating a G-igsaw transformation of M_5 in the $\delta = y^2/x^2$ direction leads to the G-equivariant $\mathbb{C}[x, y, z]$-module $M_6 = \{y^2, x, y, xy\}$ whose deformation parameters $x^2/y^2, y^2$ and yz/x cut out the cone σ_6 shown in Figure 5.3. Finally, replacing every $x \in M_6$ by yz defines the module $M_3 := \{y^2, yz, y, y^2z\}$ with deformation parameters $x/yz, y^2$ and z^2 cutting out σ_3. This sequence of 'simulated G-igsaw transformations' illustrated in Figure 5.4 determines the fan shown in Figure 5.3.

This example suggests that it may be possible to generalise Nakamura's G-igsaw transformations in order to calculate flops of G-Hilb $\mathbb{C}^3$ for finite Abelian subgroups $G \subset \mathrm{SL}(3, \mathbb{C})$.

5.3 Moduli of θ-stable G-constellations

Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite subgroup (nothing is gained by restricting to the case $G \subset \mathrm{SL}(3, \mathbb{C})$ at this stage). In this section we define moduli of G-equivariant $\mathbb{C}[x_1, \ldots, x_n]$-modules in terms of representations of the McKay quiver introduced in §1.3.

Definition 5.6 A *G-constellation* is a G-equivariant $\mathbb{C}[x_1, \ldots, x_n]$-module which is isomorphic as a G-module to the regular representation R of G.

For $Z \in G$-Hilb $\mathbb{C}^n$, the ring $\mathbb{C}[x_1, \ldots, x_n]/I_Z \cong R$ may be regarded as a G-equivariant $\mathbb{C}[x_1, \ldots, x_n]$-module with generator 1 mod I_Z. Thus, as the terminology suggests, 'G-constellation' is a generalisation of 'G-cluster'.

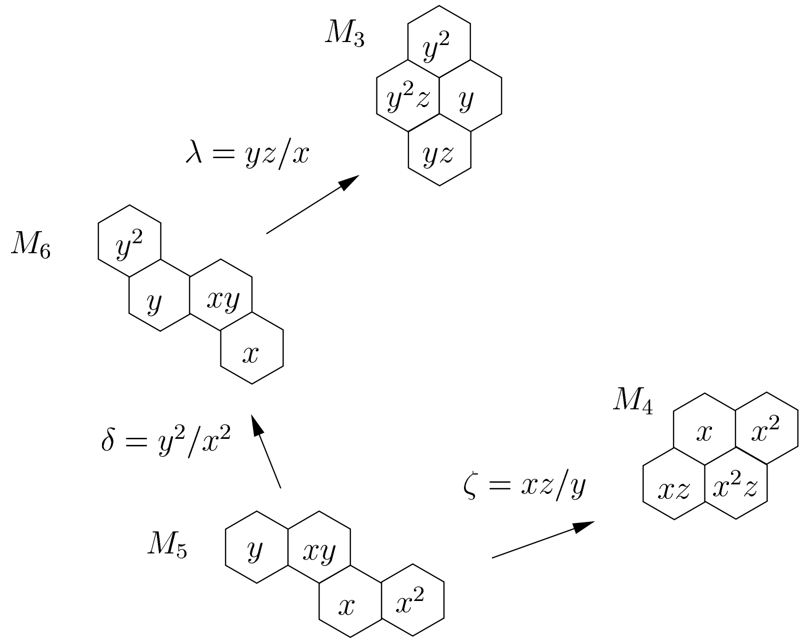


Figure 5.4: A new G-igsaw puzzle for $\mathbb{Z}/2 \times \mathbb{Z}/2$

A G-constellation M has a quiver-theoretic interpretation as follows. Let $Q = \mathbb{C}^n$ denote the *given* representation of the group G induced by the inclusion $G \subset \mathrm{GL}(n, \mathbb{C})$. The G-equivariant $\mathbb{C}[x_1, \ldots, x_n]$-module structure on M defines a G-equivariant map $B = (B_1, \ldots, B_n) \colon M \to Q \otimes M$ given by

$$B_i(m) = x_i \cdot m \quad \text{for} \quad i = 1, \ldots, n;$$

here $x_1, \ldots, x_n$ is a basis of Q. Clearly $[B_i, B_j] = 0$ for $i, j = 1, \ldots, n$ or, more invariantly, $B \wedge B = 0$. Since $M \cong R$ as G-modules, B defines a point of the affine variety introduced in (1.18):

$$\big\{B \in \mathrm{Hom}_G(R, Q \otimes R) \bigm| B \wedge B = 0\big\} \cong X \subset \mathrm{Rep}(\mathcal{Q}, \underline{r}).$$

Conversely, a point $V \in X \subset \mathrm{Rep}(\mathcal{Q}, \underline{r})$ determines a G-equivariant map $B \colon R \to Q \otimes R$ satisfying $B \wedge B = 0$ or, equivalently, maps $B_i \colon R \to R$ for $i = 1, \ldots, n$ such that $[B_i, B_j] = 0$ for $i, j = 1, \ldots, n$. By regarding B_i as multiplication by x_i, the maps B_i define a G-equivariant $\mathbb{C}[x_1, \ldots, x_n]$-module structure on R and hence a G-constellation M. Thus we get:

Proposition 5.7

$$\{G\text{-constellations}\}/\text{isom} \cong X/\,\mathrm{PGL}(\underline{r}).$$

It follows immediately that the moduli $\mathcal{M}_\theta$ of θ-stable representations described in §1.3 may also be regarded as moduli of G-constellations. For convenience we translate the notion of stability for representations into the language of G-modules: for a parameter $\theta = (\theta_0, \dots, \theta_N) \in \mathbb{Q}^{N+1}$ and a G-equivariant $\mathbb{C}[x_1, \dots, x_n]$-module V whose decomposition into irreducible G-modules is $V = \bigoplus_{i \in I} v_i \rho_i$ where $v_i = \dim_{\mathbb{C}} \operatorname{Hom}_G(\rho_i, V)$, set

$$\theta(V) := \sum \theta_i \cdot v_i.$$

Definition 5.8 A G-constellation V is said to be *θ-stable* if $\theta(V) = 0$ and every proper $\mathbb{C}[x_1, \dots, x_n]$-submodule $0 \subsetneq W \subsetneq V$ has $\theta(W) > 0$. The notion of *θ-semistable* is the same with $\geq$ replacing $>$.

By construction, the moduli space of θ-stable G-constellations coincides with the moduli space $\mathcal{M}_\theta$ of θ-stable points of $X \subset \operatorname{Rep}(\mathcal{Q}, \underline{r})$ introduced in Definition 1.35. As a result, all of the results described in §1.3.2 apply to the moduli of G-constellations.

For a finite subgroup $G \subset \mathrm{SL}(2, \mathbb{C})$, Ito and Nakamura [IN99] proved that G-Hilb $\mathbb{C}^2$ is the minimal resolution of $\mathbb{C}^2/G$. Combined with Kronheimer's Theorem 1.36, it follows that $G\text{-Hilb}\ \mathbb{C}^2 \cong \mathcal{M}_\theta$ for generic $\theta \in \Pi$ (see (1.19) for the definition of Π). Ito and Nakajima subsequently observed that $G\text{-Hilb}\ \mathbb{C}^n \cong \mathcal{M}_\theta$ for some $\theta \in \Pi$; the appropriate parameters $\theta \in \Pi$ were calculated by King using his own direct interpretation of stability.

Proposition 5.9 ([IN00],[Kin00]) *For a finite subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$,*

$$G\text{-Hilb}\ \mathbb{C}^n = \mathcal{M}_\theta$$

for $\theta \in \Pi$ satisfying $\theta_0 < 0$ and $\theta_i > 0$ for $i = 1, \dots, N$.

Proof. Write $M := \mathbb{C}[x_1, \dots, x_n]/I_Z$ for the G-constellation determined by a G-cluster $Z \in G\text{-Hilb}\ \mathbb{C}^n$. The condition $\theta \in \Pi$ gives $\theta_0 = -\sum \theta_i \cdot r_i$ so $\theta(M) = 0$ because $M \cong R$ as a G-module. The element 1 mod I_Z generates M as a $\mathbb{C}[x_1, \dots, x_n]$-module, so no proper submodule $S \subsetneq M$ may contain 1 mod I_Z. Hence the decomposition $S = \bigoplus_{i \in I} s_i \rho_i$ must have $s_0 = 0$. Then $\theta(S) > 0$ as required because $\theta_i > 0$ for $i = 1, \dots, N$.

Conversely, a θ-stable orbit of X is (up to the action of $\mathrm{PGL}(\underline{r})$) a map $B = (B_1, \dots, B_n)$ satisfying $[B_i, B_j] = 0$ for $i, j = 1, \dots, n$. Define a map $\phi\colon \mathbb{C}[x_1, \dots, x_n] \to R$ by $\phi(f) = f(B_1, \dots, B_n)\rho_0$. Then $S := \operatorname{im} \phi$ is a submodule of R containing $\phi(1) = \rho_0$ so the G-module decomposition $S = \bigoplus_{i \in I} s_i \rho_i$ has $s_0 = 1$. The stability assumption ensures that $S = R$. Hence $\mathbb{C}[x_1, \ldots, x_n]/\ker\phi \cong R$, so $\ker\phi$ defines a G-cluster $Z \in G\text{-Hilb}\,\mathbb{C}^n$. $\square$

Remark 5.10 For a finite subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$, Bridgeland, King and Reid established that $\varphi\colon G\text{-Hilb}\,\mathbb{C}^3 \to \mathbb{C}^3/G$ is a crepant resolution by showing that certain full subcategories of the derived category of $G\text{-Hilb}\,\mathbb{C}^3$ have trivial Serre functor (see Bridgeland et al. [BKR99, Lemma 3.1]). The same method shows that $\varphi_\theta\colon \mathcal{M}_\theta \to \mathbb{C}^3/G$ is a crepant resolution for generic $\theta \in \Pi$, but this approach yields little explicit information about $\mathcal{M}_\theta$.

For a finite Abelian subgroup $G \subset \mathrm{GL}(n,\mathbb{C})$ acting freely away from the origin, Sardo Infirri [SI94, SI96b] constructed the moduli spaces $\mathcal{M}_\theta$ as n-dimensional toric varieties. This description led to the following simple procedure to compute the parameters $\theta \in \Pi$ with respect to which a given G-constellation M is θ-stable. First, regard a G-constellation M as a representation of the McKay quiver with vertices $m_0, \ldots, m_N \in M$. The module structure on M is determined by the linear maps of the representation and hence by the arrows in the quiver. For $G \subset \mathrm{SL}(3,\mathbb{C})$, multiplication by x (resp. y, z) is represented say by an arrow pointing east (resp. north-west, south-west).

Definition 5.11 A *flow* f on M is the assignment of a non-negative rational number $f(a) \in \mathbb{Q}_{\geq 0}$ to each arrow a of the quiver. The *support* of a flow $\mathrm{Supp}(f)$ is the set of arrows a for which $f(a) \neq 0$. For a vertex m, write $\mathrm{Head}(m)$ (resp. $\mathrm{Tail}(m)$) for the arrows which have their head (resp. tail) at m. The net contribution of f at m is the rational number

$$\partial f(m) = \sum_{a \in \mathrm{Head}(m)} f(a) - \sum_{a \in \mathrm{Tail}(m)} f(a).$$

Define $\partial f := \big(\partial f(m_0), \partial f(m_1), \ldots, \partial f(m_N)\big) \in \mathbb{Q}^{N+1}$.

Proposition 5.12 ([SI96b]) *A G-constellation M is θ-stable if and only if there exists a flow f on M such that $\partial f = \theta$ whose support connects any two vertices of M.*

Example 5.13 A flow f on the G-constellation M_5 introduced in §5.2 is shown in Figure 5.5(a) for $a, b, c \in \mathbb{Q}_{\geq 0}$. Assume that the support of f connects any two vertices of M_5, so in fact $a, b, c > 0$. List the elements of M_5 as $m_{0,0} = x^2, m_{1,0} = x, m_{0,1} = y$ and $m_{1,1} = xy$, so $\partial f = (c, -b-c, -a, a+b)$. Then M_5 is θ-stable with respect to parameters in the set

$$\begin{aligned} \theta(M_5) \;\; &:= \;\; \{\theta \in \mathbb{Q}^4 \,\big|\, \theta = (c, -b-c, -a, a+b) \text{ for some } a, b, c > 0\} \\ &= \;\; \{\theta \in \Pi \,\big|\, \theta_{0,0} > 0;\; \theta_{0,1} < 0;\; \theta_{1,0} + \theta_{0,0} < 0;\; \theta_{1,1} + \theta_{0,1} > 0\}. \end{aligned}$$

Observe that $\theta(M_5) \subset \{\theta \in \Pi \,\big|\, \theta_{1,0}, \theta_{0,1} < 0;\; \theta_{0,0}, \theta_{1,1} > 0\}$.

$$\begin{tikzcd} y \arrow[r, "a"] & xy & \\ & x \arrow[u, "b"'] \arrow[r, "c"] & x^2 \end{tikzcd}$$

(a)

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d, "\beta"'] & x^2 \arrow[d, "\gamma"] \\ xz \arrow[r, "\delta"] & x^2 z \end{tikzcd}$$

(b)

Figure 5.5: Flows on the G-constellations (a) M_5; (b) M_6

Similarly, a flow f on M_6 is shown in Figure 5.5(b) for $\alpha, \beta, \gamma, \delta \in \mathbb{Q}_{\geq 0}$. Assume that the support of f connects any two vertices of M_6, so at most one of $\alpha, \beta, \gamma, \delta$ is zero. The set of parameters

$$\theta(M_6) := \{\theta \in \Pi \,\big|\, \theta = (\alpha - \gamma, -\alpha - \beta, \beta - \delta, \gamma + \delta)\}$$

for which M_6 is θ-stable is a cone lying in $\{\theta \in \Pi \,\big|\, \theta_{1,0} < 0;\; \theta_{1,1} > 0\}$.

Definition 5.14 Define $\theta(M) := \{\theta \in \Pi \,\big|\, M \text{ is } \theta\text{-stable}\}$.

The set $\theta(M)$ is an open (in the Euclidean topology) cone which may be hard to calculate explicitly in general. Nevertheless, it is easy to see that

$$\theta(M) \subseteq \begin{cases} \{\theta \in \Pi \,\big|\, \theta_i < 0\} & \text{if } m_i \text{ is a generator of } M, \\ \{\theta \in \Pi \,\big|\, \theta_i > 0\} & \text{if } m_i \text{ is annihilated by } x_1, \ldots, x_n. \end{cases} \tag{5.3}$$

Indeed, let f be a flow on M which connects any two vertices. If $m_i \in M$ is a $\mathbb{C}[x_1, \ldots, x_n]$-module generator then $\mathrm{Head}(m_i) = \emptyset$ so $\partial f(m_i) < 0$. Similarly, if m_i is annihilated by $x_1, \ldots, x_n$ then $\mathrm{Tail}(m_i) = \emptyset$ so $\partial f(m_j) > 0$.

5.4 The mysterious first step

Hereafter, let $G \subset \mathrm{SL}(3,\mathbb{C})$ be a finite Abelian subgroup. Our goal is to generalise the procedure from §5.1 to construct a flop $X_{L,\Sigma'}$ of G-Hilb $\mathbb{C}^3$ as a moduli space $\mathcal{M}_\theta$ for some $\theta \in \Pi$. The first step is the construction of a G-constellation M' which corresponds to a $\mathbb{T}^3$-fixed point of $X_{L,\Sigma'}$. Our somewhat mysterious 'trial and error' method is motivated by the easiest case $G = \mathbb{Z}/2 \times \mathbb{Z}/2$:

Example 5.15 The sequence of simulated G-igsaw transformations from §5.2 leading to the fan Σ' shown in Figure 5.3 begins with the creation of the G-constellation $M_5 = \{x^2, x, y, xy\}$ from the G-graph $\Gamma_1 = \{1, x, y, xy\}$. The process of passing from Γ_1 to M_5 determines a jump[3] from one chamber to another inside the parameter space Π. Indeed, Γ_1 (when regarded as a G-constellation) is generated by 1 over $\mathbb{C}[x,y,z]$ so, by (5.3), it is θ-stable with respect to parameters in the set

$$\theta(\Gamma_1) \subset \{\theta \in \Pi \mid \theta_{0,0} < 0\}.$$

Compare this with the calculation of $\theta(M_5)$ from Example 5.13 to see that $\theta(M_5) \cap \theta(\Gamma_1) = \emptyset$. Thus the process of creating M_5 from Γ_1 necessitates a jump from $\theta(\Gamma_1)$ to $\theta(M_5)$ inside Π.

This example can be generalised as follows. Suppose that Σ' ($\neq \Sigma$) is a basic triangulation of the junior simplex Δ of $\mathbb{C}^3/G$ so that the toric variety $X_{L,\Sigma'}$ is a flop of $X_{L,\Sigma} \cong G$-Hilb $\mathbb{C}^3$. Choose 3-dimensional cones $\sigma' \in \Sigma' \setminus \Sigma$ and $\sigma \in \Sigma \setminus \Sigma'$ such that $\sigma \cap \sigma' \cap \Delta \neq \emptyset$. Let Γ be the G-graph of the cone σ and let $S \subset \Gamma$ be a connected subset of monomials including 1. Replace each monomial $s \in S$ by a monomial s' of the same weight to form a set $S' := \{s' \mid s \in S\}$ so that the resulting set of monomials $(\Gamma \setminus S) \cup S'$ defines a G-constellation $M' = \{m'_0, \dots m'_N\}$ which is endowed with the obvious $\mathbb{C}[x,y,z]$-module structure.

Remark 5.16 The process of choosing the sets S and S' so that the deformation parameters of M' cut out the cone σ' is done by trial and error. In fact it is not even clear whether S and S' can always be chosen so that the deformation parameters of M' cut out σ'. Our understanding of this process is limited so we label it the *mysterious first step*.

[3]We deliberately avoid the phrase 'wall crossing' which suggests that the chambers involved share a common wall.

Example 5.17 Consider the action of $G \cong \mathbb{Z}/6$ on $\mathbb{C}^3$ defining the quotient singularity $\frac{1}{6}(1,2,3)$. The fans Σ and Σ' defining G-Hilb $\mathbb{C}^3$ and a flop $X_{L,\Sigma'}$ of G-Hilb $\mathbb{C}^3$ respectively are shown in Figure 5.6. The cones σ_7, σ_8 lie in $\Sigma' \setminus \Sigma$ so choose one, say $\sigma' = \sigma_8$. Both $\sigma_2, \sigma_6 \in \Sigma \setminus \Sigma'$ intersect σ_7 in the junior simplex Δ so choose one, say $\sigma = \sigma_6$.

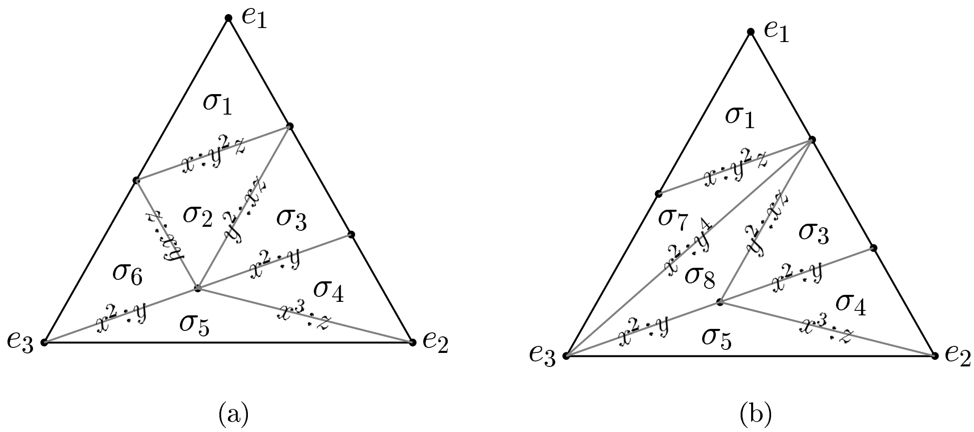


Figure 5.6: (a) Σ of G-Hilb $\mathbb{C}^3$; (b) Σ' of a flop of G-Hilb $\mathbb{C}^3$

By convention we list the elements of a monomial G-constellation in 'cyclic order', beginning with the G-invariant monomial. Thus, write the monomial G-cluster defining the cone σ as $\Gamma = \{1, x, y, xy, y^2, xy^2\}$. Choose $S = \{1\}$ and $S' = \{y^3\}$ to give $M' = \{y^3, x, y, xy, y^2, xy^2\}$. However, the deformation parameters $x^2/y, y^3, z/xy$ of M' define the cone σ_6 rather than σ_8, so the first attempt has failed. As a second attempt, choosing $S = \{1, y\}$ and $S' = \{y^3, x^2\}$ leads to $M' = \{y^3, x, x^2, xy, y^2, xy^2\}$ whose deformation parameters $x^2/y, y^4/x^2, xz/y^2$ cut out the supporting hyperplanes of the cone $\sigma' = \sigma_8$ as required.

Remark 5.18 There are many choices to be made in the mysterious first step. Nevertheless, this process is successfully implemented in each of the worked examples in §5.8. Note that particular care is taken in choosing the cone $\sigma' \in \Sigma' \setminus \Sigma$ – see Remarks 5.36 and 5.37.

5.5 Ishii's local coordinates on $\mathcal{M}_\theta$

For a finite Abelian subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$, Ishii [Ish00] proved directly that the moduli $\mathcal{M}_\theta$ are smooth for generic $\theta \in \Pi$ by calculating explicit coordinates on the affine toric charts as we now describe.

The representation $Q = \mathbb{C}^3$ induced by the inclusion $G \subset \mathrm{SL}(3,\mathbb{C})$ decomposes into a sum of irreducibles $Q = \rho_1 \oplus \rho_2 \oplus \rho_3$ for which $\rho_1 \otimes \rho_2 \otimes \rho_3 = \rho_0$. Recall that $\mathrm{m}\colon R \to M$ is the G-module isomorphism.

Theorem 5.19 (Ishii) If a G-constellation M gives a $(\mathbb{C}^*)^3$-fixed point of $\mathcal{M}_\theta$ then there exists a unique irreducible representation $\sigma \in R$ such that

$$x \cdot \mathrm{m}(\sigma) = 0 \quad y \cdot \mathrm{m}(\sigma \otimes \rho_1) = 0 \quad z \cdot \mathrm{m}(\sigma \otimes \rho_1 \otimes \rho_2) = 0.$$

The three arrows of the oriented triangle $\sigma \to \sigma \otimes \rho_1 \to \sigma \otimes \rho_1 \otimes \rho_2 \to \sigma$ give an affine toric chart isomorphic to $\mathbb{C}^3$ centred at the fixed point.

The oriented triangle $\sigma \to \sigma \otimes \rho_1 \to \sigma \otimes \rho_1 \otimes \rho_2 \to \sigma$ appears where three fundamental domains of M adjoin in the universal cover of the McKay quiver. We illustrate this point with a simple example.

Example 5.20 The G-constellation $M_5 = \{x^2, x, y, xy\}$ introduced in §5.2 tiles the universal cover of the McKay quiver; three fundamental domains are illustrated in Figure 5.7:

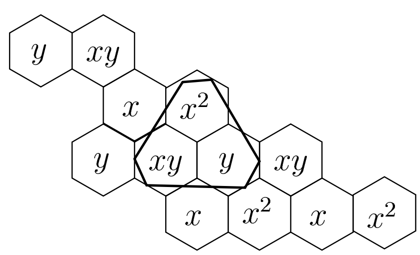


Figure 5.7: Three fundamental domains adjoin in the tiling of the plane

Where three fundamental domains adjoin there is a tripod of monomials xy, y and x^2, one from each fundamental domain. In this case $\mathrm{m}(\sigma) = xy$, $\mathrm{m}(\sigma \otimes \rho_1) = y$ and $\mathrm{m}(\sigma \otimes \rho_1 \otimes \rho_2) = x^2$. The oriented triangle determines the deformation parameters: $x \cdot xy = \alpha y$, $y \cdot y = \beta x^2$ and $z \cdot x^2 = \gamma xy$. Ishii's theorem states that the corresponding Laurent monomials

$$\alpha = x^2, \quad \beta = y^2/x^2 \quad \text{and} \quad \gamma = xz/y$$

form coordinates on the affine toric chart centred at M_5. It is easy to verify all of the deformation parameter calculations performed in §5.2 in this way.

5.6 Procedure to calculate flops of G-Hilb $\mathbb{C}^3$

Let Σ' denote a fan defining a flop $X_{L,\Sigma'}$ of $X_{L,\Sigma} = G$-Hilb $\mathbb{C}^3$. Suppose that the 'mysterious first step' has been performed successfully, i.e., we have constructed a G-constellation M' whose elements are monomials and whose deformation parameters cut out a 3-dimensional cone $\sigma' \in \Sigma' \setminus \Sigma$.

Starting with M', or more generally any G-constellation M, our goal is to construct a new G-constellation N from M by deforming in the v-direction, for some $v = v_{\mathrm{num}}/v_{\mathrm{den}} \in \mathrm{Def}(M)$. Example 5.5 illustrated a first attempt, where a G-igsaw transform in the v-direction was simulated by replacing every occurence of $v_{\mathrm{den}} \in M$ by v_{num} to define N. However, this approach fails for groups of larger order as the next example shows.

Example 5.21 For the $G = \mathbb{Z}/6$-action introduced in Example 5.17, the deformation parameters of the G-constellation $M_8 := \{y^3, x, x^2, xy, y^2, xy^2\}$ cut out the cone σ_8 in Figure 5.6(b). Simulating a G-igsaw transformation in the $v = x^2/y$-direction leads to $M_5 := \{x^6, x, x^2, x^3, x^4, x^5\}$ whose deformation parameters $x^6, y/x^2, z/x^3$ cut out the cone σ_5 in Figure 5.6(b) as we might expect. The problem arises when we transform back: simulating a G-igsaw transformation in the y/x^2-direction from M_5 defines the G-constellation $M_6 := \{y^3, x, y, xy, y^2, xy^2\}$ rather than M_8. In fact, M_6 deforms to give the cone σ_6 in Figure 5.6(a) which doesn't even lie in the fan Σ'.

The problem here is that one of the G-igsaw transformations has made us jump between chambers in Π. To see this, note that $y^4 \in M_8$ is annihilated by x, y and z. It lies in the ε^2 character space (for $\varepsilon^6 = 1$ primitive) so it follows from (5.3) shows that $\theta(M_8) \subset \{\theta \in \Pi \,|\, \theta_2 > 0\}$. However, $y \in M_6$ also lies in the ε^2 character space and is a generator of M_6, so (5.3) gives $\theta(M_6) \subset \{\theta \in \Pi \,|\, \theta_2 < 0\}$. Clearly $\theta(M_6) \cap \theta(M_8) = \emptyset$, so an unwanted jump from $\theta(M_8)$ to $\theta(M_6)$ has occurred as we transform from $M_8 \to M_5 \to M_6$.

Our solution to this problem is to replace only a subset of the occurences of $v_{\mathrm{den}} \in M$ by v_{num} to define the G-igsaw transform of M in the v-direction. For the module $M' = \{m'_0, \dots, m'_N\}$ constructed in the first step, define

$$I_{\mathrm{gen}} := \{i \in I \,|\, m'_i \text{ is a generator of } M'\}.$$

Let M be a G-constellation whose generators m_i are indexed by a subset of I_{gen}. Given $v = v_{\text{num}}/v_{\text{den}} \in \operatorname{Def}(M)$ such that $v \notin \mathbb{C}[x,y,z]^G$ (this ensures that $v_{\text{den}} \neq 1$), we now introduce an algorithm to construct a G-constellation N from M by replacing as many occurences of $v_{\text{den}} \in M$ by v_{num} as possible, with the proviso that the following statement[4] holds:

$$\textit{the generators } n_i \textit{ of } N \textit{ are indexed by a subset of } I_{\text{gen}}. \qquad (5.4)$$

Algorithm 5.22 (generalised G-igsaw transformation) *Begin with*

$$N := \left\{ v^{c_{\max}(m_i)} \cdot m_i \,\middle|\, m_i \in M \right\},$$

for $c_{\max}(m_i) := \max\left\{ c \in \mathbb{Z} \,\middle|\, v^c \cdot m_i \in \mathbb{C}[x,y,z] \right\}$. *If condition (5.4) fails to hold,* $\exists\, i \notin I_{\text{gen}}$ *such that* n_i *is a generator of* N. *Replace* n_i *by* $v^{c_{\max}(m_i)-1} \cdot m_i$ *to define a new* G*-constellation which we also denote* N. *If the new module satisfies (5.4) then we're done, otherwise repeat the process.*

Eventually this algorithm terminates. Indeed, after sufficiently many loops, every element $n_i = v^c \cdot m_i$ of N is replaced by m_i so the resulting G-constellation is M itself which, by assumption, satisfies (5.4) as required.

Example 5.23 Recall from Example 5.17 that the mysterious first step for the fan Σ' of Figure 5.6(b) defines $M' = M_8 = \{y^3, x, x^2, xy, y^2, xy^2\}$, so $I_{\text{gen}} = \{1, 4\}$. The generalised G-igsaw transformation in the x^2/y-direction is $M_5 := \{x^6, x, x^2, x^3, x^4, x^5\}$, where condition (5.4) is satisfied first time.

Next, apply the algorithm to M_5 in the y/x^2-direction as follows. Begin with $N = \{y^3, x, y, xy, y^2, xy^2\}$, but $n_2 = y$ is a generator of N and $2 \notin I_{\text{gen}}$. Now, $y = v^1 x^2$ for $v = y/x^2$, so the algorithm replaces y by x^2 to give a new module $N = \{y^3, x, x^2, xy, y^2, xy^2\}$ which satisfies (5.4). Thus, the algorithm for M_5 in the y/x^2-direction leads to $N = M_8$ rather than to the module M_6 of Example 5.21.

Apply 5.22 to M_5 in the z/x^3-direction: $N = \{z^2, x, x^2, z, xz, x^2z\}$, but $n_3 = z$ is a generator of N and $3 \notin I_{\text{gen}}$ so the algorithm replaces z by x^3. The resulting module still has $n_0 = z^2$ as a generator. Since $0 \notin I_{\text{gen}}$, replace z^2 by x^3z to give the new module $M_4 := \{x^3z, x, x^2, x^3, xz, x^2z\}$ satisfying (5.4). Now, M_4 has deformation parameters $x^3/z, y/x^2, z^2$. Apply 5.22 in the y/x^2-direction to construct $M_3 := \{xyz, x, x^2, xy, xz, x^2z\}$ after twice

[4] See §5.7 for the significance of statement (5.4).

failing condition (5.4). Continuing in this way leads to the modules M_i listed in Table 5.1 (the module generators are underlined). We strongly urge the reader to draw a G-igsaw puzzle similar to that shown in Figure 5.4 using the information from the table. The resulting cones σ_i form the fan Σ' of Figure 5.6(b).

i	M_i	$\mathrm{Def}(M_i)$	σ_i
1	$\{y^3, y^2z, y^4, y^3z, \underline{y^2}, y^4z\}$	$x/y^2z, y^3, z^2$	σ_1
7	$\{y^3, \underline{x}, y^4, xy, \underline{y^2}, xy^2\}$	$x^2/y^4, y^2z/x, y^3$	σ_7
8	$\{y^3, \underline{x}, x^2, xy, \underline{y^2}, xy^2\}$	$x^2/y, y^4/x^2, xz/y^2$	σ_8
3	$\{xyz, \underline{x}, x^2, xy, xz, x^2z\}$	$x^2/y, y^2/xz, z^2$	σ_3
4	$\{x^3z, \underline{x}, x^2, x^3, xz, x^2z\}$	$x^3/z, y/x^2, z^2$	σ_4
5	$\{x^6, \underline{x}, x^2, x^3, x^4, x^5\}$	$x^6, y/x^2, z/x^3$	σ_5

Table 5.1: G-constellations M_i and cones σ_i for the fan Σ' from Figure 5.6(b)

This example suggests that it may be possible to generalise Nakamura's construction of G-Hilb $\mathbb{C}^3$ to calculate fans Σ' of other minimal models of $\mathbb{C}^3/G$ using the following procedure:

Procedure 5.24 *Let Σ' be a basic triangulation of the junior simplex Δ of $\mathbb{C}^3/G$ for a finite Abelian subgroup $G \subset \mathrm{SL}(3, \mathbb{C})$.*

Step 1 *Perform the* mysterious first step *to get a G-constellation M' and a set I_{gen} indexing the generators of M'. Relabel $M := M'$.*

Step 2 *Deform M according to Theorem 5.19 to produce deformation parameters $v_1, v_2, v_3 \in \mathrm{Def}(M)$ cutting out a cone $\sigma(M)$.*

Step 3 *Run Algorithm 5.22 on M in the v_k-direction (for some $k = 1, 2, 3$) to produce a G-constellation N satisfying condition (5.4).*

Step 4 *Set $M := N$ and return to* Step 2.

Given our limited understanding of the mysterious first step it is not yet clear when Step 1 can be performed. Moreover, even if the procedure does begin it is not clear a priori that it stabilises after finitely many steps, i.e., that after producing finitely many G-constellations, the only remaining deformation parameters are those $v \in \mathbb{C}[x, y, z]^G$.

Nevertheless, in Example 5.23 the procedure does stabilise and produces $|G| = 6$ modules M_i such that the corresponding cones $\sigma_i = \sigma(M_i)$ define the fan Σ'. In §5.8 the procedure is successfully implemented for every basic triangulation of the junior simplex of the singularities $\frac{1}{6}(1,2,3)$ and $\frac{1}{11}(1,2,8)$.

5.7 θ-stable G-igsaw transformations

To understand the significance of statement (5.4) we pick out a distinguished orthant in the parameter space $\mathbb{Q}^{N+1}$ by defining

$$\mathrm{Orth}(M') := \{\theta \in \mathbb{Q}^{N+1} \,\big|\, \theta_i < 0 \text{ if } i \in I_{\mathrm{gen}};\ \theta_i > 0 \text{ otherwise}\}.$$

Lemma 5.25 *For the G-constellation M' constructed in the mysterious first step, we have $\theta(M') \cap \mathrm{Orth}(M') \neq \emptyset$.*

Proof. Draw a flow f on M' whose support connects any two vertices of M'. Set $f(a) := 1$ if the head of a is annihilated by x, y and z. Next, consider the arrows b whose head m'_i is the tail of an arrow a with $f(a) = 1$. Choose $f(b) \geq 2$ sufficiently large to ensure that $\partial f(m'_i) > 0$. Repeat for the arrows whose head is the tail of an arrow b, and so on. Eventually every vertex m'_i with $\mathrm{Head}(m'_i) \neq \emptyset$ satisfies $\partial f(m'_i) > 0$. The remaining vertices are the generators for which $\partial f(m'_i) < 0$. Then M' is θ-stable with respect to f and $\partial f \in \mathrm{Orth}(M')$ as required. □

Example 5.26 For the fan Σ' of Example 5.23 we have $I_{\mathrm{gen}} = \{1, 4\}$, hence $\mathrm{Orth}(M') := \{\theta \in \mathbb{Q}^6 \,\big|\, \theta_1, \theta_4 < 0;\ \theta_0, \theta_2, \theta_3, \theta_5 > 0\}$. Lemma 5.25 constructs a flow f on $M' = M_8$ where $f(a) = 1$ for every arrow a except the arrow from x to xy for which, say, $f(b) = 2$. Clearly $\partial f = (1, -3, 1, 1, -2, 2) \in \mathrm{Orth}(M')$.

As a result, M' is stable with respect to some $\theta \in \mathrm{Orth}(M')$. Statement (5.4) is a necessary condition for a G-constellation N to be stable with respect to a parameter $\theta \in \mathrm{Orth}(M')$ – this follows immediately from (5.3). This condition is not sufficient in general: to see this, cook up N satisfying (5.4) where an element $n_i \in N$ with $i \in I_{\mathrm{gen}}$ is annihilated by x, y, z, so (5.3) ensures that $\theta(N) \subset \{\theta \in \Pi \,\big|\, \theta_i > 0\}$, hence $\theta(N) \cap \mathrm{Orth}(M') = \emptyset$.

Nevertheless, our examples suggest that the following statement holds:

Conjecture 5.27 *For Σ' as in Procedure 5.24, suppose that the procedure has been successfully implemented to produce $|G|$ modules M_i such that the fan Σ' is determined by the 3-dimensional cones $\sigma_i = \sigma(M_i)$. Then there exists an open subset of parameters θ in the orthant $\mathrm{Orth}(M')$ such that every M_i is θ-stable, i.e., $\theta(M_1) \cap \cdots \cap \theta(M_{|G|}) \cap \mathrm{Orth}(M') \neq \emptyset$.*

It follows from this conjecture that the statement (5.4) is sufficient to ensure that N is stable with respect to some $\theta \in \mathrm{Orth}(M')$ whenever N is obtained from M' by finitely many generalised G-igsaw transformations. Given that this is not true a priori, it suggests that θ-stability has been encoded into the generalised G-igsaw transformation.

Theorem 5.28 *When Conjecture 5.27 holds, $X_{L,\Sigma'} = \mathcal{M}_\theta$ for each of the parameters $\theta \in \mathrm{Orth}(M')$ given by the conjecture.*

Proof. The conjecture shows that $X_{L,\Sigma'} \subseteq \mathcal{M}_\theta$. On the other hand, the toric variety $\mathcal{M}_\theta$ is projective over $\mathbb{C}^3/G$ by Theorem 1.37 (and Remark 1.38) so the support of its fan is exactly the positive octant in the vector space $L \otimes \mathbb{R} \cong \mathbb{R}^3$. Hence the inclusion $X_{L,\Sigma'} \subseteq \mathcal{M}_\theta$ must be equality. □

Definition 5.29 When Conjecture 5.27 holds, every application of Step 3 in Procedure 5.24 is called a *θ-stable G-igsaw transformation.*

Example 5.30 For Σ defining G-Hilb $\mathbb{C}^3 = X_{L,\Sigma}$, Proposition 5.9 states that $X_{L,\Sigma} = \mathcal{M}_\theta$ for $\theta \in \{\theta \in \Pi \,|\, \theta_0 < 0;\ \theta_i > 0 \text{ for } i = 1, \ldots, N\}$ and Nakamura's G-igsaw transformations are θ-stable.

Example 5.31 For the fan Σ' shown in Figure 5.2, the procedure constructs the G-constellations illustrated in Figure 5.3. The intersection (in $\mathrm{Orth}(M')$) of the sets $\theta(M_i)$ for $i = 3, 4, 5, 6$ is the nonempty open cone

$$\{\theta \in \Pi \,|\, \theta_{0,0} > 0;\ \theta_{0,1} < 0;\ \theta_{1,0} + \theta_{0,0} < 0; \theta_{1,1} + \theta_{0,1} > 0; \theta_{0,0} + \theta_{0,1} < 0\}.$$

Hence[5] $X_{L,\Sigma'} = \mathcal{M}_\theta$ for any parameter in this cone, e.g., $\theta = (1, -2, -2, 3)$.

[5] The junior simplex admits two other basic triangulations obtained by rotating Figure 5.3 by $2\pi/3$ and $4\pi/3$. The appropriate G-igsaw puzzles are found by permuting $y \leftrightarrow z$ (or $x \leftrightarrow z$) in the modules shown in Figure 5.4. This simply has the effect of permuting the parameters $\theta_{0,1} \leftrightarrow \theta_{1,1}$ (or $\theta_{1,0} \leftrightarrow \theta_{1,1}$), hence every toric flop of G-Hilb $\mathbb{C}^3$ is of the form $\mathcal{M}_\theta$.

Example 5.32 For Σ' shown in Figure 5.6(b), Procedure 5.24 determines the G-constellations listed in Table 5.1. To calculate the intersection of the sets $\theta(M_i)$ for $i = 1,3,4,5,7,8$, draw a flow on the quiver corresponding to each G-constellation. This computation is lengthy so we do not reproduce it here. Suffice to say that each G-constellation is stable with respect to $\theta = (1,-4,1,1,-3,4) \in \mathrm{Orth}(M')$. Thus $X_{L,\Sigma'} = \mathcal{M}_\theta$ for this choice of θ.

In §5.8.1 we exhibit a parameter θ for which Conjecture 5.27 holds for every basic triangulation Σ' of the junior simplex of the singularities $\frac{1}{6}(1,2,3)$ and $\frac{1}{11}(1,2,8)$. It follows that every minimal model of these cyclic quotient singularities can be described as a moduli space $\mathcal{M}_\theta$ for some θ.

We conclude this section by proposing an appropriate generalisation of Nakamura's Key Lemma 5.1. First we generalise the notion of G-graph to account for the fact that G-constellations may have more than one generator.

Definition 5.33 Given a pair of ideals $I \subset J \subset \mathbb{C}[x,y,z]$, each of which is generated by monomials, write $\Gamma(I,J)$ for the set of monomials lying in $J\setminus I$. There is a map $\mathrm{wt}\colon \Gamma(I,J) \to G^\vee$ and, as in §5.1, we call the set $\Gamma(I,J)$ a *G-graph* if the map wt is one-to-one.

Every G-graph $\Gamma(I,J)$ defines a G-constellation M and the generators of the ideal J form the $\mathbb{C}[x,y,z]$-module generators of M. In particular, a G-graph defines a G-cluster if and only if $J = \langle 1 \rangle$.

Let M be a G-constellation whose elements can be written in the form $\Gamma(I,J)$ for some $J \neq \langle 1 \rangle$, where the ideal generators of J are indexed by a subset of I_{gen}. Each codimension 1 face τ of the cone $\sigma(M)$ is of the form $\tau = \sigma(M) \cap v^\perp$ for some $v \in \mathrm{Def}(M)$. Simultaneously deform I and J in the v-direction to produce ideals $I(v)$ and $J(v)$ for each $v \in \mathbb{A}^1$. Assume that $v \notin \mathbb{C}[x,y,z]^G$, so Algorithm 5.22 may be applied to M in the v-direction giving rise to a G-constellation N.

Conjecture 5.34 *With notation as above, suppose the generalised G-igsaw transformation from M to N is θ-stable, i.e., $\exists\, \theta \in \theta(M) \cap \theta(N) \cap \mathrm{Orth}(M')$. Then there is a notion of $\lim_{v\to\infty}$, determined in some way by θ, such that the ideals $I' := \lim_{v\to\infty} I(v)$ and $J' := \lim_{v\to\infty} J(v)$ are monomial ideals for which the set $\Gamma(I',J')$ is the G-graph of N. Moreover, the cones $\sigma(M)$ and $\sigma(N)$ share the codimension 1 face $\tau = \sigma(M) \cap \sigma(N)$ so lie adjacent in $L \otimes \mathbb{R} \cong \mathbb{R}^3$.*

Remark 5.35 As with Nakamura's calculation of G-Hilb $\mathbb{C}^3$, much of the work described in this chapter can be generalised to the case of a finite Abelian subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$. Of course, Ishii's Theorem 5.19 no longer applies in this case so the affine toric varieties $X_{L,\sigma(M)}$ defined by deforming a G-constellation M are not necessarily smooth. Moreover, the computations become considerably more difficult because the universal cover of the McKay quiver is no longer planar. Still, these calculations may provide a deeper understanding of, say, the relation between G-Hilb $\mathbb{C}^4$ and other partial resolutions Y of $\mathbb{C}^4/G$, for a finite Abelian subgroup $G \subset \mathrm{SL}(4, \mathbb{C})$.

5.8 Further examples

The monomials in a G-graph correspond one-to-one with the characters of G. For a cyclic group $G \cong \mathbb{Z}/r$, it is convenient to list the monomials in 'cyclic order', beginning with the G-invariant monomial.

5.8.1 Flops of G-Hilb $\mathbb{C}^3$ for the G-action $\frac{1}{6}(1, 2, 3)$

Consider the action of $G \cong \mathbb{Z}/6$ on $\mathbb{C}^3$ giving rise to the quotient singularity $\frac{1}{6}(1, 2, 3)$. The junior simplex of $\mathbb{C}^3/G$ admits 5 basic triangulations; two are shown in Figure 5.6; the remaining three are shown in Figures 5.8 and 5.9.

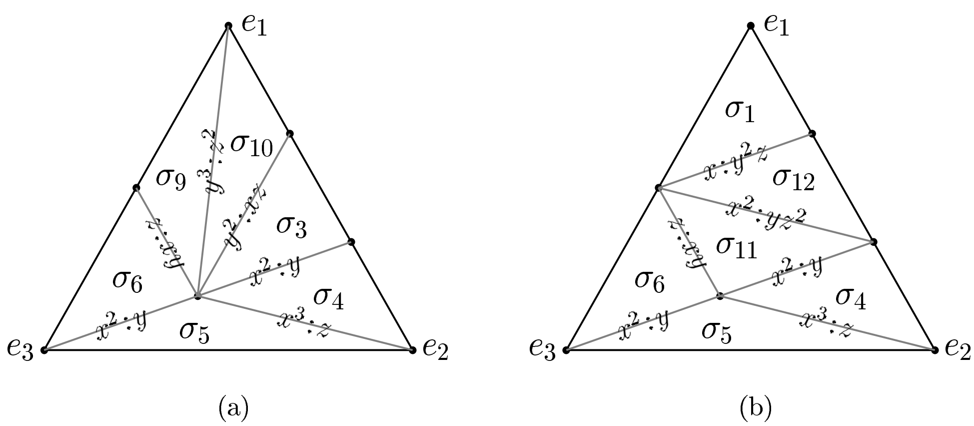


Figure 5.8: (a) Σ_2 of Flop 2; (b) Σ_3 of Flop 3 for $\frac{1}{6}(1, 2, 3)$

The mysterious first step for $\Sigma' := \Sigma_2$ uses the cones $\sigma' = \sigma_9 \in \Sigma' \setminus \Sigma$ and $\sigma = \sigma_1 \in \Sigma \setminus \Sigma'$ to construct $M' := M_9 = \{y^3, y^2z, y, z, y^2, yz\}$. The module generators y, z lie in the $\varepsilon^2, \varepsilon^3$ character spaces so $I_{\text{gen}} := \{2, 3\}$. Carry out a sequence of generalised G-igsaw transformations according to Procedure 5.24 gives rise to the G-constellations listed in Table 5.2; these modules are θ-stable with respect to (an open subset of Π containing) $\theta = (1, 1, -3, -4, 1, 4)$.

i	Generators	M_i	$\mathrm{Def}(M_i)$	σ_i
9	$\langle y, z\rangle$	$\{y^3, y^2z, y, z, y^2, yz\}$	$xy/z, y^3, z^2/y^3$	σ_9
10	$\langle y, z\rangle$	$\{z^2, y^2z, y, z, y^2, yz\}$	$xz/y^2, y^3/z^2, z^2$	σ_{10}
3	$\langle y, z\rangle$	$\{z^2, xz^2, y, z, xz, yz\}$	$x^2/y, y^2/xz, z^2$	σ_3
4	$\langle x^2, z\rangle$	$\{z^2, xz^2, x^2, z, xz, x^2z\}$	$x^3/z, y/x^2, z^2$	σ_4
5	$\langle x^2\rangle$	$\{x^6, x^7, x^2, x^3, x^4, x^5\}$	$x^6, y/x^2, z/x^3$	σ_5
6	$\langle y\rangle$	$\{y^3, xy^3, y, xy, y^2, xy^2\}$	$x^2/y, y^3, z/xy$	σ_6

Table 5.2: G-constellations M_i and cones σ_i for Flop 2 of $\frac{1}{6}(1,2,3)$

For Flop 3 with fan $\Sigma' := \Sigma_3$, the mysterious first step begins with $\sigma' = \sigma_{12}$ and $\sigma = \sigma_3$ to give $M' := M_{12} = \{z^2, x, yz^2, z, xz, yz\}$. The generators x, z lie in the $\varepsilon, \varepsilon^3$ character spaces so $I_{\text{gen}} := \{1, 3\}$. Procedure 5.24 produces the modules and cones listed in Table 5.3. These G-constellations are θ-stable with respect to (an open subset of Π containing) $\theta = (1, -2, 1, -4, 3, 1)$.

i	Generators	M_i	$\mathrm{Def}(M_i)$	σ_i
1	$\langle z\rangle$	$\{z^2, y^2z, yz^2, z, y^2z^2, yz\}$	$x/y^2z, y^3, z^2$	σ_1
12	$\langle x, z\rangle$	$\{z^2, x, yz^2, z, xz, yz\}$	$x^2/yz^2, y^2z/x, z^2$	σ_{12}
11	$\langle x, z\rangle$	$\{z^2, x, x^2, z, xz, yz\}$	$x^2/y, xy/z, yz^2/x$	σ_{11}
4	$\langle x, z\rangle$	$\{z^2, x, x^2, z, xz, x^2z\}$	$x^3/z, y/x^2, z^2$	σ_4
5	$\langle x\rangle$	$\{x^6, x, x^2, x^3, x^4, x^5\}$	$x^6, y/x^2, z/x^3$	σ_5
6	$\langle x\rangle$	$\{x^2y^2, x, x^2, xy, x^2y, xy^2\}$	$x^2/y, y^3, z/xy$	σ_6

Table 5.3: G-constellations M_i and cones σ_i for Flop 3 of $\frac{1}{6}(1,2,3)$

The fourth fan $\Sigma' := \Sigma_4$ shown in Figure 5.9 contains three cones σ_{12}, σ_{13}, σ_{14} which do not lie in the fan Σ of G-Hilb $\mathbb{C}^3$. To perform the mysterious

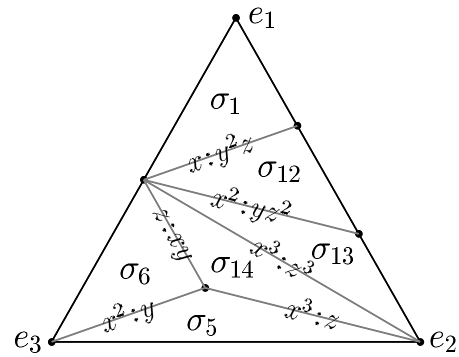


Figure 5.9: The toric fan Σ' of Flop 4 of G-Hilb $\mathbb{C}^3$

first step set σ' to be either σ_{13} or σ_{14} (beginning with σ_{12} leads via Procedure 5.24 to the fan Σ_3 shown in Figure 5.8(b); see Remark 5.36). Choosing $\sigma' = \sigma_{13}$ and $\sigma = \sigma_6$ gives $M' := M_{13} = \{z^2, xz^2, x^2, z^3, xz, x^2z\}$. The module generators x^2, xz, z^2 lie in the $\varepsilon^0, \varepsilon^2, \varepsilon^4$ character spaces so $I_{\text{gen}} := \{0, 2, 4\}$. Procedure 5.24 produces the G-constellations listed in Table 5.4; these modules are θ-stable with respect to $\theta = (-3, 4, -2, 1, -3, 3)$.

i	Generators	M_i	$\text{Def}(M_i)$	σ_i
1	$\langle z^2\rangle$	$\{z^2, y^2z^3, yz^2, z^3, y^2z^2, yz^3\}$	$x/y^2z, y^3, z^2$	σ_1
12	$\langle xz, z^2\rangle$	$\{z^2, xz^2, yz^2, z^3, xz, yz^3\}$	$x^2/yz^2, y^2z/x, z^2$	σ_{12}
13	$\langle x^2, xz, z^2\rangle$	$\{z^2, xz^2, x^2, z^3, xz, x^2z\}$	$x^3/z^3, yz^2/x, z^2$	σ_{13}
14	$\langle x^2, xz, z^2\rangle$	$\{z^2, xz^2, x^2, x^3, xz, x^2z\}$	$x^3/z, xy/z, z^3/x^3$	σ_{14}
5	$\langle x^2\rangle$	$\{x^6, x^7, x^2, x^3, x^4, x^5\}$	$x^6, y/x^2, z/x^3$	σ_5
6	$\langle x^2\rangle$	$\{x^2y^2, x^3y^2, x^2, x^3, x^2y, x^3y\}$	$x^2/y, y^3, z/xy$	σ_6

Table 5.4: G-constellations M_i and cones σ_i for Flop 4 of $\frac{1}{6}(1,2,3)$

Remark 5.36 The cones $\sigma_{13}, \sigma_{14} \in \Sigma_4$ are distinguished in the sense that they arise only after performing two flops of G-Hilb $\mathbb{C}^3$, whereas $\sigma_{12} \in \Sigma_4$ also appears in Σ_3 which can be reached from G-Hilb $\mathbb{C}^3$ by a single flop. For more on this point see Remark 5.37.

5.8.2 Flops of G-Hilb $\mathbb{C}^3$ for the G-action $\frac{1}{11}(1,2,8)$

Consider the action of $G \cong \mathbb{Z}/11$ on $\mathbb{C}^3$ defining the singularity $\frac{1}{11}(1,2,8)$. The fan Σ defining G-Hilb $\mathbb{C}^3$ is shown in Figure 5.10.

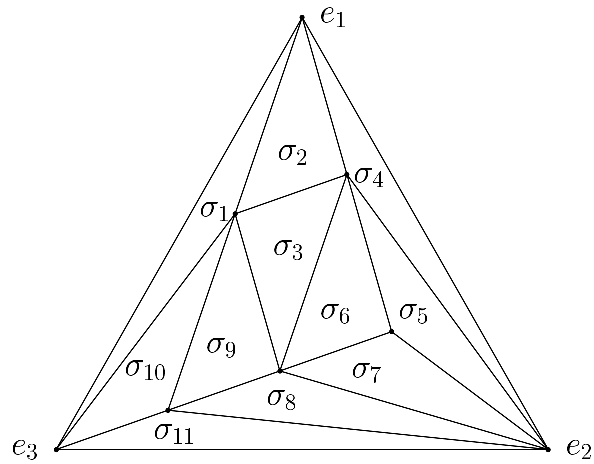


Figure 5.10: The fan Σ of G-Hilb $\mathbb{C}^3$ for the $G = \mathbb{Z}/11$-action

The junior simplex of $\mathbb{C}^3/G$ admits 4 other basic triangulations shown in Figures 5.11 and 5.12. For the first of these fans $\Sigma' := \Sigma_1$ shown in Figure 5.11(a), choose cones $\sigma' = \sigma_{12}$ and $\sigma = \sigma_2 \in \Sigma$. The mysterious first step defines $M' := M_{12} = \{y^3z^2, y^2z, z^3, y^3z, y^2, z^2, y^3, yz^2, y^4, y^2z^2, y^5\}$ generated by the monomials y^2 and z^2, so $I_{\text{gen}} = \{4,5\}$. Procedure 5.24 produces the G-constellations and cones listed in Table 5.5 (module generators are underlined). It can be shown that these G-constellations are θ-stable with respect to (an open subset of Π containing) the parameter $\theta = (1,1,1,1,-7,-9,1,1,1,8,1)$.

The second fan $\Sigma' := \Sigma_2$ is shown in Figure 5.11(b). For $\sigma' = \sigma_{14}$ and $\sigma = \sigma_6$ we get $M' := M_{14} = \{xyz, x, z^3, xy, yz^3, z^2, xz^2, yz^2, z, xz, yz\}$ with generators x, z, so $I_{\text{gen}} = \{1,8\}$. Procedure 5.24 produces the G-constellations and cones listed in Table 5.6. These G-constellations are θ-stable with respect to (an open subset of Π containing) the parameter $\theta = (1,-9,1,1,1,1,1,1,-6,7,1)$.

The third fan $\Sigma' := \Sigma_3$ is shown in Figure 5.12(a). Choosing $\sigma' = \sigma_{16}$ and $\sigma = \sigma_9$ gives $M' := M_{16} = \{xyz, x, x^2, xy, y^2, xy^2, y^3, xy^3, y^4, xz, y^5\}$ with generators x, y^2, so $I_{\text{gen}} = \{1,4\}$. Procedure 5.24 produces the G-constellations and cones listed in Table 5.7. These G-constellations are θ-

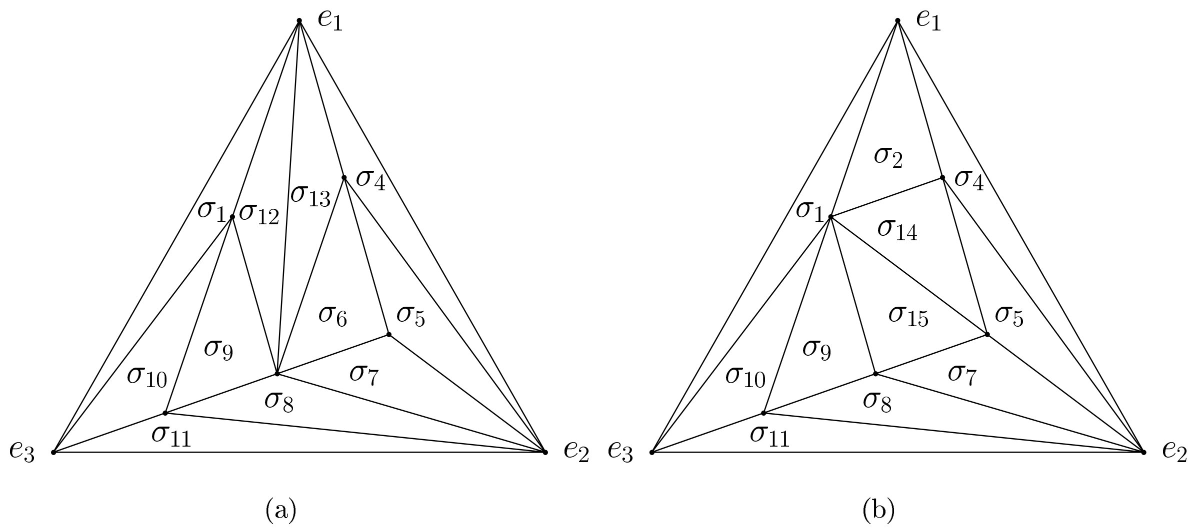


Figure 5.11: (a) Σ_1 of Flop 1; (b) Σ_2 of Flop 3 for $\frac{1}{11}(1,2,8)$

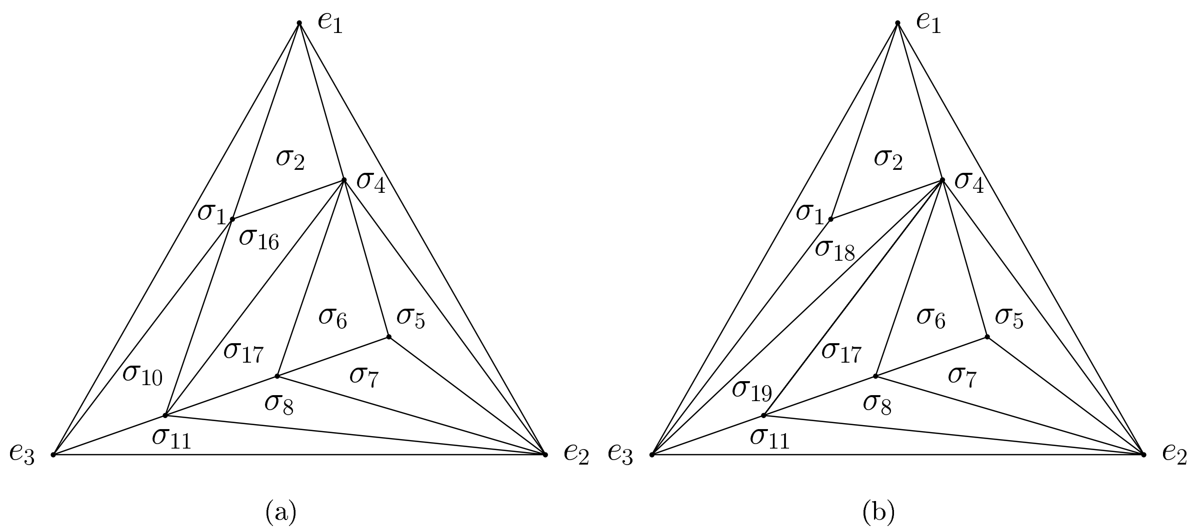


Figure 5.12: (a) Σ_3 of Flop 3; (b) Σ_4 of Flop 4 for $\frac{1}{11}(1,2,8)$

stable with respect to (an open subset of Π containing) the parameter $\theta = (1,-9,1,1,-4,5,1,1,1,1,1)$.

G-constellations M_i	$\mathrm{Def}(M_i)$	σ_i
$\{y^3z^2, y^2z, z^3, y^3z, \underline{y^2}, \underline{z^2}, y^3, yz^2, y^4, y^2z^2, y^5\}$	$xy^2/z^2, y^4/z, z^4/y^5$	σ_{12}
$\{y^3z^2, y^2z, z^3, y^3z, \underline{y^2}, \underline{z^2}, y^3, yz^2, y^4, y^2z^2, z^4\}$	$xz^2/y^3, y^5/z^4, z^3/y$	σ_{13}
$\{y^{11}, y^6, y^{12}, y^7, \underline{y^2}, y^8, y^3, y^9, y^4, y^{10}, y^5\}$	$x/y^6, y^{11}, z/y^4$	σ_1
$\{z^{11}, z^7, z^3, z^{10}, z^6, \underline{z^2}, z^9, z^5, z^{12}, z^8, z^4\}$	$x/z^7, y/z^3, z^{11}$	σ_4
$\{xz^4, z^7, z^3, xz^3, z^6, \underline{z^2}, xz^2, z^5, xz^5, z^8, z^4\}$	$x^2/z^3, y/z^3, z^7/x$	σ_5
$\{xz^4, y^2z, z^3, xz^3, \underline{y^2}, \underline{z^2}, xz^2, yz^2, xyz^2, y^2z^2, z^4\}$	$x^2/y, y^3/xz^2, z^3/y$	σ_6
$\{xz^4, x^4z, z^3, xz^3, \underline{x^4}, \underline{z^2}, xz^2, x^2z^2, x^3z^2, x^4z^2, z^4\}$	$x^5/z^2, y/x^2, z^3/x^2$	σ_7
$\{x^{11}, x^4z, x^5z, x^6z, \underline{x^4}, x^5, x^6, x^7, x^8, x^9, x^{10}\}$	$x^8/z, y/x^2, z^2/x^5$	σ_8
$\{xy^5, y^2z, xy^2z, y^3z, \underline{y^2}, xy^2, y^3, xy^3, y^4, xy^4, y^5\}$	$x^2/y, y^4/z, z^2/xy^2$	σ_9
$\{xy^5, y^6, xy^6, y^7, \underline{y^2}, xy^2, y^3, xy^3, y^4, xy^4, y^5\}$	$x^2/y, y^6/x, z/y^4$	σ_{10}
$\{x^{11}, x^{12}, x^{13}, x^{14}, \underline{x^4}, x^5, x^6, x^7, x^8, x^9, x^{10}\}$	$x^{11}, y/x^2, z/x^8$	σ_{11}

Table 5.5: G-constellations M_i and cones σ_i for Flop 1 of $\frac{1}{11}(1,2,8)$

G-constellations M_i	$\mathrm{Def}(M_i)$	σ_i
$\{xyz, \underline{x}, z^3, xy, yz^3, z^2, xz^2, yz^2, \underline{z}, xz, yz\}$	$x^2/z^3, y^2z/x, z^3/y$	σ_{14}
$\{xyz, \underline{x}, x^2, xy, x^2y, z^2, xz^2, yz^2, \underline{z}, xz, yz\}$	$x^2/y, xy^2/z^2, z^3/x^2$	σ_{15}
$\{y^{11}, y^6, y^{12}, y^7, y^{13}, y^8, y^{14}, y^9, \underline{y^4}, y^{10}, y^5\}$	$x/y^6, y^{11}, z/y^4$	σ_1
$\{y^3z^2, y^2z, z^3, y^3z, yz^3, z^2, y^2z^3, yz^2, \underline{z}, y^2z^2, yz\}$	$x/y^2z, y^4/z, z^3/y$	σ_2
$\{z^{11}, z^7, z^3, z^{10}, z^6, z^2, z^9, z^5, \underline{z}, z^8, z^4\}$	$x/z^7, y/z^3, z^{11}$	σ_4
$\{xz^4, \underline{x}, z^3, xz^3, z^6, z^2, xz^2, z^5, \underline{z}, xz, z^4\}$	$x^2/z^3, y/z^3, z^7/x$	σ_5
$\{x^3z, \underline{x}, x^2, x^3, x^4, z^2, xz^2, x^2z^2, z, xz, x^2z\}$	$x^5/z^2, y/x^2, z^3/x^2$	σ_7
$\{x^3z, \underline{x}, x^2, x^3, x^4, x^5, x^6, x^7, z, xz, x^2z\}$	$x^8/z, y/x^2, z^2/x^5$	σ_8
$\{xyz, x, x^2, xy, x^2y, xy^2, x^2y^2, xy^3, \underline{z}, xz, yz\}$	$x^2/y, y^4/z, z^2/xy^2$	σ_9
$\{xy^5, \underline{x}, x^2, xy, x^2y, xy^2, x^2y^2, xy^3, \underline{y^4}, xy^4, y^5\}$	$x^2/y, y^6/x, z/y^4$	σ_{10}
$\{x^{11}, \underline{x}, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}\}$	$x^{11}, y/x^2, z/x^8$	σ_{11}

Table 5.6: G-constellations M_i and cones σ_i for Flop 2 of $\frac{1}{11}(1,2,8)$

The fourth fan $\Sigma' := \Sigma_4$ is shown in Figure 5.12(b). For $\sigma' = \sigma_{18}$ and $\sigma = \sigma_{10}$ we get $M' := M_{18}$ (see below) with generators x^2, xy^2, y^4, so $I_{\mathrm{gen}} = \{2, 5, 8\}$. Procedure 5.24 produces the modules and cones listed in Table 5.8 (deformation parameters are omitted for lack of space, but you've got the idea by now). These G-constellations are θ-stable with respect to (an open subset of Π containing) the parameter $\theta = (1, 1, -8, 1, 1, -7, 8, 1, -4, 5, 1)$.

G-constellations M_i	$\mathrm{Def}(M_i)$	σ_i
$\{xyz, \underline{x}, x^2, xy, \underline{y^2}, xy^2, y^3, xy^3, y^4, xz, y^5\}$	$x^2z/y^5, y^4/z, y^2z/x$	σ_{16}
$\{xyz, \underline{x}, x^2, xy, \underline{y^2}, xy^2, y^3, xy^3, y^4, xz, x^2z\}$	$x^2/y, y^5/x^2z, xz^2/y^3$	σ_{17}
$\{y^{11}, y^6, y^{12}, y^7, \underline{y^2}, y^8, y^3, y^9, y^4, y^{10}, y^5\}$	$x/y^6, y^{11}, z/y^4$	σ_1
$\{y^3z^2, y^2z, y^4z^2, y^3z, \underline{y^2}, y^4z, y^3, y^5z, y^4, y^2z^2, y^5\}$	$x/y^2z, y^4/z, z^3/y$	σ_2
$\{z^{11}, z^7, z^{14}, z^{10}, \underline{z^6}, z^{13}, z^9, z^{16}, z^{12}, z^8, z^{15}\}$	$x/z^7, y/z^3, z^{11}$	σ_4
$\{xz^4, \underline{x}, x^2, xz^3, \underline{z^6}, xz^6, xz^2, x^2z^2, xz^5, xz, x^2z\}$	$x^2/z^3, y/z^3, z^7/x$	σ_5
$\{xyz, \underline{x}, x^2, xy, \underline{y^2}, xy^2, xz^2, x^2z^2, xyz^2, xz, x^2z\}$	$x^2/y, y^3/xz^2, z^3/y$	σ_6
$\{x^3z, \underline{x}, x^2, x^3, x^4, x^5, xz^2, x^2z^2, x^3z^2, xz, x^2z\}$	$x^5/z^2, y/x^2, z^3/x^2$	σ_7
$\{x^3z, \underline{x}, x^2, x^3, x^4, x^5, x^6, x^7, x^8, xz, x^2z, \}$	$x^8/z, y/x^2, z^2/x^5$	σ_8
$\{xy^5, \underline{x}, x^2, xy, \underline{y^2}, xy^2, y^3, xy^3, y^4, xy^4, y^5\}$	$x^2/y, y^6/x, z/y^4$	σ_{10}
$\{x^{11}, \underline{x}, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}\}$	$x^{11}, y/x^2, z/x^8$	σ_{11}

Table 5.7: G-constellations M_i and cones σ_i for Flop 3 of $\frac{1}{11}(1,2,8)$

Remark 5.37 The fan Σ_4 contains three cones $\sigma_{17}, \sigma_{18}, \sigma_{19}$ which do not lie in the fan Σ of G-Hilb $\mathbb{C}^3$ shown in Figure 5.10. As with Flop 4 for $\frac{1}{6}(1,2,3)$ in §5.8.1, two of these cones, namely σ_{18}, σ_{19}, are distinguished in the sense that they arise only after performing two flops of G-Hilb $\mathbb{C}^3$, whereas $\sigma_{17} \in \Sigma_4$ also appears in Σ_3 which can be reached from G-Hilb $\mathbb{C}^3$ by a single flop.

This observation may lead to an obstruction to successful implementation of the mysterious first step, or it may provide further evidence that the Procedure 5.24 can always be implemented. For instance, what if two varieties $X_{L,\Sigma'}, X_{L,\Sigma''}$ can be reached from G-Hilb $\mathbb{C}^3$ by the same number of flops and yet their fans Σ', Σ'' contain a common cone σ' which does not lie in the fan Σ of G-Hilb $\mathbb{C}^3$? It may be that the mysterious first step breaks down when applied to this cone. On the other hand, there are choices involved in the mysterious first step and it may be that different choices of the sets S, S' introduced in §5.4 give different modules M' and M'' leading to Σ' and Σ'' respectively. This point requires further investigation.

G-constellation M_i	σ_i
$\{xy^5, y^6, \underline{x^2}, y^7, x^2y, \underline{xy^2}, x^2y^2, xy^3, \underline{y^4}, xy^4, y^5\}$	σ_{18}
$\{xy^5, y^6, \underline{x^2}, x^3, x^2y, \underline{xy^2}, x^2y^2, xy^3, \underline{y^4}, xy^4, y^5\}$	σ_{19}
$\{y^{11}, y^6, y^{12}, y^7, y^{13}, y^8, y^{14}, y^9, \underline{y^4}, y^{10}, y^5\}$	σ_1
$\{y^7z, y^6, y^4z^2, y^7, y^5z^2, y^4z, y^6z^2, y^5z, \underline{y^4}, y^6z, y^5\}$	σ_2
$\{z^{22}, z^{18}, z^{14}, z^{21}, z^{17}, z^{13}, z^{20}, z^{16}, \underline{z^{12}}, z^{19}, z^{15}\}$	σ_4
$\{x^3z, x^2z^4, \underline{x^2}, x^3, x^2z^3, xz^6, x^2z^6, x^2z^2, \underline{xz^5}, x^2z^5, x^2z\}$	σ_5
$\{x^3z, x^2yz, \underline{x^2}, x^3, x^2y, \underline{xy^2}, x^2y^2, x^2z^2, \underline{xyz^2}, x^2yz^2, x^2z\}$	σ_6
$\{x^3z, x^4z, \underline{x^2}, x^3, x^4, x^5, x^6, x^2z^2, x^3z^2, x^4z^2, x^2z\}$	σ_7
$\{x^3z, x^4z, \underline{x^2}, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^2z\}$	σ_8
$\{x^{11}, x^{12}, \underline{x^2}, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}\}$	σ_{11}
$\{x^3z, x^2yz, \underline{x^2}, x^3, x^2y, \underline{xy^2}, x^2y^2, xy^3, \underline{y^4}, xy^4, x^2z\}$	σ_{17}

Table 5.8: G-constellations M_i and cones σ_i for Flop 4 of $\frac{1}{11}(1, 2, 8)$

Appendix A

Why 'motivic' integration?

In this appendix we investigate the motivic nature of the integral introduced in Chapter 2. We also justify the notation $\mathbb{L}$ for the class of the complex line $\mathbb{C}$ in the Grothendieck ring of algebraic varieties.

The category $\mathcal{M}_{\mathbb{C}}$ of Chow motives over $\mathbb{C}$ is defined as follows (see [Sch94]): an object is a triple (X, p, m) where X is a smooth, complex projective variety of dimension d, p is an element of the Chow ring $A^d(X \times X)$ which satisfies $p^2 = p$ and $m \in \mathbb{Z}$. If (X, p, m) and (Y, q, n) are motives then

$$\mathrm{Hom}_{\mathcal{M}_{\mathbb{C}}}\left((X, p, m), (Y, q, n)\right) = qA^{d+n-m}(X, Y)p$$

where composition of morphisms is given by composition of correspondences. $\mathcal{M}_{\mathbb{C}}$ is additive, $\mathbb{Q}$-linear and pseudo-abelian. Tensor product of motives is defined as $(X, p, m) \otimes (Y, q, n) = (X \times Y, p \otimes q, m + n)$. There is a functor

$$h\colon \mathcal{V}_{\mathbb{C}}^{\circ} \to \mathcal{M}_{\mathbb{C}}$$

which sends X to $(X, \Delta_X, 0)$, the Chow motive of X, where the diagonal $\Delta_X \subset X \times X$ is the identity in $A^*(X \times X)$. The motive of a point $1 = h(\mathrm{Spec}\ \mathbb{C})$ is the identity with respect to tensor product. The Lefschetz motive $\mathbb{L}$ is defined implicitly via the relation $h(\mathbb{P}^1_{\mathbb{C}}) = 1 \oplus \mathbb{L}$.

Definition A.1 The Grothendieck group of $\mathcal{M}_{\mathbb{C}}$ is the free abelian group generated by isomorphism classes of objects in $\mathcal{M}_{\mathbb{C}}$ modulo the subgroup generated by elements of the form $[(X, p, m)] - [(Y, q, n)] - [(Z, r, k)]$ whenever $(X, p, m) \simeq (Y, q, n) \oplus (Z, r, k)$. Tensor product of motives induces a ring structure and the resulting ring, denoted $K_0(\mathcal{M}_{\mathbb{C}})$, is the Grothendieck ring of Chow Motives (over $\mathbb{C}$).

Gillet and Soulé [GS96] exhibit a map

$$M\colon \mathcal{V}_{\mathbb{C}} \longrightarrow K_0(\mathcal{M}_{\mathbb{C}})$$

which sends a smooth, projective variety X to the class $[h(X)]$ of the motive of X. Furthermore the map is additive on disjoint unions of locally closed subsets and satisfies $M(X \times Y) = M(X) \cdot M(Y)$.

We now play the same game as we did in §2.3. Namely, M factors through $K_0(\mathcal{V}_{\mathbb{C}})$ inducing

$$M\colon K_0(\mathcal{V}_{\mathbb{C}}) \longrightarrow K_0(\mathcal{M}_{\mathbb{C}}).$$

Observe that the image of $[\mathbb{C}]$ under M is the class of the Lefschetz motive $\mathbb{L}$; this explains why we use the notation $\mathbb{L}$ to denote the class of $\mathbb{C}$ in $K_0(\mathcal{V}_{\mathbb{C}})$ in § 2. Sending $\mathbb{L}^{-1} \in K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]$ to $\mathbb{L}^{-1} \in K_0(\mathcal{M}_{\mathbb{C}})$ produces a map

$$M\colon K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}] \longrightarrow K_0(\mathcal{M}_{\mathbb{C}}).$$

At present it is unknown whether or not M annihilates the kernel of the natural completion map $\phi\colon K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}] \to R$. Denef and Loeser conjecture that it does (see [DL98, Remark 1.2.3]). If this is true, extend M to a ring homomorphism

$$M_{\mathrm{st}}\colon \phi\left(K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]\right)\left[\left\{\frac{1}{\mathbb{L}^i-1}\right\}_{i\in\mathbb{N}}\right] \longrightarrow K_0(\mathcal{M}_{\mathbb{C}})\left[\left\{\frac{1}{\mathbb{L}^i-1}\right\}_{i\in\mathbb{N}}\right]$$

such that the image of $[D_J^\circ]$ under M_{st} is equal to $M(D_J^\circ)$.

Definition A.2 Let X denote a complex algebraic variety with at worst canonical, Gorenstein singularities and let $\varphi\colon Y \to X$ be any resolution of singularities for which the discrepancy divisor $D = \sum a_i D_i$ has only simple normal crossings. The *stringy motive* of X is

$$\begin{aligned} M_{\mathrm{st}}(X) &:= M_{\mathrm{st}}\left(\int_{J_\infty(Y)} F_D \,\mathrm{d}\mu \cdot \mathbb{L}^n\right) \\ &= \sum_{J\subseteq\{1,\ldots,r\}} M(D_J^\circ)\cdot\left(\prod_{j\in J}\frac{\mathbb{L}-1}{\mathbb{L}^{a_j+1}-1}\right) \end{aligned}$$

where we sum over all subsets $J \subseteq \{1,\ldots,r\}$ including $J = \emptyset$. As with the definition of the stringy E-function (see Definition 2.25) we multiply by $\mathbb{L}^n$ for convenience.

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