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Convex and Algebraic Geometry

Organised by
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ABSTRACT. The subjects of convex and algebraic geometry meet primarily in the theory of toric varieties. Toric geometry is the part of algebraic geometry where all maps are given by monomials in suitable coordinates, and all equations are binomial. The combinatorics of the exponents of monomials and binomials is sufficient to embed the geometry of lattice polytopes in algebraic geometry. Recent developments in toric geometry that were discussed during the workshop include applications to mirror symmetry, motivic integration and hypergeometric systems of PDE's, as well as deformations of (unions of) toric varieties and relations to tropical geometry.

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Introduction by the Organisers

The workshop *Convex and Algebraic Geometry* was organized by Klaus Altmann (Berlin), Victor Batyrev (Tübingen), and Bernard Teissier (Paris). Both title subjects meet primarily in the theory of toric varieties. These constitute the part of algebraic geometry where all maps are given by monomials in suitable coordinates, and all equations are binomial. The combinatorics of the exponents of monomials and binomials is sufficient to embed the geometry of lattice polytopes in algebraic geometry. Thus, toric geometry and its several generalizations provide a kind of section from polyhedral into algebraic geometry. While this reflects only a thin slice of algebraic geometry, it is general enough to display many important phenomena, techniques, and methods. It serves as a wonderful testing ground for general theories such as the celebrated mirror symmetry in its different flavours. In particular, much of the popularity of toric geometry originates in mathematical physics.

The meeting was attended by almost 50 participants from many European countries, Canada, the USA, and Japan. The program consisted of talks by 23 speakers, among them many young researchers. Most subjects fit more or less into the following main areas:

- *Derived categories, quivers, and (homological) mirror symmetry* (Bondal, Craw, Horja, Maclagan, Perling, Siebert, Ueda)

One of the major discussions during the meeting concerned the existence of strongly exceptional sequences on toric varieties which consist of line bundles. A full exceptional sequence provides a kind of “basis” for the derived category. While Hille and Perling presented an example that does not carry such a sequence of full length, Bondal suggested a method to link this question to sheaves on the dual real torus that are constructible with respect to a certain stratification.

In general, one expects to gain exceptional sequences from the universal bundles on moduli spaces. Using this method, Craw constructs those sequences on smooth toric Fano threefolds. In this context, Maclagan and Ueda consider the case of three-dimensional abelian quotient singularities. Ueda investigates the Fukaya category of the corresponding potential on the dual torus explicitly.

Using mirror symmetry, Horja establishes a connection between the orbifold K -theory of toric Deligne-Mumford stacks and solutions to GKZ-hypergeometric D -modules.

- *Degenerations and deformations* (Brown, Hausen, Siebert, Süß, Vollmert)

Gross and Siebert have developed a program to understand mirror symmetry as the duality of certain degeneration data. The special fibers split into toric components, and the degeneration is encoded in a topological manifold B with an affine and a polytopal structure. Duality is now inherited from discrete geometry, and the topology of B reflects the topology of the general fiber. In particular, if B is a (\mathbb{Q} -homology) $\mathbb{P}_{\mathbb{C}}^n$, then this construction might lead to (compact) Hyperkähler varieties.

Considering, in a special case, a certain contraction of the total space of these families leads to a description of torus actions on algebraic varieties via divisors on their Chow quotients. These divisors carry polytopes or even polyhedral complexes as their coefficients, compare the talks of Hausen, Süß, and Vollmert. In a similar setting, but with an explicit manipulation of Pfaffians, Brown and Reid construct smoothings of certain non-isolated singularities giving rise to four-dimensional flips.

- *Tropical geometry and Welschinger invariants* (Itenberg, Shustin, Siebert)

The most rigorous degeneration of a variety is the tropical one. Here, everything takes place over the so-called tropical semiring, and one ends up with piecewise linear spaces. In fact, Siebert’s degeneration data mentioned above correspond to these objects.

Itenberg and Shustin use this approach to calculate the Welschinger invariants, which are a kind of real version of Gromov-Witten invariants. Along the lines of the method of Gathmann and Markwig, there is a recursive formula for these

invariants. In the case of del Pezzo surfaces, it turns out that both invariants are (log-)asymptotically equivalent.

- *Commutative algebra, GKZ-systems, and polytopes* (Bruns, Haase, Hering, Horja, Miller, Pasquier, Stienstra)

A generalization of toric varieties in a different direction from the torus actions mentioned above is given by the notion of spherical varieties. Pasquier considers horospherical Fano varieties and comes up with an adapted notion of (generalized, coloured) reflexive polytopes. Bruns, Haase, and Hering deal with ordinary polytopes and their relations to syzygies of toric varieties.

For an integral matrix A one obtains a semigroup algebra $\mathbb{C}[NA]$ (leading to the usual affine toric variety) and a GKZ-hypergeometric system of differential equations. The latter depends on a parameter β , and Miller has reported on a result that relates the set of β where the rank of the system jumps to the set of those multidegrees where the semigroup algebra $\mathbb{C}[NA]$ carries local cohomology. In particular, the Cohen-Macaulay property is equivalent to the constant rank condition, answering an old question of Sturmfels.

One of the nighttime discussions gave rise to the suggestion to not include normality in the definition of a toric variety, thus overcoming the cumbersome term of a “not necessarily normal toric variety”.

The workshop was closed on Friday night by an informal piano recital by Benjamin Nill and Milena Hering featuring Strawinsky, Liszt, and Chopin.

Workshop: Convex and Algebraic Geometry

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Abstracts

Polyhedral Divisors and Algebraic Torus Actions

JÜRGEN HAUSEN

(joint work with Klaus Altmann)

This is a report on the paper [1]. There we present a complete description of normal affine varieties X with an effective action of an algebraic torus in terms of what we call proper polyhedral divisors on semiprojective varieties. Our approach extends classical cone constructions of Dolgachev [3], Demazure [2] and Pinkham [5], and it comprises the theory of affine toric varieties.

Let Y be a normal semiprojective variety, where “semiprojective” merely means that Y is projective over some affine variety. In order to introduce the notion of a proper polyhedral divisor on Y , consider a linear combination

$$\mathfrak{D} = \sum \Delta_i \otimes D_i$$

where the D_i are prime divisors on Y , the coefficients Δ_i are convex polyhedra in a rational vector space $N_{\mathbb{Q}} = \mathbb{Q} \otimes N$ with a free finitely generated abelian group N , and all Δ_i have a common pointed cone $\sigma \subset N_{\mathbb{Q}}$ as their recession cone.

Let $M := \text{Hom}(N, \mathbb{Z})$ be the dual of N , and write $\sigma^{\vee} \subset M_{\mathbb{Q}}$ for the dual cone. Then the above \mathfrak{D} defines an evaluation map into the group of rational Weil divisors on Y :

$$\sigma^{\vee} \rightarrow \text{WDiv}(Y), \quad u \mapsto \mathfrak{D}(u) := \sum_{v \in \Delta_i} \min \langle u, v \rangle D_i.$$

We say that \mathfrak{D} is a *proper polyhedral divisor* if any evaluation $\mathfrak{D}(u)$ is a semiample rational Cartier divisor, being big whenever u belongs to the relative interior of the cone σ^{\vee} .

The evaluation map $u \mapsto \mathfrak{D}(u)$ turns out to be piecewise linear and convex in the sense that the difference $\mathfrak{D}(u + u') - \mathfrak{D}(u) + \mathfrak{D}(u')$ is always effective. This convexity property enables us to define a graded algebra of global sections:

$$A := \bigoplus_{u \in \sigma^{\vee} \cap M} \Gamma(Y, \mathcal{O}(\mathfrak{D}(u))).$$

This ring turns out to be normal and finitely generated. Thus, it gives rise to a normal affine variety $X := \text{Spec}(A)$, and the M -grading of A defines an effective action of the torus $T := \text{Spec}(\mathbb{C}[M])$ on X .

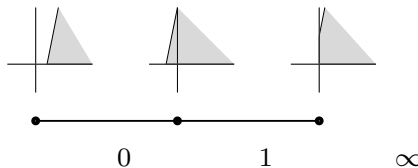
Example. Let $Y = \mathbb{P}^1$ and $N = \mathbb{Z}^2$. The vectors $(1, 0)$ and $(1, 12)$ generate a pointed convex cone σ in $N_{\mathbb{Q}} = \mathbb{Q}^2$, and we consider the polyhedra

$$\Delta_0 = \left(\frac{1}{3}, 0 \right) + \sigma, \quad \Delta_1 = \left(-\frac{1}{4}, 0 \right) + \sigma, \quad \Delta_{\infty} = (\{0\} \times [0, 1]) + \sigma.$$

Attaching these polyhedra as coefficients to the points $0, 1, \infty$ on the projective line, we obtain a proper polyhedral divisor

$$\mathfrak{D} = \Delta_0 \otimes \{0\} + \Delta_1 \otimes \{1\} + \Delta_\infty \otimes \{\infty\}.$$

In this situation, we may even represent our proper polyhedral divisor by a little picture as follows:



The proper polyhedral divisor \mathfrak{D} describes the affine threefold $X = V(z_1^3 + z_2^4 + z_3 z_4)$ in \mathbb{C}^4 with the action of $T = (\mathbb{C}^*)^2$ given by

$$t \cdot z = (t_1^4 z_1, t_1^3 z_2, t_2 z_3, t_1^{12} t_2^{-1} z_4).$$

Assigning to the pp-divisor \mathfrak{D} the affine T -variety X , as indicated, turns out to be functorial. Moreover, a canonical construction, based on the chamber structure of the set of GIT-quotients of X , shows that in fact every normal affine variety with effective torus action arises from a proper polyhedral divisor. These results can be summarized as follows.

Theorem. *The assignment $\mathfrak{D} \mapsto X$ defines an essentially surjective faithful covariant functor from the category of proper polyhedral divisors on semiprojective varieties to the category of normal affine varieties with effective torus action.*

After localizing the category of proper polyhedral divisors by the maps coming from (birational) modifications of the semiprojective base varieties, we even arrive at an equivalence of categories. This allows in particular to decide whether two given proper polyhedral divisors define (equivariantly) isomorphic varieties.

First applications of the above result are a description of the orbit decomposition of an affine T -variety X in terms of its defining pp-divisor \mathfrak{D} , see [1, Sec. 10] and a description of the collection of all open T -invariant subsets $U \subset X$ admitting a complete orbit space U/T , see [4].

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Codimension one torus actions

ROBERT VOLLMERT

Consider an effective action of $T = (\mathbb{C}^*)^{n-1}$ on an n -dimensional normal affine variety X . Mumford shows how to associate a toroidal embedding with this situation [2]. (In fact, for this approach, X need not be affine.) On the other hand, Altmann and Hausen give a description of X by a polyhedral divisor on a curve [1]. We will compare the fan of the toroidal embedding with this polyhedral divisor.

A toroidal embedding is a pair (U, X) of a normal variety X and an open subset $U \subset X$ such that for each point $x \in X$, there exists a toric variety (T, Z) with embedded torus $T \subset Z$ which is locally formally isomorphic near some point $z \in Z$ to (U, X) near x . Furthermore, we will assume that the components E_i of $X \setminus U$ are normal.

The components of the sets $\cap_{i \in I} E_i \setminus \cup_{i \notin I} E_i$ give a stratification of X . The star of a stratum Y is defined to be the union of strata Z with $Y \subset \overline{Z}$. Given a stratum Y , we have the lattice M_Y of Cartier divisors on the star of Y with support in $\text{star}(Y) \setminus U$. The submonoid of effective divisors is dual to a polyhedral cone σ_Y in the dual lattice N_Y .

If $Z \subset \text{star}(Y)$ is a stratum, its cone σ_Z is a face of σ_Y . The toroidal fan of the embedding (U, X) is the union of the cones σ_Y glued along common faces.

We now assume $T = (\mathbb{C}^*)^{n-1}$ acts effectively on the n -dimensional affine normal variety X and describe Mumford's approach. There is a canonically defined rational quotient map $p: X \dashrightarrow C$ to a complete nonsingular curve C . Sufficiently small invariant open sets $W \subset X$ split as $W \cong U \times T$ for some open set $U \subset C$, where the first projection $U \times T \rightarrow U$ corresponds to p . We will identify $U \times T$ with W .

We define \tilde{X} to be the normalization of the closure of the graph of the rational map p in $X \times C$. The action of T on X extends to \tilde{X} . We may consider $U \times T$ as an open subset of \tilde{X} , and the projection to U now extends to a regular map $\pi: \tilde{X} \rightarrow C$.

After possibly replacing U by an open subset, we are in the following situation: Let $P \in C \setminus U$ be a point in the complement of U . The sets U , $U' = U \cup \{P\}$ and $\pi^{-1}(U')$ are affine with coordinate rings R , R' and S , respectively. We may regard S as a subring of $R \otimes \mathbb{C}[M]$ which is generated by homogeneous elements with respect to the M -grading. Denoting by s a local parameter at P , the ring S is generated over R' by a finite number of monomials $s^{k_\nu} \chi^{u_\nu}$.

The corresponding semigroup in $\mathbb{Z} \times M$ defines a toric variety Z ; denote its dual cone $\mathbb{Z} \times N$ by δ_P . There is an étale map $\pi^{-1}(U) \rightarrow Z$, showing that $(U \times T, \tilde{X})$ is a toroidal embedding. Its fan consists of the cones δ_P glued along the common face $\delta_P \cap (\{0\} \times N_{\mathbb{Q}})$.

Given $U \subset C$, the constructed toroidal fan $\Delta(X, U)$ is independent of the choice of equivariant isomorphism $U \times T \cong W$. It does however depend on the choice of U .

On the other hand, we can associate with the T -variety X a proper polyhedral divisor $\mathfrak{D} \in \text{PPDiv}(Y, \sigma)$ on a nonsingular curve Y . Denoting by N the lattice of one-parameter subgroups of T , the cone σ in N is the dual of the weight cone of the given action. The divisor \mathfrak{D} can be represented as a sum $\sum \Delta_P \otimes [P]$ where $P \in Y$ are points and $\Delta_P \subset N_{\mathbb{Q}}$ are polyhedra with tail cone σ , subject to some positivity conditions.

For every weight u in the weight monoid, \mathfrak{D} gives a divisor $\mathfrak{D}(u)$ on the curve Y . We can define the sheaf $\mathcal{A} = \bigoplus_u \mathcal{O}_Y(\mathfrak{D}(u))$ of M -graded \mathcal{O}_Y -algebras. Then we can recover X as $\text{Spec } \Gamma(Y, \mathcal{A})$.

Now to compare the two descriptions: Consider the toroidal embedding $(U \times T, \tilde{X})$ for a suitable $U \subset Y$. As before, let C be the complete nonsingular curve containing U (and Y). The fan $\Delta(X, U)$ is glued from cones δ_P for $P \in C \setminus U$. We express the polyhedral divisor \mathfrak{D} as $\sum_{P \in Y \setminus U} \Delta_P \otimes [P]$.

It turns out that for $P \in C \setminus Y$, we have $\delta_P \cong \{0\} \times \sigma \subset \mathbb{Q} \times N_{\mathbb{Q}}$. For $P \in Y$, the cone δ_P is isomorphic to the homogenization of Δ_P , i.e., to the cone in $\mathbb{Q} \times N_{\mathbb{Q}}$ generated by $\{0\} \times \sigma$ and $\{1\} \times \Delta_P$. Thus, we can describe $\Delta(X, U)$ in terms of \mathfrak{D} :

Theorem. *The toroidal fan obtained by gluing the homogenizations of the coefficient polyhedra Δ_P of points $P \in Y \setminus U$ along their common face $\{0\} \times \sigma$ is isomorphic to $\Delta(X, U)$.*

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Glueing polyhedral divisors

HENDRIK SÜSS

(joint work with Klaus Altmann and Jürgen Hausen)

We consider normal varieties X with effective torus action. For affine T -varieties there exists a partially combinatorial description, namely by polyhedral divisors introduced in [AH06]. A polyhedral divisor on a variety Y is a finite formal sum $\sum_{i=1}^k \Delta_i \otimes D_i$ where the D_i are prime divisors of Y and the Δ_i are polyhedra in a fixed \mathbb{Q} vector space $N \otimes \mathbb{Q}$ over some lattice $N \cong \mathbb{Z}^r$. We generalise the notion of these polyhedral divisors in order to cover also the nonaffine case.

In the case of a torus T of same dimension as X we have the well known language of fans consisting of polyhedral cones describing (normal) toric varieties. In this language the fan structure reflects the unique open affine T -invariant covering of X .

For tori of lower dimension we use the same strategy. We introduce a fan-like structure consisting of polyhedral divisors and reflecting a (no longer unique) T -invariant open affine covering of X together with its glueing isomorphisms.

As an example figure 1 shows such a fan-like structure over $Y = \mathbb{P}^1$ consisting of six polyhedral divisors $\mathfrak{D}^1, \dots, \mathfrak{D}^6$. The three pictures are associated to the prime divisors $\{\infty\}$, $\{0\}$ and $\{1\}$ of Y . For every polyhedral divisor \mathfrak{D}^i we can read off the polyhedral coefficient of the considered prime divisors directly from the picture.

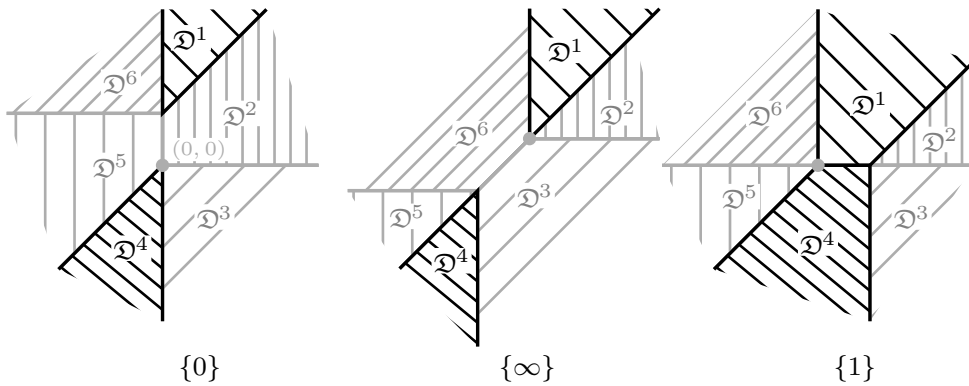


FIGURE 1. a fan of polyhedral divisors

We can define the intersection $\mathfrak{D}^i \cap \mathfrak{D}^j$ of two polyhedral divisors by intersecting their coefficients. In our example we get for instance

$$\mathfrak{D}^1 \cap \mathfrak{D}^4 = \emptyset \otimes \{0\} + \emptyset \otimes \{\infty\} + \overline{(0,0)(1,0)} \otimes \{1\},$$

where the divisor is considered on the complement in Y of the points with coefficient \emptyset . For every prime divisor the resulting coefficient of the intersection is a face of those of \mathfrak{D}^1 and \mathfrak{D}^4 . It turns out that if some extra conditions are satisfied, such a face relation of the polyhedral divisors defines an open inclusion of the corresponding affine T -varieties.

We call a set of polyhedral divisors a fan if they intersect in such a way. In this case the corresponding affine T -varieties glue and we get a T -scheme.

Criteria for separatedness and completeness of the resulting scheme can be expressed in terms of convex geometry. For instance, it is a necessary condition for completeness, that for every prime divisor the polyhedral coefficients cover the whole space. For the fan in figure 1 this condition holds.

As an application of our language we describe the projectivations $\mathbb{P}(\mathcal{E}) \rightarrow X_\Sigma$ of equivariant rank 2 bundles \mathcal{E} on toric varieties X_Σ . In this case, Y is \mathbb{P}^1 . For a cone $\sigma \in \Sigma$ the restriction $\mathcal{E}|_{U_\sigma}$ is generated by two global sections g_0^σ, g_1^σ . We can consider them as elements of $\bigoplus_{m \in M} k^2$ [Per02, Kly90], thus as elements of k^2 with associated weights m_0^σ, m_1^σ . For every maximal cone σ we obtain two polyhedral divisors by cutting σ with an affine hyperplane orthogonal to $m_0^\sigma - m_1^\sigma$ and using the resulting pieces as coefficients of the prime divisors $g_0^\perp, g_1^\perp \in \mathbb{P}^1$. Here, g_i^\perp is

the one dimensional subspace of k^2 orthogonal to g_i and thus an element of \mathbb{P}^1 . To be precise we define the polyhedra

$$\begin{aligned}\Delta_{\sigma}^{+}(m) &= \{n \in N \mid \langle 2m - m_0^{\sigma} - m_1^{\sigma}, n \rangle \geq 1\} \cap \sigma \\ \Delta_{\sigma}^{-}(m) &= \{n \in N \mid \langle 2m - m_0^{\sigma} - m_1^{\sigma}, n \rangle \leq 1\} \cap \sigma\end{aligned}$$

and the polyhedral divisors $\mathfrak{D}_{\sigma}^{+}, \mathfrak{D}_{\sigma}^{-}$ on $Y = \mathbb{P}^1$ with

$$\begin{aligned}\mathfrak{D}_{\sigma}^{+} &= \{g_0^{\perp}\} \otimes \Delta^{+}(m_0) + \{g_1^{\perp}\} \otimes \Delta^{+}(m_1) \\ \mathfrak{D}_{\sigma}^{-} &= \{g_0^{\perp}\} \otimes \Delta^{-}(m_0) + \{g_1^{\perp}\} \otimes \Delta^{-}(m_1).\end{aligned}$$

The set $\{\mathfrak{D}_{\sigma}^{\pm} \mid \sigma \in \Sigma^{\max}\}$ of all these polyhedral divisors is in fact a fan and encodes $\mathbb{P}(\mathcal{E})$. For the cotangent bundle $\Omega_{\mathbb{P}^1/k}$ on \mathbb{P}^1 we get the fan of figure 1.

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Moduli of representations of the McKay quiver

DIANE MACLAGAN

(joint work with Alastair Craw, Rekha R. Thomas)

The McKay correspondence originated in the observation of McKay [9] that there is a tight connection between the representation theory of a finite group $G \subseteq \mathrm{SL}(2, \mathbb{C})$ and the minimal resolution of the quotient singularity \mathbb{C}^n/G . Specifically, if $\sigma: G \rightarrow \mathrm{SL}(2, \mathbb{C})$ is the given faithful two-dimensional representation of G then we define the *McKay quiver* to be the quiver (oriented graph) that has vertices the irreducible representations of G , and an arrow from representation ρ to representation ρ' if ρ appears in $\sigma \otimes \rho'$. For finite $G \subseteq \mathrm{SL}(2, \mathbb{C})$, this quiver is a doubly-oriented extended Dynkin diagram of type A, D, or E. McKay's observation is that this is the extended Dynkin diagram of the graph that arises in the consideration of resolutions of the DuVal singularity \mathbb{C}^2/G . In this graph the vertices are the irreducible components of the exceptional divisor of the resolution $Y \rightarrow \mathbb{C}^2/G$, and the edges represent intersection of components.

One philosophy that has emerged over the past two decades from consideration of this unexpected connection is that if $G \subseteq \mathrm{SL}(n, \mathbb{C})$, the geometry of a crepant resolution of \mathbb{C}^n/G should be determined by the representation theory of G , particularly as evidenced in the McKay quiver. An example of this is the result of Batyrev [1] that the Euler number of any crepant resolution of \mathbb{C}^n/G is equal to the number of vertices of the McKay quiver. See [11] for details.

For $G \subseteq \mathrm{SL}(2, \mathbb{C})$ or $G \subseteq \mathrm{SL}(3, \mathbb{C})$ it follows from [2], [7], [8] that Nakamura's G -Hilbert scheme [10] is always a crepant resolution of \mathbb{C}^3/G . In [2] the stronger result

is shown that the derived category of coherent sheaves on G -Hilb is equivalent to the derived category of G -equivariant sheaves on \mathbb{C}^3 . For $n > 3$, \mathbb{C}^n/G need not have a crepant resolution, and even when it does G -Hilb need not be one. See [6] for more information about families of G where a crepant resolution is known to exist. This motivates the search for other sources of resolutions of \mathbb{C}^n/G .

One source of such examples are the moduli spaces M_θ of θ -stable representations of the McKay quiver. While these are not smooth or irreducible in general, when G is abelian they do have a distinguished irreducible component that is birational to \mathbb{C}^n/G . In joint work with Alastair Craw and Rekha R. Thomas ([4], [5]) we give an explicit description of this distinguished component Y_θ .

In this case the McKay quiver has $n|G|$ arrows, n from each vertex labeled $1, \dots, n$, so representations of the McKay quiver of dimension vector $(1, \dots, 1)$ correspond to points of $\mathbb{A}^{n|G|}$. We consider representations satisfying certain commuting relations, which correspond to points of a subscheme $Z \subseteq \mathbb{A}^{n|G|}$. The moduli of such representations is then the GIT quotient $M_\theta = Z //_\theta T$, where $T \subseteq (\mathbb{C}^*)^{n|G|}$ is a $(|G| - 1)$ -dimensional torus acting on Z , and θ is a character of T corresponding to a choice of linearization.

Let C be the augmented vertex-edge incidence matrix of the McKay quiver. This is a $(|G| + n) \times (n|G|)$ matrix whose columns correspond to arrows in the quiver. The first $|G|$ rows are the vertex-edge incidence matrix, and the last n rows record the label of the edge.

Theorem 1 (Craw–Maclagan–Thomas [4]). *The not-necessarily-normal toric variety $V = \text{Spec } \mathbb{C}[\text{NC}]$ is a T -invariant irreducible component of the scheme $Z \subset \mathbb{A}^{n|G|}$. In addition:*

- (1) *For $\theta \in \Theta$, the GIT quotient $Y_\theta := V //_\theta T$ is a not-necessarily-normal toric variety that admits a projective birational morphism $\tau_\theta: Y_\theta \rightarrow \mathbb{C}^n/G$ obtained by variation of GIT quotient.*
- (2) *For generic $\theta \in \Theta$, the variety Y_θ is the unique irreducible component of M_θ containing the T -orbit closures of the points of $Z \cap (\mathbb{C}^*)^{nr}$.*
- (3) *Let $\pi: \mathbb{Z}^{|G|+n} \rightarrow \mathbb{Z}^{|G|}$ be the projection onto the first $|G|$ coordinates. The toric fan of Y_θ is the inner normal fan of the polyhedron P_θ obtained as the convex hull of the set $P \cap \pi^{-1}(\theta)$.*

One application of Theorem 1 is an algorithm to determine whether there is a parameter θ for which M_θ is a crepant resolution of \mathbb{C}^n/G . Another consequence is local equations for G -Hilb, which lets us construct an example of a nonnormal G -Hilb, answering a question of Nakamura [10].

Theorem 2. *There is a $G \subseteq \text{GL}(6, \mathbb{C})$ isomorphic to $(\mathbb{Z}/5\mathbb{Z})^4$ for which G -Hilb is not normal.*

The example of Theorem 2 was found after extensive computer search, but can be verified without a computer.

Finally, the theorem allows us to describe explicitly the quiver representations corresponding to torus-fixed points of Y_θ , using Gröbner bases (see [5]).

In [3] Craw and Ishii show that for abelian $G \subseteq \mathrm{SL}(3, \mathbb{C})$ every relatively projective crepant resolution of \mathbb{C}^3/G is isomorphic to Y_θ for some θ . We are hopeful that the technology developed above will allow us to decide whether the same is true for arbitrary $G \subseteq \mathrm{SL}(n, \mathbb{C})$ when \mathbb{C}^n/G admits a crepant resolution.

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Toric varieties are fine moduli spaces

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(joint work with Gregory G. Smith)

For a smooth projective toric variety X , the bounded derived category of coherent sheaves $D^b(\mathrm{Coh}(X))$ has been the focus of tremendous interest in recent years, yet it has been calculated explicitly for relatively few examples. For projective space, the line bundles $\mathcal{O}_{\mathbb{P}^n}, \dots, \mathcal{O}_{\mathbb{P}^n}(n)$ freely generate $D^b(\mathrm{Coh}(\mathbb{P}^n))$ by the celebrated result of Beilinson [1]. More generally, Bondal [2] observed that if line bundles L_0, \dots, L_r freely generate $D^b(\mathrm{Coh}(X))$, then the functor

$$(1) \quad \mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(\oplus_i L_i, -): D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{mod}\text{-}A)$$

is an equivalence of triangulated categories, where $\mathrm{mod}\text{-}A$ is the category of finite-dimensional right modules over the algebra $A = \mathrm{End}(\oplus_i L_i)$. King [5] exhibited appropriate collections of line bundles on the Hirzebruch surfaces \mathbb{F}_n and the smooth Fano toric surfaces, and hence established derived equivalences as in (1) above. As part of this construction, King showed that each of these toric surfaces

is a fine moduli space of θ -stable representations of a quiver with relations, Q whose path algebra is the endomorphism algebra A . The tautological bundles of the moduli construction are precisely the line bundles L_0, \dots, L_r that freely generate $D^b(\text{Coh}(X))$.

Here we describe the results from Craw–Smith [3] generalising King’s moduli construction to projective toric varieties of arbitrary dimension. As an application, we construct an equivalence as in (1) for each smooth Fano toric 3-fold.

To construct the quivers that arise, let L_0, L_1, \dots, L_r be a collection of effective line bundles on X . The *Bondal quiver* of the collection is the quiver Q with one vertex for each L_i , and one arrow from L_i to L_j for each element of $\text{Hom}(L_i, L_j)$ that does not factor through some other L_k . The relations in Q are introduced to ensure that the path algebra of the quiver modulo the ideal of relations is isomorphic to the algebra

$$A = \bigoplus_{0 \leq i, j \leq r} \text{Hom}(L_i, L_j).$$

This quiver is conveniently encoded in the *quiver matrix* $C = C(Q)$: for the arrow from L_i to L_j arising from $s \in \text{Hom}(L_i, L_j)$, the corresponding column records the head with $+1$, the tail with -1 , and the effective divisor of zeroes $\text{div}(s)$ of the defining section $s \in \Gamma(L_j \otimes L_i^{-1})$. Thus, C is the incidence matrix of the quiver augmented with additional rows that record the divisors labeling the arrows.

The matrix C naturally defines a toric ideal $I_C = (z^u - z^v : u - v \in \ker(C))$ in the polynomial ring R with one variable for each arrow, and hence defines an affine toric variety¹ $V := \text{Spec}(R/I_C)$. Let T denote the algebraic torus whose character lattice is generated by the columns of the incidence matrix of Q . Then T acts naturally by change of basis on $\text{Spec}(R)$, and restricts to give an action on the affine variety V .

For any character $\theta \in T^*$, the GIT quotient $V //_{\theta} T$ is a toric variety, but it need not coincide with X in general. However, this proves to be the case if we restrict to quivers arising from *geometric collections* of line bundles. This means that there exist $\theta_1, \dots, \theta_r \in \mathbb{Z}_{>0}$ such that, for $L := L_1^{\theta_1} \otimes \dots \otimes L_r^{\theta_r}$, we have:

- (1) L is ample and normally generated; and
- (2) the multiplication map

$$H^0(L_1)^{\theta_1} \otimes \dots \otimes H^0(L_r)^{\theta_r} \longrightarrow H^0(L)$$

is surjective.

If we set $\theta_0 := -\theta_1 - \dots - \theta_r$, then the parameter $\theta := (\theta_0, \theta_1, \dots, \theta_r)$ is a character of T , so the GIT quotient $V //_{\theta} T$ is well defined.

Theorem 1. *Let L_0, \dots, L_r be any geometric collection of line bundles on a projective toric variety X . Then X is isomorphic to the geometric quotient $V //_{\theta} T$.*

¹Here we adopt the convention agreed upon by the participants of this Oberwolfach conference, whereby toric varieties are not assumed to be normal!

Since it is straightforward to construct geometric collections on projective toric varieties, this gives a wealth of new GIT constructions of toric varieties.

The second result relates $V//_{\theta}T$ to moduli of quiver representations. The relations in the quiver cut out a binomial subscheme $Z \subset \text{Spec}(R)$, and the GIT quotient $\mathcal{M}_{\theta} := Z//_{\theta}T$ parametrises S-equivalence classes of θ -semistable quiver representations satisfying the relations. The parameters $\theta \in T^*$ arising from geometric collections are generic, so the work of King [4] shows that \mathcal{M}_{θ} is the *fine* moduli space of θ -stable representations satisfying the given relations. The quotient $V//_{\theta}T$ is isomorphic to a closed subscheme of \mathcal{M}_{θ} , but the inclusion may be strict even within a given component of the moduli space.

We introduce a complex of modules F^{\bullet} over the homogeneous coordinate ring $\text{Cox}(X \times X)$ of the product $X \times X$, and prove that the variety V is an irreducible component of the scheme Z whenever the complex is exact at the F^1 -term. This leads to the second main result.

Theorem 2. *Let L_0, \dots, L_r be a geometric collection on X for which the complex F^{\bullet} is exact at F^1 . Then $X \cong V//_{\theta}T$ is isomorphic to a component of the fine moduli space \mathcal{M}_{θ} , and the induced tautological bundle on X is $\bigoplus_{i=0}^r L_i$.*

By sheafifying F^{\bullet} , we obtain a complex of sheaves $\mathcal{F}^{\bullet} \rightarrow \mathcal{O}_{\Delta}$, where $\Delta \subset X \times X$ is the diagonal. For collections satisfying the hypotheses of Theorem 2, we have exactness at \mathcal{F}^1 , and it is natural to ask whether the complex is actually a resolution of \mathcal{O}_{Δ} . For smooth Fano threefolds X , we construct strong exceptional, geometric collections of line bundles L_0, \dots, L_r for which \mathcal{F}^{\bullet} is a resolution. This shows that, in addition to the moduli construction of Theorem 2, we obtain the derived equivalence (1) in these cases.

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Toward a new, “tropical” construction of Calabi-Yau varieties

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(joint work with Mark Gross)

A classical construction, due to Mumford, produces a degeneration of toric varieties out of an integral polyhedral decomposition \mathcal{P} of \mathbb{R}^n . This works by simply taking the fan Σ in \mathbb{R}^{n+1} defined by the closures of the cones $\mathbb{R}_{\geq 0} \cdot (\Xi \times \{1\})$ over the cells $\Xi \in \mathcal{P}$; the projection to the last coordinate defines a map of this fan to the fan of \mathbb{A}^1 , hence a toric morphism of toric varieties $\pi : X = X(\Sigma) \rightarrow \mathbb{A}^1$. General fibers of π are isomorphic to the toric variety given by the *asymptotic fan* $\{\lim_{t \rightarrow 0} t \cdot \Xi \mid \Xi \in \mathcal{P}\}$ of the polyhedral decomposition. The central fiber $X_0 = \pi^{-1}(0)$ can also readily be read off from \mathcal{P} as follows. To a vertex v of the polyhedral decomposition is associated the complete, n -dimensional fan with cones $\mathbb{R}_{\geq 0} \cdot (\Xi - v)$ generated by corners of the cells $\Xi \in \mathcal{P}$ containing v . The corresponding toric varieties are the irreducible components of X_0 . They are glued pairwise by identifying toric prime divisors torically.

This construction was one of the motivations for my joint program with Mark Gross (UCSD) for a comprehensive explanation of the mirror phenomenon [GrSi1]. The central idea is to view mirror symmetry as a duality of degeneration data associated to *maximal, toric degenerations*. These exist for large classes of varieties, including Calabi-Yau varieties. The central fiber of a toric degeneration is a union of toric varieties glued along toric divisors just as in the Mumford construction above. \mathbb{R}^n is now replaced by a topological manifold B , which is still obtained by gluing polyhedra $\Xi \in \mathcal{P}$ by integral affine maps. Moreover, the fans defining the toric components of X_0 provide a neighbourhood of each vertex with an integral affine structure that is compatible with the affine structures on the cells. In agreement with recent developments in “tropical algebraic geometry” this discrete part of our degeneration data is what one may call a “tropical Calabi-Yau variety”, in case B is compact. This is the arena where according to our program mirror symmetry should be understood. The mirror transformation itself is a discrete version of the Legendre transformation. This requires a polarization on X_0 , giving rise to a (multi-valued) convex, piecewise affine function φ on B . The triple $(B, \mathcal{P}, \varphi)$ is the complete set of discrete degeneration data. Results on cohomology and base change [GrSi2] relate the topology of B with the topology of the general fiber of the degeneration. For example, for a toric degeneration of Calabi-Yau varieties B is a \mathbb{Q} -homology sphere, while for a degeneration of Hyperkähler manifolds B is a \mathbb{Q} -homology complex projective space (these statements require a technical hypothesis on maximality of the degeneration).

While possibly not absolutely necessary for the investigation of mirror symmetry, it is natural to ask if one can revert the logic and construct a toric degeneration for any given (B, \mathcal{P}) ? The paper [GrSi1] already goes a long way toward this aim by establishing a one-to-one correspondence between certain cohomological data on B and spaces X_0 glued from complete toric varieties together with a *log-smooth*

structure. Such log-structures allow to work with non-normal spaces in many respects as if they were smooth [Ka]. Log-smooth deformation theory (or Friedman's smoothability result for "d-semistable K3-surfaces" [Fr]) now implies that the answer to our question in dimension two is affirmative. In higher dimensions there is first a problem in applying known results off the shelves because the log structures itself have singularities and because many of the results are only available for the normal crossings case. These problems can be overcome [GrSi2]. However, in the non-normal crossings case there is still a problem with non-vanishing obstruction groups that does not seem to have a solution within this technique.

This talk was about a different idea that we first pursued in the first half of 2004. Deformation theory usually relies on patching thickenings of an open affine cover to produce a k -th order deformation X_k over $\text{Spec } \mathbb{C}[t]/(t^{k+1})$. However, for reducible, reduced X_0 the (unique) primary decomposition for X_k exhibits the deformation by gluing k -th order thickenings of the irreducible components of X_0 . This suggests a closed cover approach to deformation theory as opposed to the traditional open cover approach. These thickenings are not flat deformations itself, so much care has to be taken. Nevertheless, the Mumford construction above does have a neat interpretation in these terms; in this case the naive thickenings provided by the polyhedral decomposition are indeed consistent. In the more "non-linear" Calabi-Yau world this is not true anymore and corrections become necessary. This means that we have to change the gluings of the thickenings by (log-) automorphisms. Unfortunately, the automorphisms tend to affect also the other gluings, with the affected number of gluings increasing with k . The natural setup to study these effects is the dual picture, where $(B, \mathcal{P}, \varphi)$ is interpreted as (polarized) *intersection complex*. The rings to be glued can then be interpreted as monoid rings given locally by integral points in \mathbb{R}^{n+1} lying above the graph of φ , in an integral affine chart.

At this point Kontsevich and Soibelman gave a rigid-analytic version of our two-dimensional reconstruction theorem mentioned above [KoSo]. While technically a very different approach, which works on the affine manifold with singularities B alone and with a choice of Riemannian metric, it developed a picture of automorphisms propagating along (curvilinear) rays and producing new rays whenever two rays intersect. Possibly infinitely many new rays are inserted in such a way that the product of automorphisms along a small loop about the intersection point is the identity.

Translated to our language the relevant automorphisms fix the rings over a local affine hyperplane (a *wall*) in B . These automorphisms propagate along the walls, with the affected order measured by the change of slope of φ . The rule to generate new walls follows from a decomposition lemma for a group of automorphisms of the relevant ring just as in [KoSo]. The picture that one obtains is a refinement of the polyhedral decomposition of B by parts of such hyperplanes, which becomes more and more complicated when we increase the order. It is then possible to construct a k -th order thickening of X_0 by taking a copy of the relevant ring for

each cell of this refined decomposition and glue them via the automorphisms of those hyperplanes that a connecting path crosses.

While there is a convergence issue presently left it seems that we will soon have a complete solution to our reconstruction problem in any dimensions along these lines. Among other things, this gives the exciting perspective of constructing new Hyperkähler manifolds in dimension 4 by the construction of affine structures on $\mathbb{C}\mathbb{P}^2$ minus a codimension two locus.

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Tropical enumeration of real rational curves

ILIA ITENBERG

(joint work with Viatcheslav Kharlamov and Eugeniï Shustin)

The talk is devoted to an enumeration of real rational curves interpolating fixed collections of real points in a real surface Σ , more precisely, to the following question: *given a real divisor D and a generic collection \mathbf{w} of $c_1(\Sigma) \cdot D - 1$ real points in Σ , how many of the complex rational curves in the linear system $|D|$ passing through \mathbf{w} are real?* By rational curves we mean irreducible genus zero curves and their degenerations, so that they form in $|D|$ a projective subvariety $S(\Sigma, D)$; this subvariety is called the *Severi variety*. A curve on a real surface Σ is called real, if the curve is invariant under the involution $c : \Sigma \rightarrow \Sigma$ defining the real structure of Σ .

While, under mild conditions on Σ and D , the number of complex curves in question is the same for all generic collections \mathbf{w} (it is equal to the degree of $S(\Sigma, D)$), it is no more the case for real curves (except few very particular situations).

J.-Y. Welschinger [7, 8] discovered a way to attribute weights ± 1 to real solutions so that the number of solutions counted with weights is independent of the configuration of real points. As an immediate consequence, the absolute value of the Welschinger invariant $W_{\Sigma, D}$ provides a lower bound on the number $R_{\Sigma, D}(\mathbf{w})$ of real solutions: $R_{\Sigma, D}(\mathbf{w}) \geq |W_{\Sigma, D}|$.

In some cases (for example, in the case of toric Del Pezzo surfaces) the Welschinger invariant can be calculated or estimated using Mikhalkin's approach [4, 5] which deals with a corresponding count of tropical curves. In tropical geometry, complicated non-linear algebro-geometric objects are replaced by simpler piecewise-linear ones. For example, tropical plane curves are piecewise-linear graphs whose edges have rational slopes. Tropical curves can be seen as algebraic curves over the tropical semiring $(\max, +)$.

Tropical (idempotent) semirings have been used since the 1990s in optimization, control theory, and max-plus operators. The recent rapid development of tropical geometry was initiated in much extent by O. Viro [6] who linked the tropical semirings with real algebraic geometry, by M. Kapranov [2] who introduced non-Archimedean amoebas and showed that they represent tropical varieties, and by M. Kontsevich (see [3]) who predicted applications of tropical geometry in enumerative geometry. Kontsevich's prediction was confirmed by G. Mikhalkin [4, 5] who established a correspondence theorem and found a combinatorial algorithm for computing Gromov-Witten type invariants of toric surfaces. Informally speaking, Mikhalkin's correspondence theorem deals with complex nodal curves in a given linear system and of a given genus which pass through some fixed points in general position in a toric surface, and states, in particular, that the number of these curves can be calculated via the count (with appropriate multiplicities) of their tropical analogs passing through certain points in general position in \mathbb{R}^2 .

Using the tropical approach, we prove (see [1]) the logarithmic equivalence for the Welschinger and Gromov-Witten invariants of any toric Del Pezzo surface equipped with its standard real structure, i.e., the real structure that comes together with the toric one.

Theorem 1 (see [1]). *Let Σ be a toric Del Pezzo surface equipped with its standard real structure, and D an ample divisor on Σ . The sequences $\log W_{\Sigma, nD}$ and $\log GW_{\Sigma, nD}$, $n \in \mathbb{N}$, of the Welschinger invariants and the corresponding Gromov-Witten invariants are asymptotically equivalent. More precisely, $\log W_{\Sigma, nD} = \log GW_{\Sigma, nD} + O(n)$ and $\log GW_{\Sigma, nD} = (c_1(\Sigma) \cdot D) \cdot n \log n + O(n)$.*

In particular, Welschinger's bound implies that asymptotically in the logarithmic scale all the complex rational curves of degree n which pass through given $3n - 1$ points in general position in the real projective plane are real.

We also prove the logarithmic equivalence for the Welschinger and Gromov-Witten invariants in the case of several toric Del Pezzo surfaces equipped with a non-standard real structure. In particular, we proved the following statement.

Theorem 2. *Let Σ be $\mathbb{P}^1 \times \mathbb{P}^1$ equipped with the real structure $(z_1, z_2) \mapsto (\bar{z}_2, \bar{z}_1)$. The sequences $\log W_{\Sigma, (n, n)}$ and $\log GW_{\Sigma, (n, n)}$, $n \in \mathbb{N}$, of the Welschinger and Gromov-Witten invariants in bi-degree (n, n) are asymptotically equivalent.*

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Recursive formulas for tropical and algebraic Welschinger invariants

EUGENII SHUSTIN

(joint work with Ilia Itenberg and Viatcheslav Kharlamov)

Let Σ be a real Del Pezzo surface with a connected real part. The Welschinger invariant $W_m(\Sigma, D)$, where $m \geq 0$, $D \subset \Sigma$ is an ample divisor, is the number of real rational curves C in the linear system $|D|$ on Σ , passing through $-DK_\Sigma - 1 - 2m$ generic real points and through m pairs of conjugate imaginary points, and counted with weights $w(C) = (-1)^s$, where s is the number of isolated real nodes of C [10]. In spite of a clear geometric analogy with Gromov-Witten invariants, so far no formulas for the Welschinger invariants have been found, and the tropical enumerative geometry is the only way to compute them [4, 5, 6, 7, 8, 9], and as results the Welschinger invariants were expressed as the total weights of certain combinatorial objects. We present here the first recursive formula for Welschinger invariants, which is a version of the Caporaso-Harris formula [1].

Let Σ be a real toric Del Pezzo surface with the standard real structure (i.e., the plane, or the quadric as the product of two real lines, or the plane blown up at $k = 1, 2, 3$ real generic points), L one of the toric divisors. Denote by $\text{Pic}(\Sigma, L)$ the set of positive-dimensional base-point free complete linear systems $|D|$ on Σ such that D is nef and $DL > 0$. An element $|D| \in \text{Pic}(\Sigma, L)$ is called terminal if $D^2 \leq 1$. If $|D|$ is not terminal, we define $|D : L| \in \text{Pic}(\Sigma, L)$ to be the base-point free part of the linear system $|D - L|$. Introduce the integers $N_{\Sigma, |D|}(\alpha, \beta, \delta)$, where $|D| \in \text{Pic}(\Sigma, L)$, $\delta \in \mathbb{Z}$, and

$$(1) \quad \begin{cases} \alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{Z}^\infty, \beta = (\beta_1, \beta_2, \dots) \in \mathbb{Z}^\infty, & \alpha_i, \beta_i \geq 0, \\ I\alpha + I\beta := \sum_{i \geq 1} (2i - 1)\alpha_i + \sum_{i \geq 1} (2i - 1)\beta_i = DL, \end{cases}$$

defined by the following conditions:

- (i) $N_{\Sigma, |D|}(\alpha, \beta, \delta) = 0$ if $\delta < 0$,
- (ii) if $|D|$ is terminal, then $N_{\Sigma, |D|}(\alpha, \beta, 0) = 1$ as $\sum_i \beta_{2i} = 0$, in other cases $N_{\Sigma, |D|}(\alpha, \beta, \delta) = 0$,

(iii) if $|D|$ is not terminal, then

$$(2) \quad N_{\Sigma, |D|}(\alpha, \beta, \delta) = \sum_{\substack{k \geq 1 \\ \beta_k > 0}} N_{\Sigma, |D|}(\alpha + e_k, \beta - e_k, \delta) + \sum_{\alpha', \beta', \delta'} \binom{\alpha}{\alpha'} \binom{\beta'}{\beta} N_{\Sigma, |D:L|}(\alpha', \beta', \delta'),$$

where $e_k \in \mathbb{Z}^\infty$ is the k -th unit vector,

$$\binom{\alpha}{\alpha'} = \prod_{i \geq 1} \binom{\alpha_i}{\alpha'_i}, \quad \binom{\beta'}{\beta} = \prod_{i \geq 1} \binom{\beta'_i}{\beta_i}, \quad I^{\beta' - \beta} = \prod_{i \geq 1} i^{\beta'_i - \beta_i},$$

and α', β', δ' are subject to the restrictions

$$\alpha'_i \leq \alpha_i, \quad \beta'_i \geq \beta_i, \quad i \geq 1, \quad I\alpha' + I\beta' = (D : L)L, \quad \delta' \leq \delta, \\ \delta - \delta' + \sum_{i \geq 1} (\beta'_i - \beta_i) = (D : L)L.$$

The initial conditions (i), (ii), and the recursive formula (2) determine all the numbers $N_{\Sigma, |D|}(\alpha, \beta, \delta)$ uniquely.

Now, on the set \mathcal{S} of quadruples $(|D|, \alpha, \beta, \delta)$, where $|D| \in \text{Pic}(\Sigma, L)$, $\delta \geq 0$, and α, β satisfy (2), we define the operation

$$(|D|, \alpha, \beta, \delta) + (|D'|, \alpha', \beta', \delta') := (|D + D'|, \alpha + \alpha', \beta + \beta', \delta + \delta' + DD'),$$

and then introduce the numbers $N_{\Sigma, |D|}^{irr}(\alpha, \beta, \delta)$, where $|D| \in \text{Pic}(\Sigma, L)$, $\delta \geq 0$, and α, β satisfy (1), by the following relations:

$$(3) \quad N_{\Sigma, |D|}(\alpha, \beta, \delta) = \sum \frac{\alpha!}{\alpha^{(1)}! \dots \alpha^{(s)}!} \cdot \frac{n!}{n_1! \dots n_s!} \prod_{k=1}^s N_{\Sigma, |D_k|}^{irr}(\alpha^{(k)}, \beta^{(k)}, \delta^{(k)}),$$

where the sum is taken over all unordered splittings in \mathcal{S}

$$(|D|, \alpha, \beta, \delta) = \sum_{k=1}^s (|D_k|, \alpha^{(k)}, \beta^{(k)}, \delta^{(k)}),$$

and

$$n = \frac{D^2 - DK_\Sigma}{2} - \delta - DL + |\beta|, \quad n_k = \frac{D_k^2 - D_k K_\Sigma}{2} - \delta^{(k)} - D_k L + |\beta^{(k)}|, \quad k = 1, \dots, s.$$

The equations (3) uniquely determine the numbers $N_{\Sigma, |D|}^{irr}(\alpha, \beta, \delta)$ out of the sequence $N_{\Sigma, |D|}(\alpha, \beta, \delta)$.

Theorem 1. *Let Σ be a toric Del Pezzo surface with the standard real structure, D an ample divisor on Σ , L one of the toric divisors of Σ , then*

$$W_0(\Sigma, D) = N_{\Sigma, |D|}^{irr}(0, (DL), \delta_0), \quad \delta_0 = (D^2 + DK_\Sigma)/2 + 1.$$

The proof is based on the correspondence between real rational curves and rational tropical curves as defined in [6, 7, 8], and uses the work by Gathmann and Markwig [2, 3], who suggested a tropical version of the Caporaso-Harris formula.

Recursive formulas (2) and (3) allow one to compute the Welschinger invariants much easier than by means of the Mikhalkin's algorithm of counting tropical curves.

Corollary 1. *For the projective plane,*

$$W_0(\mathbb{P}^2, 1) = W_0(\mathbb{P}^2, 2) = 1, \quad W_0(\mathbb{P}^2, 3) = 8, \quad W_0(\mathbb{P}^2, 4) = 240,$$

$$W_0(\mathbb{P}^2, 5) = 18264, \quad W_0(\mathbb{P}^2, 6) = 2845440.$$

The first five values have been computed using the lattice path algorithm [5], and the last value as a consequence of formulas (2), (3) and Theorem 1.

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Quantum marginal problem, flag varieties, and representations of the symmetric group

ALEXANDER KLYACHKO

The quantum marginal problem is about the relation between reduced states ρ_A , ρ_B , ρ_C of a pure state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ of a three (or multi) component quantum system. In plain language it can be stated as follows:

Under what conditions do three Hermitian matrices ρ_A , ρ_B , ρ_C of orders ℓ , m , n coincide with the Gram matrices formed by the Hermitian dot products of parallel slices of a complex cubic matrix $\psi = [\psi_{\alpha\beta\gamma}]$ of the format $\ell \times m \times n$?

Clearly the compatibility depends only on the spectra

$$\lambda^A = \text{Spec}(\rho_A), \quad \lambda^B = \text{Spec}(\rho_B), \quad \lambda^C = \text{Spec}(\rho_C).$$

An equivalent version of the problem seeks for a relation between the spectra of the Hermitian operator $\rho_{AB}: \mathcal{H}_A \otimes \mathcal{H}_B \rightarrow \mathcal{H}_A \otimes \mathcal{H}_B$ and its reduced operators $\rho_A: \mathcal{H}_A \rightarrow \mathcal{H}_A$ and $\rho_B: \mathcal{H}_B \rightarrow \mathcal{H}_B$. The reduction $\rho_{AB} \mapsto \rho_A$ is known to mathematicians as *contraction*, e.g. the Ricci curvature operator $\text{Ric}: T \rightarrow T$ is the contraction of the Riemann curvature $R: T \wedge T \rightarrow T \wedge T$.

The problem has a long history. Its fermionic version dealing with a skew symmetric state $\psi \in \wedge^N \mathcal{H}$ is known in quantum chemistry since the early 60s as the N -representability problem. A couple of years ago it came into focus again, now in the framework of quantum information theory.

Here I outline a solution of the problem given by linear inequalities governed by the topology of flag varieties and establish its connection with representations of the symmetric group. As an example I write down explicitly inequalities between the spectra of the Riemann and Ricci curvatures of a 4-manifold. One can find more details in [2, 3] and references therein.

Theorem. *All constraints on the spectra*

$$\lambda^{AB} = \text{Spec}(\rho_{AB}), \quad \lambda^A = \text{Spec}(\rho_A), \quad \lambda^B = \text{Spec}(\rho_B),$$

are given by linear inequalities of the form

$$\sum_i a_i \lambda_{u(i)}^A + \sum_j b_j \lambda_{v(j)}^B \leq \sum_k (a+b)_k^\downarrow \lambda_{w(k)}^{AB},$$

where

$$a : a_1 \geq a_2 \geq \cdots \geq a_m, \quad b : b_1 \geq b_2 \geq \cdots \geq b_n, \quad \sum a_i = \sum b_j = 0,$$

are “test” spectra, $(a+b)^\downarrow$ is the sequence $a_i + b_j$ arranged in decreasing order, and $u \in S_m$, $v \in S_n$, $w \in S_{mn}$ are permutations subject to the topological condition $c_{uv}^w(a, b) \neq 0$ explained below.

Consider the flag variety $\mathcal{F}_a(\mathcal{H}_A)$, understood as set of Hermitian operators $X_A: \mathcal{H}_A \rightarrow \mathcal{H}_A$ of spectrum a , and the natural morphism

$$\begin{aligned} \varphi_{ab}: \mathcal{F}_a(\mathcal{H}_A) \times \mathcal{F}_b(\mathcal{H}_B) &\rightarrow \mathcal{F}_{(a+b)}(\mathcal{H}_A \otimes \mathcal{H}_B), \\ X_A \times X_B &\mapsto X_a \otimes 1 + 1 \otimes X_B. \end{aligned}$$

The coefficients $c_{uv}^w(a, b)$ are defined in terms of the induced morphism of cohomology

$$\varphi_{ab}^*: H^*(\mathcal{F}_{(a+b)}(\mathcal{H}_A \otimes \mathcal{H}_B)) \rightarrow H^*(\mathcal{F}_a(\mathcal{H}_A)) \otimes H^*(\mathcal{F}_b(\mathcal{H}_B))$$

written in the basis of Schubert cocycles σ^w

$$\varphi_{ab}: \sigma^w \mapsto \sum_{u,v} c_{uv}^w(a, b) \sigma^u \otimes \sigma^v.$$

Remark. The coefficients $c_{uv}^w(a, b)$ depend only on the order in which the quantities $a_i + b_j$ appear in the spectrum $(a + b)^\downarrow$. The order changes when the pair (a, b) crosses one of the hyperplanes

$$H_{ij|kl} : a_i + b_j = a_k + b_l$$

which cut the set of pairs (a, b) into a finite number of pieces called *cubicles*. For each cubicle one has to check the inequalities of the theorem for its extremal edges only. Hence the marginal constraints amount to a *finite* system of inequalities.

Example. For two qubits $\dim \mathcal{H}_A = \dim \mathcal{H}_B = 2$ the theorem amounts to the *Bravyi inequalities* [1]

$$\begin{aligned} \lambda_A &\geq \lambda_3^{AB} + \lambda_4^{AB}, & \lambda_B &\geq \lambda_3^{AB} + \lambda_4^{AB}, \\ \lambda_A + \lambda_B &\geq \lambda_2^{AB} + \lambda_3^{AB} + 2\lambda_4^{AB}, \\ |\lambda_A - \lambda_B| &\leq \min(\lambda_3^{AB}, \lambda_2^{AB} - \lambda_4^{AB}), \end{aligned}$$

where λ_A and λ_B are minimal eigenvalues of ρ_A and ρ_B , respectively.

Marginal constraints in the next dimension $\dim \mathcal{H}_A = \dim \mathcal{H}_B = 3$ are given by a system of 387 independent inequalities which can't be reproduced here.

The QM problem can be restated in terms of decomposition of the tensor product of irreducible representations of the symmetric group S_N

$$\mathcal{S}_\lambda \otimes \mathcal{S}_\mu = \sum_{\nu} g(\lambda, \mu, \nu) \mathcal{S}_\nu,$$

which are parameterized by Young diagrams treated here as integral spectra

$$\lambda : \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0, \quad \sum_i \lambda_i = N.$$

The multiplicities $g(\lambda, \mu, \nu)$ are known as *Kronecker's coefficients*. There is no simple way to calculate them and some authors consider this as the last unsolved problem in the complex representation theory of S_N .

Theorem. *The following conditions on Young diagrams λ, μ, ν are equivalent*

- (1) $g(n\lambda, n\mu, n\nu) \neq 0$ for some $n > 0$.
- (2) *There exists $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ with reduced matrices ρ_A, ρ_B, ρ_C of spectra λ, μ, ν .*

The tensor product of irreducible representations with two row diagrams can be described pretty explicitly, and one may recover Bravyi inequalities in this way. On the other hand, as we've seen above, a similar decomposition for three row diagrams is shaped by 387 inequalities. This may be the reason why all attempts to understand the combinatorics of this decomposition failed.

A similar result holds in the fermionic settings for spectra of the operator $\rho : \wedge^n \mathcal{H}_r \rightarrow \wedge^N \mathcal{H}_r$ and its one point reduced operator $\rho^{(1)} : \mathcal{H}_r \rightarrow \mathcal{H}_r, \dim \mathcal{H}_r = r$.

Theorem. All constraints on the spectra $\nu = \text{Spec } \rho$ and $\lambda = \text{Spec } \rho^{(1)}$ are given by inequalities

$$\sum_i \lambda_{v(i)} \leq \frac{1}{n} \sum_j (\wedge^n a)_j \nu_{w(j)}$$

where $a : a_1 \geq a_2 \geq \dots \geq a_r$, $\sum_i a_i = 0$ is the test spectrum, $\wedge^n a = \{a_{i_1} + a_{i_2} + \dots + a_{i_n} \mid i_1 < i_2 < \dots < i_n\}^\downarrow$, and $v \in S_r$, $w \in S_{\binom{r}{n}}$ are permutations subject to the topological condition $c_v^w(a) \neq 0$ described below.

The coefficients $c_v^w(a)$ are defined via the morphism of flag varieties

$$\varphi_a : \mathcal{F}_a(\mathcal{H}_r) \rightarrow \mathcal{F}_{\wedge^n a}(\wedge^n \mathcal{H}_r), \quad X \mapsto X^{(n)},$$

where $X^{(n)} : \alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_n \mapsto \sum_i \alpha_1 \wedge \alpha_2 \wedge \dots \wedge X \alpha_i \wedge \dots \wedge \alpha_n$, and the induced morphism of cohomology is

$$\varphi_a^* : H^*(\mathcal{F}_{\wedge^n a}(\wedge^n \mathcal{H}_r)) \rightarrow H^*(\mathcal{F}_a(\mathcal{H}_r)), \quad \sigma^w \mapsto \sum_v c_v^w(a) \sigma^v$$

written in the basis of Schubert cocycles σ^w .

Example. For the system $\wedge^2 \mathcal{H}_4$ one gets the following inequalities, which can be seen as constraints on the spectra of the Riemann and Ricci curvatures of a 4-manifold.

$$\begin{aligned} 2\lambda_1 &\leq \nu_1 + \nu_2 + \nu_3 \\ 2\lambda_4 &\geq \nu_4 + \nu_5 + \nu_6 \\ 2(\lambda_1 - \lambda_4) &\leq \nu_1 + \nu_2 - \nu_5 - \nu_6 \\ \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 &\leq \nu_1 - \nu_6 \\ \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 &\leq \min(\nu_1 - \nu_5, \nu_2 - \nu_6) \\ |\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4| &\leq \min(\nu_1 - \nu_4, \nu_2 - \nu_5, \nu_3 - \nu_6) \\ 2 \max(\lambda_1 - \lambda_3, \lambda_2 - \lambda_4) &\leq \min(\nu_1 + \nu_3 - \nu_5 - \nu_6, \nu_1 + \nu_2 - \nu_4 - \nu_6) \\ 2 \max(\lambda_1 - \lambda_2, \lambda_3 - \lambda_4) &\leq \min(\nu_1 + \nu_3 - \nu_4 - \nu_6, \nu_2 + \nu_3 - \nu_5 - \nu_6, \\ &\quad \nu_1 + \nu_2 - \nu_4 - \nu_5). \end{aligned}$$

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Crystals, quivers and dessins d'enfants

JAN STIENSTRA

Recently, physicists (in particular A. Hanany and co-authors) working on the AdS/CFT correspondence found a way to associate quivers with certain planar lattice polygons [2]. At closer inspection, the method, which they describe through examples, turns out to start like N.G. de Bruijn's method of constructing Penrose tilings and quasi-crystals [4]. While the latter uses grids 'parallel' to the sides of a regular pentagon, the former uses grids 'parallel' to the sides of the given lattice polygon. The method yields crystals (periodic rhombus tilings) if the slopes of the polygon's sides are rational, and quasi-crystals (non-periodic rhombus tilings) if the slopes are irrational. Our present interest is in lattice polygons.

So, let \mathcal{P} be a polygon in \mathbb{R}^2 with vertices in \mathbb{Z}^2 . Let $\mathbf{v}_1, \dots, \mathbf{v}_N$ be a collection of (distinct) lattice points on the boundary $\partial\mathcal{P}$ of \mathcal{P} , including all vertices, ordered according to their appearance as one walks along $\partial\mathcal{P}$ counterclockwise. Let $\mathbf{b}_i = \mathbf{v}_{i+1} - \mathbf{v}_i$ for $i = 1, \dots, N$, $\mathbf{v}_{N+1} = \mathbf{v}_1$. Think of these as column vectors. Let $\mu = \frac{1}{6}N(N-1)(N-2)$ and index the coordinates of \mathbb{R}^μ by strictly increasing triples $i < j < k \in \{1, \dots, N\}$. Associated with these data is the linear map

$$P: \mathbb{R}^N \rightarrow \mathbb{R}^\mu, \quad P(\gamma_1, \dots, \gamma_N) = \left(\det \begin{pmatrix} \mathbf{b}_i & \mathbf{b}_j & \mathbf{b}_k \\ \gamma_i & \gamma_j & \gamma_k \end{pmatrix} \right)_{ijk}.$$

The kernel of P is the linear 2-plane in \mathbb{R}^N spanned by the rows of the matrix $(\mathbf{b}_1, \dots, \mathbf{b}_N)$. The matrix for P is just made from the Plücker coordinates of this linear 2-plane in \mathbb{R}^N . For every \mathbf{c} in the image of P the set $P^{-1}(\mathbf{c})$ is an affine 2-plane in \mathbb{R}^N . In \mathbb{R}^N one also has the standard N -grid formed by the hyperplanes $H_{i,m} = \{x_i = m\}$ for $1 \leq i \leq N$, $m \in \mathbb{Z}$. Intersecting this standard N -grid with $P^{-1}(\mathbf{c})$ gives an N -grid in the plane. Next consider the map

$$F: \mathbb{R}^N \rightarrow \mathbb{Z}^N, \quad F(\gamma_1, \dots, \gamma_N) = ([\gamma_1], \dots, [\gamma_N]),$$

where $[x]$ for a real number x denotes the largest integer $\leq x$. The map F is constant on the connected components of the grid complement in $P^{-1}(\mathbf{c})$ and takes different values on different components. For \mathbf{c} in the image of P we now let $S_{\mathbf{c}}$ denote the piecewise linear surface in \mathbb{R}^N which is the union of the 2-dimensional squares with sides of length 1 and vertices in the set $F(P^{-1}(\mathbf{c}))$. An appropriate projection map $W: \mathbb{R}^N \rightarrow \mathbb{R}^2$ makes $S_{\mathbf{c}}$ appear as a rhombus tiling of the plane \mathbb{R}^2 ; for this the columns of W must be vectors of length 1 and for the angles between the columns there are some restrictions. The choice of W corresponds to the 'isradial embeddings and R-charges' in [2] §3. The piecewise linear surface $S_{\mathbf{c}}$ is invariant under translations by vectors from the lattice $\mathbb{L} = \mathbb{Z}^N \cap \ker P$. Let $\phi: \mathbb{R}^N \rightarrow \mathbb{R}$ denote the map which assigns to a vector the sum of its coordinates. It is invariant under translations by vectors from \mathbb{L} , because $\sum_i \mathbf{b}_i = 0$. Summarizing: *for every \mathbf{c} in the image of P we have*

$$\text{the torus } S_{\mathbf{c}}/\mathbb{L}, \quad \text{the map } \phi: S_{\mathbf{c}}/\mathbb{L} \rightarrow \mathbb{R};$$

the torus is equipped with a piecewise linear structure given by a tiling with squares and the map is piecewise linear.

A very special and beautiful situation arises when ϕ takes only three values on the set $F(P^{-1}(\mathbf{c}))$. These values are then three consecutive integers, but since the numerical values are irrelevant for what follows, we call them R, W, B, with R being the maximum and B the minimum. Every square in the tiling on $S_{\mathbf{c}}/\mathbb{L}$ then has one R, two W and one B vertex and its diagonals are R-B and W-W. We give the W-W diagonal an orientation so that the R vertex is on its right hand side.

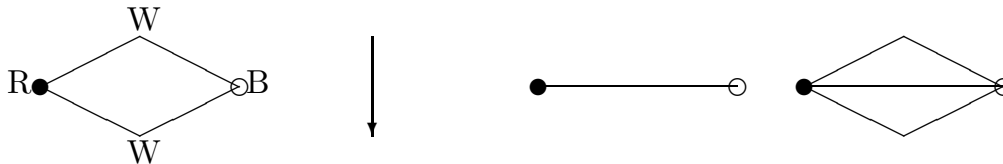


FIGURE 1. *Rhombus and its diagonals.*

The oriented W-W diagonals from the tiling on the torus $S_{\mathbf{c}}/\mathbb{L}$ form the quiver, mentioned in the title of this report. The R-B diagonals in the tiling of the plane, on the other hand, form a periodic bipartite graph. Let me emphasize here that this story tells only what I think I read in the physics literature.

A new aspect I want to add are the *dessins d'enfants*: these are the triangulations of the torus $S_{\mathbf{c}}/\mathbb{L}$ one obtains from the squares-tiling by dividing each square into two triangles by cutting it along the R-B diagonal. Each triangle has one R, one W and one B vertex. According to the general theory of dessins d'enfants [5, 1] this observation implies that $S_{\mathbf{c}}/\mathbb{L}$ can be given the structure of an elliptic curve together with a morphism $\psi: S_{\mathbf{c}}/\mathbb{L} \rightarrow \mathbb{P}^1$, everything defined over some number field, such that ψ is unramified outside the R, W, B points and sends all R points to ∞ , all W points to 1, all B points to 0. In the triangulation on $S_{\mathbf{c}}/\mathbb{L}$ the cells are given as the ψ -inverse image of the upper- or lower hemisphere, the R-W edges are in $\psi^{-1}([1, \infty))$, the B-W edges are in $\psi^{-1}([0, 1])$, the R-B edges are in $\psi^{-1}((\infty, 0])$.

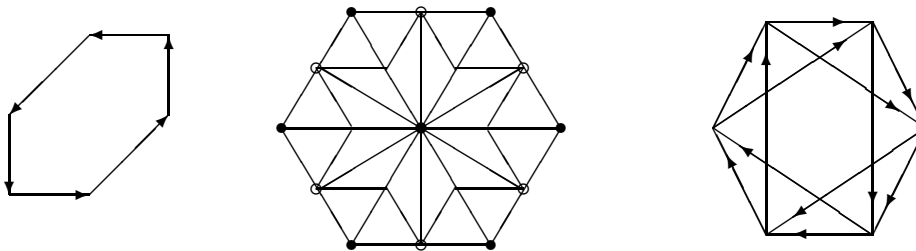


FIGURE 2. *Polygon (left), Dessin (middle) and Quiver (right).*

Figure 2 shows, as an example, the polygon, dessin and quiver for what in the physics literature is known as Model I for the Del Pezzo surface dP_3 (i.e. \mathbb{P}^2 with three points blown up). This dessin can be realized on the elliptic curve \mathcal{E} with equation $y^2 = x^3 - 1$ by the composite of the following three maps:

$$\begin{array}{ccccccc} \text{curve:} & \mathcal{E} & \longrightarrow & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \\ \text{affine coordinate:} & x, y & & x & & t & & s \\ \text{branched covering:} & & & y^2 = x^3 - 1 & & x^3 = \frac{t+1}{t-1} & & t^2 = \frac{s}{s-1} \end{array}$$

The elliptic curve \mathcal{E} is in fact the Fermat cubic.

Remark. I am still working on a program to construct all periodic rhombus tilings with periodicity and collection of tiles specified by the initial lattice polygon. From the thus constructed data set one can then easily select the quivers and dessins. I expect that this, for instance, will also yield Models II, III, IV of dP_3 .

Remark. In the evenings of the workshop Alastair Craw, Lutz Hille, Markus Perling, Duco van Straten and I discussed possible relations between the quivers in my talk and those in the talks of Perling and Bondal. A few days later the paper [3] by Hanany, Herzog and Vegh appeared, which deals with the same issues and (partly) answers our questions.

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Intersection cohomology of hypertoric varieties and Gale duality

TOM BRADEN

(joint work with Nicholas Proudfoot)

We study hypertoric varieties, sometimes also called toric hyperkähler varieties. They were first considered by Bielawski and Dancer [3] as hyperkähler analogues of toric varieties. If a d -dimensional quasiprojective toric variety can be expressed as a GIT quotient $\mathbb{C}^n //_{\alpha} T_0$ of an affine space by a diagonal action of an algebraic torus $T_0 \cong (\mathbb{C}^*)^{n-d}$, then the corresponding hypertoric variety is a hyperkähler quotient $(T^*\mathbb{C}^n) //_{(\alpha,0)} T_0$ of the cotangent bundle of \mathbb{C}^n by the induced action. It is $2d$ -dimensional, and carries a residual action of the torus $T = (\mathbb{C}^*)^n / T_0$.

The structure of a hypertoric variety is governed by an arrangement \mathcal{H} of hyperplanes in \mathfrak{t}^* whose angles are determined by the torus action and whose positions within their parallelism classes are determined by the auxiliary parameter α . This

is analogous to the situation for toric varieties, which are described by a polyhedron obtained by intersecting certain half-spaces bounded by the hyperplanes.

When the arrangement \mathcal{H} is simple, meaning that all hyperplanes intersect transversely, then the corresponding hypertoric variety $X_{\mathcal{H}}$ is an orbifold. If \mathcal{H} is not simple, then $X_{\mathcal{H}}$ has an orbifold resolution of singularities $X_{\tilde{\mathcal{H}}} \rightarrow X_{\mathcal{H}}$ where $\tilde{\mathcal{H}}$ is a “simplification” of \mathcal{H} obtained by moving the hyperplanes until they are transverse.

The cohomology and T -equivariant cohomology of smooth and orbifold hypertoric varieties was computed in [5, 6, 7, 9, 10]. The Betti numbers of a hypertoric orbifold $X_{\tilde{\mathcal{H}}}$ are the h -numbers of the “independence complex” $\Delta_{\tilde{\mathcal{H}}}$ – the simplicial complex whose simplices are subsets of hyperplanes in $\tilde{\mathcal{H}}$ with nonempty intersection. In fact, the cohomology ring $H^*(X_{\tilde{\mathcal{H}}}; \mathbb{R})$ is canonically isomorphic to the face ring $\mathbb{R}[\Delta_{\tilde{\mathcal{H}}}]$ modulo a set of linear forms, while the face ring itself gives the T -equivariant cohomology ring of $X_{\tilde{\mathcal{H}}}$.

In the most singular case where the arrangement \mathcal{H} is central, Proudfoot and Webster [12] showed that the intersection cohomology Betti numbers of $X_{\mathcal{H}}$ are the h -numbers of a subcomplex $\Delta_{\mathcal{H}}^{bc}$ of the independence complex of the simplification $\tilde{\mathcal{H}}$. This complex, called the “broken circuit” complex, is a familiar object in matroid theory. It depends on the choice of an ordering of the hyperplanes (although its h -numbers do not), so its face ring is not a suitable functorial model for $IH^*(X_{\mathcal{H}})$. Proudfoot and Speyer [11] defined a canonical ring which deforms flatly to each face ring $\mathbb{R}[\Delta_{\mathcal{H}}^{bc}]$ for any choice of ordering, so their ring has the right Betti numbers to give the intersection cohomology. Proudfoot and Webster [12] showed that this ring satisfies expected functorialities if and only if the deformed arrangement $\tilde{\mathcal{H}}$ is unimodular (this holds if and only if $X_{\tilde{\mathcal{H}}}$ is smooth). It is still not clear, however, how this ring relates to intersection cohomology; in particular intersection cohomology usually does not carry a canonical ring structure.

We present a canonical functorial computation of the T -equivariant intersection cohomology $IH_T^*(X_{\mathcal{H}})$, along the lines of the theory discovered by Barthel, Brasselet, Fieseler, and Kaup [1, 2] and Bressler and Lunts [4]. Let $L = L_{\mathcal{H}}$ be the lattice of flats of \mathcal{H} , whose elements are all possible intersections of hyperplanes, endowed with the order topology. We can put a sheaf of rings \mathcal{A} on it by declaring the stalk \mathcal{A}_F at a flat F to be the symmetric algebra over the quotient of V by the linear span of F . A *minimal extension sheaf* \mathcal{L} is a sheaf of modules over \mathcal{A} satisfying

- $\mathcal{L}_V \cong \mathbb{R}$
- \mathcal{L}_F is a free \mathcal{A}_F -module for each flat F ,
- \mathcal{L} is flabby: $\mathcal{L}(L) \rightarrow \mathcal{L}(U)$ is surjective for any open set $U \subset L$,

and which is minimal with respect to these conditions. In fact, the construction for toric varieties in [1, 2, 4] is exactly the same, except with the lattice of flats replaced by the lattice of faces of the corresponding polyhedron.

Theorem. *A minimal extension sheaf \mathcal{L} exists and is unique up to a scalar automorphism. Fixing a generator of $\mathcal{L}_V \cong \mathbb{R}$, there are canonical isomorphisms*

$$\mathcal{L}(L_{\mathcal{H}}) \cong IH_T^*(X_{\mathcal{H}})$$

and

$$\mathcal{L}(L_{\mathcal{H}})/V\mathcal{L}(L_{\mathcal{H}}) \cong IH^*(X_{\mathcal{H}}).$$

The definition of \mathcal{L} makes sense even for non-rational arrangements, which do not define a hypertoric variety. The resulting groups still have the expected dimensions, and in general they behave exactly as if they were the intersection cohomology of the non-existent variety $X_{\mathcal{H}}$. For instance, the decomposition theorem for the map $X_{\tilde{\mathcal{H}}} \rightarrow X_{\mathcal{H}}$ can be stated and proved purely combinatorially, with no rationality hypothesis on \mathcal{H} . This is analogous to the Hard Lefschetz theorem that Karu proved for minimal extension sheaves on fans in [8], although the hypertoric story seems to be much less deep.

As an application, we use this to construct a canonical dual pairing

$$H^d(X_{\tilde{\mathcal{H}}}; \mathbb{R}) \times IH^{n-d}(X_{\mathcal{H}^\vee}, X_\rho^+) \rightarrow \mathbb{R}.$$

Here \mathcal{H}^\vee is the central arrangement in \mathfrak{t}_0 which is Gale dual to \mathcal{H} , and $X_\rho^+ \subset X_{\mathcal{H}^\vee}$ is the open subset of points x for which the limit $\lim_{t \rightarrow 0} \rho(t)x$ does not exist, where ρ is a generic cocharacter $\mathbb{C}^* \rightarrow T^\vee$ of the torus which acts on $X_{\mathcal{H}^\vee}$.

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Derived categories of toric varieties

ALEXEY BONDAL

Our purpose is to relate toric geometry to topology of some stratification \mathcal{S} on a real torus \mathbb{T} . The real torus is, basically, dual to the one that acts on a toric variety X . We adopt the point of view in which the geometry of X is captured by the bounded derived category, $D(X)$, of coherent sheaves on X . We establish, under some restrictions on the defining fan of the toric variety, an equivalence between $D(X)$ and the derived category $D(\mathbb{T}, \mathcal{S})$ of sheaves on the real torus, the sheaves being *constructible* with respect to the stratification. The basic tool is the theory of exceptional collections. We construct a canonical set of line bundles on X and show that under the restrictions mentioned above this set is an exceptional collection. The construction allows a variation which depends on an element of the Picard group. Both the the set of line bundles and the stratification of the real torus come from the action of the Frobenius on line bundles.

Let X be a smooth proper toric variety of dimension n over an algebraically closed field k . Thus, an n -dimensional torus $T \simeq k^{*n}$ acts on X . Also we assume fixed an embedding of T in X . Let M be the group of characters of T and N the group of homomorphisms $k^* \rightarrow T$. Since X is smooth, both M and N are known to be free abelian groups of finite rank, and $\text{Hom}(N, \mathbb{Z}) = M$.

Denote by $\text{Pic}_T(X)$ the T -invariant Picard group of X . This group is a free abelian group with the preferred basis of T -invariant irreducible effective divisors $\{D_i, i = 1, \dots, N\}$ on X . We have the standard exact sequence:

$$(1) \quad 0 \rightarrow M \rightarrow \text{Pic}_T(X) \rightarrow \text{Pic}(X) \rightarrow 0$$

If $D = \sum a_i D_i$ with $a_i \in \mathbb{R}$, then we use notation:

$$D/l := \sum (a_i/l) D_i$$

and

$$[D] := \sum [a_i] D_i,$$

where $[a_i]$ stands for the integer part of a_i .

Consider the Frobenius map $F_l : T \rightarrow T$, $l \in \mathbb{N}$, given in toric coordinates (x_1, \dots, x_n) by the formula:

$$F_l(x_1, \dots, x_n) = (x_1^l, \dots, x_n^l).$$

We denote by the same symbol its extension to a morphism $X \rightarrow X$. F_l is the factorization map with respect to the action of the group G_l of l -torsion in T . This implies a decomposition of the push-forward $F_{l*}E$ for an arbitrary G_l -equivariant (in particular for a T -equivariant) sheaf E :

$$F_{l*}E = \oplus E_\chi$$

where the sum is taken over the group of characters $\chi \in G_l^\wedge = M/lM$.

One finds for a T -equivariant line bundle $\mathcal{L} = \mathcal{O}(D)$, $D = \sum a_i D_i$, that \mathcal{L}_χ is a line bundle:

$$\mathcal{L}_\chi = \mathcal{O}([(D - C_\chi)/l])$$

where C_χ is an arbitrary representative for $\chi \in M/lM$ in M . M is in turn interpreted as the subgroup of principal divisors in $\text{Pic}_T(X)$.

Now we embed M/lM into the real torus $\mathbb{T} = (M \otimes \mathbb{R})/M$ by multiplying by $1/l$. Thus, characters of G_l are interpreted as points in \mathbb{T} whose coordinates are rational with denominator l .

Take $\mathcal{L} = \mathcal{O}$. The function $G_l^\wedge \rightarrow \text{Pic}(X)$ defined by $\chi \mapsto \mathcal{O}_\chi$ can be extended to a well-defined function $\Phi : \mathbb{T} \rightarrow \text{Pic}(X)$ (which does not depend on l):

$$\Phi\left(\sum a_i D_i\right) := -\sum [a_i] D_i.$$

The torus \mathbb{T} can be identified with the group of $U(1)$ -local systems on the torus T (here we assume $k = \mathbb{C}$) or, equally, with the $U(1)$ -flat connections on T . Then the map Φ has an interpretation in terms of Deligne extension of flat connections from T to X . We extend the corresponding smooth D -module on T to a D -module with logarithmic singularities on X and take the underlying line bundle on X (i.e. forgetting the connection).

The function Φ has a finite image in $\text{Pic}(X)$. Denote this image by B . One can show that B generates the category $D(X)$.

Let us see when B has no higher Ext-groups:

$$\text{Ext}^i(F_{l*}\mathcal{O}, F_{l*}\mathcal{O}) = \text{Ext}^i(F_l^*F_{l*}\mathcal{O}, \mathcal{O}) = \sum \text{Ext}^i(F_l^*\mathcal{O}_\chi, \mathcal{O}) = H^i(F_l^*(\mathcal{O}_\chi)^*).$$

The pull-back F_l^* acts on $\text{Pic}(X)$ by multiplication by l . A vanishing theorem for toric varieties claims that higher cohomology of a nef divisor are trivial. Therefore, $\text{Ext}^i(F_{l*}\mathcal{O}, F_{l*}\mathcal{O}) = 0$, if all the $(\mathcal{O}_\chi)^*$ are nef.

Let C be an irreducible toric curve in X . Denote by (D_1, \dots, D_{n-1}) the irreducible toric divisors that contain C and by (a_1, \dots, a_{n-1}) the corresponding intersection numbers with C . Then one can check that all $(\mathcal{O}_\chi)^*$ are nef if for every curve C

$$(2) \quad \text{all } a_i \text{'s are } \geq -1 \text{ and no more than one is } = -1.$$

Under this condition on the toric variety the set of line bundles B is a complete strong exceptional collection. The conditions are satisfied, in particular, for all smooth toric Fano threefolds except for two.

Examples of exceptional collections on toric varieties were constructed by Altman and Hille [1]. Kawamata proved existence of a complete exceptional collection of sheaves (but not line bundles) on an arbitrary projective toric variety by means of minimal model theory [2]. The proof is inherently implicit.

Now define the strata in \mathbb{T} as the level sets of Φ . Every stratum is given by a system of linear inequalities. In fact, the stratification encodes all the information about the algebra of homomorphisms among the line bundles in B . Consider the path algebra of the stratification: choose a point x_S on each stratum S and define morphisms from one such a point x to another one y as the homotopy classes of *oriented* paths from x to y which are compatible with the stratification in the following sense: if z is a point in a stratum which belongs to its boundary, then a path can meet this point on the way inside the stratum and not the other way around.

Denote by L_S the line bundle in B corresponding to a stratum S . Then $\mathrm{Hom}_X(L_{S_1}, L_{S_2})$ is identified with the morphisms between x_{S_1} and x_{S_2} in the path algebra A of the stratification. The transfer from a stratum to another stratum corresponds to an effective toric divisor (which is not always irreducible). More precisely, under the condition (2) we have an equivalence of triangulated categories:

$$(3) \quad D(X) = D(\mathrm{mod} - A),$$

where $D(\mathrm{mod} - A)$ is the bounded derived category of finite dimensional right modules over A .

Consider the derived category $D(\mathbb{T}, \mathcal{S})$ of sheaves on the torus which are constructible with respect to the stratification. One can easily see that all the strata are contractible and have a good behavior near the boundary of the other strata. It follows that the category $D(\mathbb{T}, \mathcal{S})$ has a complete exceptional collection whose elements are constructible sheaves numerated by strata. These exceptional sheaves correspond to the irreducible projective modules over the path algebra of the stratification. This implies that under the condition (2) we have an equivalence:

$$(4) \quad D(X) = D(\mathbb{T}, \mathcal{S}),$$

thus giving a precise description of the geometry of the toric variety X in terms of the topology of the stratification.

Note that we considered the stratification corresponding to the decomposition of $F_{l*}(\mathcal{O})$. We get a new stratification when decomposing $F_{l*}(\mathcal{L})$ for another line bundle \mathcal{L} . When we vary \mathcal{L} in the Picard group of X the real torus transforms to a torsor over \mathbb{T} and the topology of the stratification transforms too.

Further understanding of the derived equivalence (4) can be obtained in the framework of mirror symmetry. This is a joint project with Wei-Dong Ruan.

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The K -theory of toric Deligne-Mumford stacks and mirror symmetry

R. PAUL HORJA

(joint work with Lev A. Borisov)

In the approach proposed by Kontsevich in his ICM 1994 presentation [7], mirror symmetry is viewed as a categorical equivalence between “Fukaya’s A_∞ category” of a Calabi–Yau variety and the bounded derived category of coherent sheaves of the mirror Calabi–Yau variety. The general conjecture remains rather mysterious; nevertheless, it has been generalized in various directions which turned out to have remarkable implications, both in geometry, and in string theory, in connection with the physics of D-branes. In particular, a better understanding of the various

categorical structures that appear in mirror symmetry and string theory (like the derived category of coherent sheaves) is needed. The hope is that such abstract structures can be used as new methods for solving classical problems in birational and symplectic geometry, and singularity theory.

In recent joint works with Lev Borisov [3], [4], we showed how Chen-Ruan (orbifold) cohomology and the K -theory of toric Deligne-Mumford stacks provide appropriate tools for the study of Fourier-Mukai transforms and their mirror integral analytic continuation transformations. Borisov, Chen and Smith [2] defined the notion of a smooth toric DM stack as a natural generalization of a toric variety. They also gave a combinatorial description of the orbifold Chow ring which is the algebraic version of the Chen–Ruan cohomology ring. In the paper [3], we proved a Stanley–Reisner type description of the Grothendieck K -theory ring of a smooth toric DM stack and defined an analog of the Chern character. We also calculated K -theory pushforwards and pullbacks for weighted blowups of reduced smooth toric DM stacks.

In the spirit of mirror symmetry, in the work [4], we constructed series solutions to the GKZ system [5] with values in a combinatorial version of the Chen-Ruan (orbifold) cohomology and in the K -theory of the associated DM stacks. By employing algebro-geometric, combinatorial and analytic methods, we showed that the K -theory action of the Fourier-Mukai functors associated to basic toric birational maps of DM stacks (of the type first analyzed by Bondal and Orlov [1], and later by Kawamata [6]) are mirrored by analytic continuation transformations of Mellin-Barnes type. This work naturally incorporates stacky K -theory and orbifold cohomology in the mirror symmetry story. In the same way as toric varieties served as prime examples and provided tools for investigating various powerful geometric conjectures made by physicists, the machinery that we developed should considerably extend the reach of the toric methods in mirror symmetry. It also opens the possibility of performing mirror symmetry checks for equivalences of derived categories, not just self-equivalences.

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Quadratic Gröbner Bases for Smooth 3×3 Transportation Polytopes

CHRISTIAN HAASE

(joint work with Andreas Paffenholz)

A lattice polytope $P \subset \mathbb{R}^d$ defines an ample line bundle L_P on a projective toric variety X_P . If X_P is smooth (the normal fan of P is unimodular), then L_P is very ample, and provides an embedding $X_P \hookrightarrow \mathbb{P}^{r-1}$, where $r = \#(P \cap \mathbb{Z}^d)$. So we can think of X_P as canonically sitting in projective space. Let us start with two conjectures.

- The following question [Stu97, Conj. 2.9], [Cox97, Conj. 5.6] about the defining equations of $X_P \subset \mathbb{P}^{r-1}$ has been around for quite a while, but its origins are hard to track (cf. [BCF⁺05]).

Question 1. *Let P be a lattice polytope whose corresponding projective toric variety is smooth. Is the defining ideal I_P generated by quadrics?*

There are two variations of this question (of strictly increasing strength).

- Is the homogeneous coordinate ring $R_P = \mathbb{k}[\underline{X}]/I_P$ Koszul?
- Does I_P have a squarefree quadratic Gröbner basis?

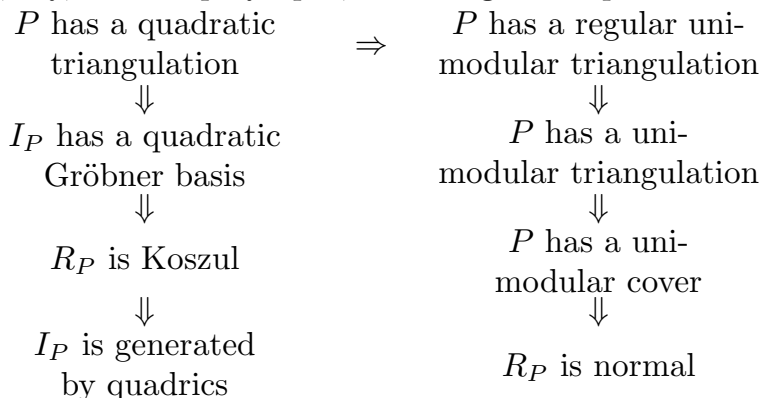
The last version has a combinatorial interpretation. It asks for the existence of very special, “quadratic” triangulations of P .

- Knudsen, Mumford, and Waterman showed that high integral multiples $n \cdot P$ have regular unimodular triangulations [KKMSD73]. We call the smallest such number the KKMS-number $n(P)$ of P .

Question 2. *Is there a constant $n = n(d)$ that works for all d -dimensional polytopes? Does every large enough factor do the trick?*

Even in dimension 3, both questions are not completely settled. Presumably, polytopes that admit unimodular triangulations are rare. One way to approach the problem is to restrict the class of polytopes one is looking at (e.g., smooth polytopes, or high multiples of polytopes). Another approach would be to vary the ‘unimodular triangulation’ property.

Using the following hierarchy of properties, one can formulate a hierarchy of conjectures for, say, smooth polytopes, or for high multiples.



A principal obstacle to theoretical progress is a serious lack of well understood examples. The examples we do have are mostly sporadic. (Compare [OH99, Ohs02, FZ99, BGT02, BG99, BGH⁺99].) These examples show that the converse to some of the above implications does not hold. But they do not tell us “why” this is case, or how we can recognize or construct more such examples.

On the positive side, we know a few classes of polytopes that admit nice triangulations: dicing polytopes, smooth 0-1-polytopes, products of nice polytopes (and more [DHZ01, HP, KS03, BG02]).

Simple transportation polytopes provide a large family of smooth polytopes. Yet, the 3×3 Birkhoff polytope B_3 is a (non-simple) transportation polytope whose ideal is **not** quadratically generated. So we thought that we might find a counterexample in this class. We do no longer think that.

Proposition. *If P is a 3×3 transportation polytope $\neq B_3$, then I_P is quadratically generated. If P is not an odd multiple of B_3 , then I_P has a squarefree quadratic Gröbner basis.*

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On the unimodality of h -vectors

WINFRIED BRUNS

(joint work with Tim Römer)

Let $P \subset \mathbb{R}^{n-1}$ be an integral convex polytope and consider the *Ehrhart-function* given by $E(P, m) = |\{z \in \mathbb{Z}^{n-1} : \frac{z}{m} \in P\}|$ for $m > 0$ and $E(P, 0) = 1$. It is well-known that $E(P, m)$ is a polynomial in m of degree $\dim(P)$ and the corresponding *Ehrhart-series* $E_P(t) = \sum_{m \in \mathbb{N}} E(P, m)t^m$ is a rational function

$$E_P(t) = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^{\dim(P)+1}}.$$

We call $h(P) = (h_0, \dots, h_d)$ (where $h_d \neq 0$) the *h -vector* of P . This vector was intensively studied in the last decades. In particular, the following questions are of interest:

- (1) For which polytopes is $h(P)$ *symmetric*, i.e. $h_i = h_{d-i}$ for all i ?
- (2) For which polytopes is $h(P)$ *unimodal*, i. e. there exists a natural number t such that $h_0 \leq h_1 \leq \cdots \leq h_t \geq h_{t+1} \geq \cdots \geq h_d$?

Let us sketch Stanley’s approach to Ehrhart functions via commutative algebra. The results we are referring to can be found in [3] or [9]. The Ehrhart function of P can be interpreted as the Hilbert function of an affine monoid algebra $K[E(P)]$ (with coefficients from an arbitrary field K). Namely, one considers the cone $C(P)$ generated by $P \times \{1\}$ in \mathbb{R}^n , and sets $E(P) = C(P) \cap \mathbb{Z}^n$. The algebra $K[E(P)]$ is graded in such a way that the degree of a monomial $x \in E(P)$ is its last coordinate, and so the Hilbert function of $K[E(P)]$ coincides with the Ehrhart function of P . Since P is integral, $K[E(P)]$ is a finite module over its subalgebra generated in degree 1.

However, in general $E(P)$ is not generated by its degree 1 elements. If it is, then we say that P is *normal*, and simplify our notation by setting $K[P] = K[E(P)]$.

The monoid $E(P)$ is always normal, and by a theorem of Hochster, $K[E(P)]$ is a Cohen–Macaulay algebra. It follows that $h_i \geq 0$ for all $i = 1, \dots, d$. Using Stanley’s Hilbert series characterization of the Gorenstein rings among the Cohen–Macaulay domains, one sees that $h(P)$ is symmetric if and only if $K[E(P)]$ is a Gorenstein ring. In terms of the monoid $E(P)$, the Gorenstein property has a simple interpretation: it holds if and only if $E(P) \cap \text{int } C(P)$ is of the form $x + E(P)$

for some $x \in E(P)$. This follows from the description of the canonical module of normal affine monoid algebras by Danilov and Stanley.

It was conjectured by Stanley that question (ii) has a positive answer for the *Birkhoff polytope* P , whose points are the real doubly stochastic $n \times n$ matrices and for which $E(P)$ encodes the *magic squares*. This long standing conjecture was recently proved by Athanasiadis [1]. (That P is normal and $K[P]$ is Gorenstein in this case is easy to see.)

Questions (i) and (ii) can be asked similarly for the combinatorial h -vector $h(\Delta(Q))$ of the boundary complex $\Delta(Q)$ of a simplicial polytope Q , and both have a positive answer. The Dehn–Sommerville equations express the symmetry, while unimodality follows from McMullen’s famous g -theorem (proved by Stanley [8]): the vector $(1, h_1 - h_0, \dots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ is an M -sequence, i. e. it represents the Hilbert function of a graded artinian K -algebra that is generated by its degree 1 elements. In particular, its entries are nonnegative, and so the h -vector is unimodal.

Athanasiadis proved Stanley’s conjecture for the Birkhoff polytope P by showing that there exists a simplicial polytope P' with $h(\Delta(P')) = h(P)$. More generally, his theorem applies to compressed polytopes, i. e. integer polytopes all of whose pulling triangulations are unimodular. (The Birkhoff polytope is compressed [7, 9].) We generalize Athanasiadis’ theorem as follows:

Theorem. *Let P be an integral polytope such that P has a regular unimodular triangulation and $K[P]$ is Gorenstein. Then the h -vector of P satisfies the inequalities $1 = h_0 \leq h_1 \leq \dots \leq h_{\lfloor d/2 \rfloor}$. More precisely, the vector $(1, h_1 - h_0, \dots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ is an M -sequence.*

Our strategy of proof (see [4] for the details) is to consider the algebra $K[M]$ of a normal affine monoid M for which $K[M]$ is Gorenstein. We relate the Hilbert series of $K[M]$ to that of a simpler affine monoid algebra $K[N]$ which we get by factoring out a suitable regular sequence of $K[M]$. In the situation of an algebra $K[P]$ for a normal polytope P , the regular sequence is of degree 1, and we obtain a normal and, up to a translation, reflexive polytope such that $h(P) = h(Q)$. (However, note that Mustața and Payne [6] have given an example of a nonnormal reflexive polytope with a nonunimodal h -vector.) If P has even a regular unimodular triangulation, we can find a simplicial polytope P' such that the h -vector of the boundary complex of P' coincides with the one of $K[P]$. Then it only remains to apply the g -theorem to P' .

Without the condition on regularity of the triangulation we can only conclude that P' can be chosen as a simplicial sphere. If the g -theorem can be generalized from polytopes to simplicial spheres, then our theorem holds for all polytopes with a unimodular triangulation.

As a side effect we show that the toric ideal of a Gorenstein polytope with a square-free initial ideal has also a Gorenstein square-free initial ideal. For a detailed discussion of special cases of this result see [5].

We are grateful to Christian Haase for pointing out to us during the Oberwolfach lecture that the hypothesis *regular* of the theorem cannot be omitted.

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Fano Horospherical Varieties

BORIS PASQUIER

I will always consider projective algebraic varieties over \mathbb{C} . Such a variety X is called Fano if X is normal and its anticanonical divisor is Cartier and ample. It is known that in each dimension, there are only finitely many families of smooth Fano varieties. These families are all known up to dimension 3 but not in higher dimension.

Toric varieties give a large number of examples of Fano varieties. In fact, Fano toric varieties correspond bijectively to reflexive polytopes, i.e., lattice polytopes with the origin in their interior and such that the dual polytope satisfies the same properties. It follows, by a theorem of D. Hensley [4], that there are only finitely many isomorphism classes of Fano toric varieties, in each dimension. For example there are 16 isomorphism classes of reflexive polytopes in dimension 2, and 5 of them correspond to smooth Fano varieties.

My work consists in generalizing the theory of Fano toric varieties to a larger class of varieties with group action. Let G be a connected reductive algebraic group, B a Borel subgroup of G and U the unipotent radical of B , then a normal variety X is called *horospherical* if G acts on X with an open orbit isomorphic to G/H where H is a subgroup of G containing U . In this case we also say that X is a G/H -embedding.

Examples include toric varieties (where $G = (\mathbb{C}^*)^n$ and $H = \{1\}$) and flag varieties G/P (where P is parabolic subgroup of G). Moreover, flag varieties are smooth Fano varieties.

For an arbitrary subgroup $H \supset U$, the normalizer of H in G is a parabolic subgroup P of G and P/H is a torus $(\mathbb{C}^*)^n$, where n is called the *rank* of G/H (and of G/H -embeddings). Such a G/H is a particular spherical homogeneous space, so by D. Luna and Th. Vust [6], the G/H -embeddings are classified with *colored fans* in \mathbb{R}^n , generalizing the fans of toric varieties.

Then I have found a smoothness criterion for G/H -embeddings in terms of colored fans.

I have also found a bijective correspondence between Fano G/H -embeddings and a new class of rational polytopes in \mathbb{R}^n called *G/H -reflexive polytopes*. In fact, there is a finite set E of rational points in \mathbb{R}^n (only dependent on G/H) such that a polytope of \mathbb{R}^n is G/H -reflexive if it satisfies the following conditions:

- (i) its vertices are lattice points or are in E ,
- (ii) its dual polytope is a lattice polytope,
- (iii) it contains E .

Using a generalization of Hensley's theorem by J. Lagarias and G. Ziegler [5], I have obtained an effective version of a result of V. Alexeev and M. Brion [1]: there are finitely many isomorphism classes of Fano G/H -embeddings.

It is interesting to note that the dimension n of the G/H -reflexive polytopes are, except in the toric case, smaller than the dimension of the associated varieties X . For example, $(SL_2 \times SL_2)/U$ -reflexive polytopes are of dimension 2 whereas the corresponding Fano varieties are of dimension 4. There are exactly 39 isomorphism classes of $(SL_2 \times SL_2)/U$ -reflexive polytopes which correspond to smooth Fano varieties. The SL_3/U -reflexive polytopes are the same as $(SL_2 \times SL_2)/U$ -reflexive polytopes, but the dimension of the corresponding Fano varieties is 5 and only 27 of these polytopes correspond to smooth Fano varieties.

Further geometric properties of a horospherical Fano variety can be read on its associated G/H -reflexive polytope, for example, the degree is linked to the volume of the polytope, and the Picard number is linked to the number of its vertices (the degree of a Fano variety X is the intersection number $(-K_X)^d$ of the anticanonical divisor, where d is the dimension of X , and the Picard number $\rho(X)$ is the rank of the Picard group).

O. Debarre has bounded the degree of smooth Fano toric varieties in terms of their dimension and Picard number [3], and C. Casagrande has recently bounded the Picard number of smooth Fano toric varieties in terms of their dimension [2]. I have obtained similar results in the case of horospherical varieties:

Theorem 1. *Let X a smooth Fano horospherical variety of dimension d , then $\rho(X) \leq 2d$.*

Theorem 2. *Let X a smooth Fano horospherical variety of dimension d , and rank n .*

*If $\rho(X) > 1$ then $(-K_X)^d \leq d! d^{d\rho(X)+n}$, hence $(-K_X)^d \leq d! d^{3d^2}$.
If $\rho(X) = 1$, then $(-K_X)^d \leq d! (d+1)^{d+n}$, hence $d! (d+1)^{2d}$.*

These theorems hold more generally for locally factorial Fano horospherical varieties.

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Homological Mirror Symmetry and McKay Correspondence

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Let A be a finite abelian subgroup of $SL_3(\mathbb{C})$. A has a natural action on \mathbb{C}^3 and the quotient \mathbb{C}^3/A has the crepant resolution $A\text{-Hilb } \mathbb{C}^3$ by Nakamura [4]. By Bridgeland, King, and Reid [3], there exists an equivalence of triangulated categories

$$D^b \text{coh}_0 A\text{-Hilb } \mathbb{C}^3 \cong D^b \text{coh}_0^A \mathbb{C}^3$$

between the derived category of coherent sheaves on $A\text{-Hilb } \mathbb{C}^3$ supported on the exceptional set and the derived category of A -equivariant coherent sheaves on \mathbb{C}^3 supported at the origin.

The quotient \mathbb{C}^3/A has a structure of a toric variety, determined by a fan whose one-dimensional cones are generated by elements of \mathbb{Z}^3 of the following forms:

$$\overline{v_1} = (v_{1,1}, v_{1,2}, 1), \quad \overline{v_2} = (v_{2,1}, v_{2,2}, 1), \quad \overline{v_3} = (v_{3,1}, v_{3,2}, 1).$$

Here, $v_i = (v_{i,1}, v_{i,2})$, $i = 1, 2, 3$, are elements of \mathbb{Z}^2 . We assume that the convex hull of $\{v_i\}_{i=1}^3$ contains at least one lattice point in its interior. Such v_i 's can be normalized by the actions of $SL_2(\mathbb{Z})$ and translations to

$$(1) \quad v_1 = (n-1, -1), \quad v_2 = (-1, n-1), \quad v_3 = (-1, -1),$$

for $n = 3, 4, \dots$, or

$$(2) \quad v_1 = (n, 0), \quad v_2 = (0, n), \quad v_3 = (-na, -nb)$$

for $a, b, n = 1, 2, \dots$. Now take a generic Laurent polynomial W in two variables x and y whose Newton polygon is the convex hull of $\{v_i\}_{i=1}^3$, and endow $(\mathbb{C}^\times)^2$ with a symplectic structure by $\omega = d \arg x \wedge d|x|/|x| + d \arg y \wedge d|y|/|y|$. Then W considered as a map from $(\mathbb{C}^\times)^2$ to \mathbb{C} is an exact Lefschetz fibration in the sense of Seidel [8], and one can define its *directed Fukaya category* $\mathfrak{Fuk}^{\rightarrow} W$ whose

objects are vanishing cycles and whose morphisms are Lagrangian intersection Floer cohomology [6].

Although $\mathfrak{Fuk}^{\rightarrow} W$ is not an honest category but merely an A_{∞} -category in general, it turns out to be a differential graded category with a trivial differential for the above W and a suitable choice of a distinguished basis of vanishing cycles. Then, one can consider its trivial extension category $\mathfrak{Fuk} W$ as in [9], (10a). Define the derived category $D^b \mathfrak{Fuk} W$ of $\mathfrak{Fuk} W$ by using twisted complexes [2].

Theorem 1. *In the above situation, we have an equivalence of triangulated categories*

$$D^b \mathfrak{Fuk} W \cong D^b \text{coh}_0^A \mathbb{C}^3.$$

The proof is given by choosing an explicit correspondence between generators of both sides and comparing morphisms between them.

The generators of $D^b \text{coh}_0^A \mathbb{C}^3$ are given by the structure sheaf of the origin tensored with irreducible representations of A , and morphisms between them can be computed by the Koszul resolution of the origin. In $D^b \mathfrak{Fuk} W$, we can draw the pictures of the vanishing cycles of W and compute their Floer cohomologies by “painting triangles.”

When v_i 's are as in (2), drawing vanishing cycles of W and computing Floer cohomologies among them can be reduced by the n^2 -fold cover $(\mathbb{C})^2 \ni (x, y) \mapsto (x^n, y^n) \in (\mathbb{C})$ to the case of $n = 1$, which are treated by Seidel [7] when $a = b = 1$ and by Auroux, Katzarkov and Orlov [1] in the general case. The case when v_i 's are as in (1) requires a separate treatment. See [5] for the case when $n = 3$.

Theorem 1 can be used to compute the Stokes matrix for certain hypergeometric series of Gelfand-Kapranov-Zelevinsky type [5].

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Searching for strongly exceptional sequences of line bundles on toric varieties

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(joint work with Lutz Hille)

We are interested in the question whether there exists a generalization of Beilinson's theorem [Bei78] for toric varieties. Such a generalization would be interesting from several points of view. On the one hand, as for projective spaces, it would be a very important practical tool for working with vector bundles over toric varieties. On the other hand, Beilinson's theorem and its generalizations (see below) have been of relevance in physics in the context of D-branes (see [Kon95], [Dou01], [Hor05], [Asp04]). The theorem by now is classical and one of the most important tools in the study of vector bundles over projective space. It states that $D^b(\mathbb{P}^n)$, the bounded derived category of coherent sheaves over \mathbb{P}^n is equivalent to $K_{[0,n]}^b(\Lambda)$, the homotopy category of free modules, which are graded in degrees $0, \dots, n$, over the exterior algebra in $n+1$ variables. The proof involves an explicit resolution of the diagonal, i.e. of the structure sheaf \mathcal{O}_Δ of the diagonal on $\mathbb{P}^n \times \mathbb{P}^n$ and the composition of functors $Rp_{2*}(\mathcal{O}_\Delta \otimes^L Lp_1^*)$. The corresponding spectral sequence [OSS80] often allows rather explicit investigation of vector bundles; this kind of analysis by now is very well understood (see [EFS03]).

Based on work of Drezet and Le Potier ([DL85]), Beilinson's theorem has been generalized as follows. Let X be some smooth projective variety, and let \mathcal{T} be a so-called *tilting sheaf* on X , which means that $\text{Ext}_X^i(\mathcal{T}, \mathcal{T}) = 0$, the endomorphism algebra $A := \text{End}(\mathcal{T})$ has finite global dimension, and \mathcal{T} generates $D^b(X)$. Then the functor

$$\mathbf{R}\text{Hom}(\mathcal{T}, \cdot) : D^b(X) \longrightarrow D^b(A\text{-mod})$$

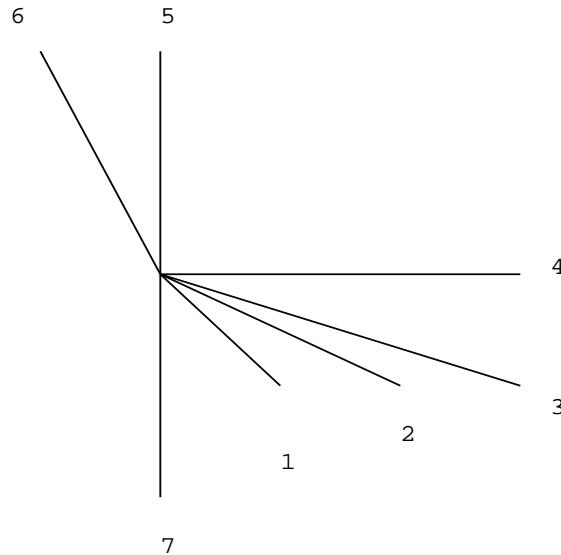
induces an equivalence of derived categories between $D^b(X)$ and $D^b(A\text{-mod})$ (see [Rud90], [Bon90]). So the general problem, from this point of view, is to find a suitable Tilting sheaf. In most of the examples considered so far, \mathcal{T} is a direct sum $\bigoplus_{i=1}^r \mathcal{T}_i$, and the sheaves \mathcal{T}_i form a so-called *strongly exceptional collection*, i.e. $\dim \text{End}(\mathcal{T}_i) = 1$ and $\text{Ext}^k(\mathcal{T}_i, \mathcal{T}_j) = 0$ for all i, j and all $k > 0$. The existence of a strongly exceptional collection, generating $D^b(X)$, implies that the Grothendieck group of X must be finitely generated and free, which excludes the existence of such collections in general. And even in the cases where the Grothendieck group is finitely generated and free, it is an open problem in whether such collections exist. For toric varieties, there is a very strong conjecture which has first been stated by King:

Conjecture ([Kin97], [AKO04]). *Let X be a smooth compact toric variety. Then X has a tilting bundle which is a direct sum of line bundles.*

Note that the summands here form a strongly exceptional sequence of line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n$, where n is the rank of the Grothendieck group of X . So far, many positive results in support of this conjecture have been obtained (see [CM04], [Hil04], [Kaw05], [BP05], [CS05]).

In our approach we try to understand the combinatorics of the cohomology regions in the Picard group of a toric variety and to increase the number of interesting examples, with help of numerical methods. The advantage of toric varieties here is that King's conjecture can be formulated as a purely combinatorial problem and thus it is straightforwardly accessible for combinatorial search. Although the complexity of the involved combinatorics makes this searching problem intractable in general, it is nevertheless possible to produce examples for strongly exceptional sequences in many interesting cases, including many toric surfaces, higher dimensional toric Fanos, as well as examples of compact, but not projective, toric varieties.

However, computer experiments do not always have positive outcome, and indeed, we have verified that there exists a counterexample to King's conjecture. This example is a toric surface which can be obtained by iteratively blowing up the Hirzebruch surface \mathbb{F}_2 three times. The picture shows the fan associated to this surface.



In coordinates, the primitive vectors of the rays are given by $(1, -1)$, $(2, -1)$, $(3, -1)$, $(1, 0)$, $(0, 1)$, $(-1, 2)$, $(0, -1)$ in counterclockwise order, starting with ray number 1. The proof is done by a somewhat elaborate but quite elementary computation. It consists of two steps. As we are dealing with line bundles here, we can assume without loss of generality that, if a strongly exceptional sequence $\mathcal{L}_1, \dots, \mathcal{L}_7$ exists, then this sequence contains the structure sheaf. This puts a strong condition of cohomology vanishing on the \mathcal{L}_i , namely it is necessary that $H^k(X, \mathcal{L}_i) = H^k(X, \mathcal{L}_i^*) = 0$ for all i and all $k > 0$ (here \mathcal{L}_i^* denotes the dual bundle). In the first step we classify all line bundles which have this condition of cohomology vanishing. In the second step we show by inspection of the obtained classification that there does not exist a strongly exceptional sequence of length 7.

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Diptych varieties and semistable Mori flips

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(joint work with Miles Reid)

We construct families of 6-dimensional Gorenstein affine varieties V_{ABLM} that admit an action of a 4-dimensional torus. These varieties contain two 4-dimensional toric varieties V_{AB} and V_{LM} on which the torus acts as the open orbit and which meet along mutual toric 2-strata. Regarding these toric subvarieties as two panels hinged along their 2-strata, we call our 6-folds *diptych varieties*. The combined combinatorics of the toric panels guide the construction of V_{ABLM} .

1. DIPTYCH VARIETIES

1.1. Tents and toric deformations. A *tent* is a union of four affine toric surfaces S_1, \dots, S_4 that meet transversely along their toric 1-strata to form a closed cycle of surfaces with a common 0-stratum. We always restrict to the case where $S_1 = S_3 = \mathbb{C}^2$, while S_2 and S_4 may, at first, be arbitrary cyclic quotients of \mathbb{C}^2 .

For example, consider the case $S_2 = S_4 = \frac{1}{5}(1, 3)$. Then S_2 can be embedded in \mathbb{C}^4 with coordinates x_0, \dots, x_3 and equations

$$(1) \quad x_0x_2 = x_1^2, \quad x_1x_3 = x_2^3 \quad \text{and} \quad x_0x_3 = x_1x_2^2.$$

As in [9], the ‘short’ equations $x_{i-1}x_{i+1} = x_i^{a_i}$ are determined by the tags a_i appearing as coefficients of a continued fraction expansion $[3, 2] = \frac{5}{5-3}$; the other ‘long’ equations can be deduced using syzygies.

Similarly using y_0, \dots, y_3 for S_4 , the tent $T = S_1 \cup S_2 \cup S_3 \cup S_4$ embeds in \mathbb{C}^8 with equations (1) (and the same in the y_i) together with ‘cross equations’

$$(2) \quad x_iy_j = 0 \quad \text{for all} \quad (i, j) \neq (0, 0), (3, 3).$$

1.2. Toric smoothings and pre-classification of diptych varieties. The tent T is singular along the four axes where the toric surfaces meet. We determine certain toric smoothings of T along these 1-strata. This is more-or-less well-known toric geometry. Along the x_3 axis, T is the hypersurface singularity $x_2y_3 = 0$. With a smoothing parameter A , this node can be smoothed by $x_2y_3 = x_3^a A$, for any $a \in \mathbb{Z}$. Picking other parameters b, B for the y_3 axis in T , one can show that there is a (Gorenstein, irreducible, normal, affine) toric variety

$$V_{AB} \subset \mathbb{C}^{10} = \mathbb{C}^8 \times \mathbb{C}^2(A, B)$$

with $T = (A = B = 0) \subset V_{AB}$ if and only if one of the two continued fraction expansions $[3, a, b, 3, 2]$ or $[2, 3, a, b, 3]$ equals 0. (The projection $V_{AB} \rightarrow \mathbb{C}^2(A, B)$ realises the deformation of T , although we only use the total space V_{AB} in what follows rather than the map.) The only solution is $\{a, b\} = \{1, 2\}$, which gives a choice of two toric 4-folds containing T .

Similarly, we can smooth the nodes along the x_0 and y_0 axes by choosing parameters there: the tags $\{1, 3\}$ together with smoothing parameters L, M determine another toric variety V_{LM} . In [2], we classify tents that admit at least two such toric smoothings, one smoothing at each end. There are four infinite families, each depending on 2 or 3 parameters.

1.3. Construction by serial unprojection. Using the tent described above, one choice of a pair of toric deformations at the corners are

$$\begin{aligned} \text{In } V_{AB}: \quad & x_2y_3 = x_3^2A, & x_3y_2 = y_3B, \\ \text{In } V_{LM}: \quad & x_1y_0 = x_0^3L, & x_0y_1 = y_0M. \end{aligned}$$

Toric calculations show that the V_{LM} smoothing of the x_0 and y_0 axes has local equations along the x_3 and y_3 axes as follows:

$$x_2y_3 = L^2M^5, \quad x_3y_2 = x_2^2LM^2.$$

The first step to construct $V = V_{ABLM}$ is to combine these equations along the x_3 and y_3 axes in what appears to be a rather naive way: simply write the equations

$$x_2y_3 = x_3^2A + L^2M^5 \quad \text{and} \quad x_3y_2 = y_3B + x_2^2LM^2.$$

Regarding this as a deformation of V_{AB} , we must deform the remaining equations of V_{AB} while preserving the syzygies. The two equations above can be mounted as two of the maximal Pfaffians of a skew 5×5 matrix (from which I omit the diagonal zeroes and the lower skew half):

$$\begin{pmatrix} x_2 & -LM^3 & -x_3A & x_1 & \\ & x_3 & -LM^2 & -B & \\ & & y_3 & -x_2^2 & \\ & & & y_2 & \end{pmatrix}.$$

The remaining 3 Pfaffians give equations that include x_1 , namely

$$x_1x_3 = x_2^3 + BLM^3, \quad x_1y_3 = x_2^2x_3A + y_2LM^3, \quad x_2y_2 = x_3AB + x_1LM^2.$$

We can generate equations involving y_1 by eliminating y_3 from this system to leave 2 equations in 6 variables, and then playing a similar Pfaffian trick to include y_1 . Continuing in this way, we compute many of the equations we need; we think of these as being the ‘short’ equations of the diptych.

The order in which we include and eliminate the variables follows from a projection argument. The step of including the variable x_1 is a kind of unprojection (in the sense of [7]). It is the first step in an induction that includes all variables one at a time. An extended example of this calculus can be seen in [8], Section 10.

1.4. Open questions. Diptych varieties are codimension 2 T-varieties in the sense of [3], and from that point of view the immediate task is to identify the corresponding surface and rank 4 lattice. Specialising V_{ABLM} as we do when constructing flips below can reduce the question to codimension 1.

We would like to know what the versal deformation space of a tent is—in the example above, is there a larger variety still containing all four toric deformations? There is the bigger problem of computing deformations of cycles of toric surfaces more generally than tents. The toric analysis seems similar to ours above, but we have little idea how extensions to larger varieties might work in the absence of straightforward projection calculus.

2. SEMISTABLE MORI FLIPS

2.1. Mori’s result. A Mori flip is a diagram $X^- \rightarrow X \leftarrow X^+$ of birational morphisms of projective 3-folds satisfying various conditions. (See [6] or [1] for details and examples.) In particular, each map contracts a curve to a point $p \in X$. Considering the flip in an analytic neighbourhood of p , the linear system $|-K_X|$ contains an element $S \ni p$ which has a DuVal (or Kleinian or ADE) singularity at p by [6]. A flip is called *semistable* when this singularity is of type A_n .

Given a Mori flip, one can construct its *canonical cover*:

$$A = \operatorname{Spec} \bigoplus_{n \in \mathbb{Z}} H^0(X, -nK_X).$$

The variety A has a \mathbb{C}^* action, determined by the grading of its defining coordinate ring, and the variation of GIT quotient of this action recovers the flip.

In [5], Mori classifies (the main case of) semistable flips by describing a Euclidean algorithm in the Picard group of X^- . He computes equations for X^+ by finding appropriate divisors using this algorithm. This both proves that X^+ exists—a motivating problem in the 1980s before [4]—and classifies these flips according to continued fraction solutions of Pell-like equations such as $3x^2 + 6xy + 2y^2 = 1$.

2.2. The link to diptych varieties. Our main claim is that certain diptych varieties V , including the example constructed above, have specialisations and a choice of \mathbb{C}^* action which are the canonical covers of certain flips.

If V is the diptych variety above, there are many \mathbb{C}^* actions on V for which x_0, y_0 have negative weight while x_3, y_3 have positive weight. The variation of such GIT quotients describes varieties and maps in the same configuration as a Mori flip. Specialising A, B, L, M equivariantly to functions of two variables reduces the dimension to 3; generic choices ensure the technical conditions for a flip.

By [5], for a general $s \in \mathcal{O}_{X,p}$ vanishing at p , the germ $p \in (s = 0)$ is also a cyclic quotient singularity. Thus the 3-fold germ $p \in X$ contains two toric surface germs. This is the motivation for the two panels V_{AB} and V_{LM} : one to cover S , the other $(s = 0)$. The choice of 4-sided tents and of \mathbb{C}^2 at the ends relate to the special case when the preimage of $p \in X$ on each side of the diagram is a \mathbb{P}^1 .

2.3. Open questions. PhD students are working on the D_n and E_6 cases. In other directions, it would be interesting to find other deformation problems for which diptych varieties are key varieties. A characteristic property of diptychs is the two toric sections, and this may hint at some candidates.

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Arcs and jets on toric singularities and quasi-ordinary singularities

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(joint work with Helena Cobo Pablos)

We denote by $(S, 0)$ a reduced irreducible equidimensional germ of algebroid singularity of dimension d defined over \mathbf{C} , an algebraically closed field of zero characteristic. A *formal arc* on the germ $(S, 0)$ is a morphism of germs $h : (D, 0) \rightarrow (S, 0)$ where $D := \text{Spec } \mathbf{C}[[t]]$. If we fix an embedding $(S, 0) \subset (\mathbf{C}^n, 0)$ and local coordinates (x_1, \dots, x_d) at 0 then the arc h is given by n power series in $\mathbf{C}[[t]]$: $x_i(t) = a_1^{(i)}t + a_2^{(i)}t^2 + \dots + a_r^{(i)}t^r + \dots$, $i = 1, \dots, n$, such that $F(x_1(t), \dots, x_n(t)) = 0$, for any F in the ideal of $(S, 0)$. The set of arcs H on $(S, 0)$ can be seen as an affine subscheme of $\text{Spec } \mathbf{C}[a_1^{(i)}, a_2^{(i)}, \dots]_{i=1}^n$. An *s-jet* on the germ $(S, 0)$ is a morphism of germs $h : (D_s, 0) \rightarrow (S, 0)$ where $D_s = \text{Spec } \mathbf{C}[[t]]/(t)^{s+1}$. The set H^s of *s-jets* on $(S, 0)$ is an affine subscheme of $\mathbf{A}_{\mathbf{C}}^{sn} := \text{Spec } \mathbf{C}[a_1^{(i)}, \dots, a_s^{(i)}]_{i=1}^n$.

Any arc $h \in H$ has a *s-jet* $j^s(h) \in H^s$. A theorem of Greenberg implies that the set $j^s(H) \subset H^s$ is a *constructible set* of H^s for every $s \geq 0$. It has an image $[j^s(H)]$ in the *Grothendieck ring* $K_0(\text{Var}_{\mathbf{C}})$ of *C-varieties*. This ring is generated by the symbols $[X]$ for X an algebraic variety, subject to relations: $[X] = [X']$ if X is isomorphic to X' , $[X] = [X - X'] + [X']$ if $[X']$ is closed in X and $[X][X'] = [X \times X']$. If X is an algebraic variety the map $X' \mapsto [X']$, for X' closed in X extends to constructible subsets W of X , $W \mapsto [W]$ in a unique way if $[W \cup W'] = [W] + [W'] - [W \cap W']$, see [2].

The *geometric Poincaré series* $P_{\text{geom}}(T) := \sum_{s \geq 0} [j^s(H)]T^s \in K_0(\text{Var}_{\mathbf{C}})[[T]]$. is an invariant of the germ $(S, 0)$. We denote by \mathbf{L} the class $\mathbf{L} = [\mathbf{A}_{\mathbf{C}}^1]$ of the affine line and by $\mathcal{M}_{\mathbf{C}}$ the ring $K_0(\text{Var}_{\mathbf{C}})[\mathbf{L}^{-1}]$. A theorem of Denef and Loeser states that the series $P_{\text{geom}}(T)$, when viewed in $\mathcal{M}_{\mathbf{C}}[[T]]$, is a rational function, i.e., it belongs to $\mathcal{M}_{\mathbf{C}}[T]$ (see [2]). The proof of this deep result concerns quantifier elimination for semi-algebraic sets of power series in zero characteristic, the theory of motivic integration introduced by Kontsevich and the existence of resolution of singularities of varieties over a field of zero characteristic. The invariants of $(S, 0)$ encoded by this series are not well understood, see Nicaise work for some particular cases [6]. Lejeune and Reguera gave an explicit description of this series in the case of an affine normal toric surface, see [4]. We describe the classes $[j^s(H)]$ associated to a *quasi-ordinary hypersurface singularity* or to a germ of affine toric variety, needless to say non necessarily normal, in terms of the convexity properties of certain monomial ideals which we associate to the singularity, if the singularity is locally unbranched along the singular locus. Rond [9] studies inductively the the series $P_{\text{geom}}(T)$ in the quasi-ordinary case.

1. TORIC CASE

Let M be a rank d lattice, N its dual lattice and $\Lambda \subset M$ a submonoid generated by e_1, \dots, e_n , such that the cone $\sigma^\vee := \mathbf{R}_{\geq 0}e_1 + \dots + \mathbf{R}_{\geq 0}e_n \subset M_{\mathbf{R}} := M \otimes \mathbf{R}$ is strictly convex of dimension d . Denote by $\sigma \subset N_{\mathbf{R}}$ the dual cone of $\sigma^\vee \subset M_{\mathbf{R}}$. If $e \in \Lambda$ we denote by $\chi^e \in \mathbf{C}[\Lambda]$ the corresponding monomial. We denote the affine toric variety $\text{Spec } \mathbf{C}[\Lambda]$ by Z^Λ . The toric morphism $Z^{\sigma^\vee \cap M} \rightarrow Z^\Lambda$ corresponding to the inclusion $\Lambda \subset \sigma^\vee \cap M$ is the normalization map. The embedding $Z^\Lambda \subset \mathbf{C}^n$, defined by $x_i = \chi^{e_i}$ for $i = 1, \dots, n$, is equivariant.

We study the class $[j^s(H)]$ if $(S, 0) = (Z^\Lambda, 0)$. An arc $h \in H$ has its generic point in the torus if and only if $\chi^m \circ h \neq 0, \forall m \in M$. We expand $\chi^m \circ h = t^{\nu_h(m)} u_h(m)$, where $u_h(m)$ is a unit in $\mathbf{C}[[t]]$. It follows that $\nu_h \in \overset{\circ}{\sigma} \cap N$. Denote by H^* the set of arcs $h \in H$ with generic point in the torus.

Lemma. *If S is locally unibranch along its singular locus and if then we have $j^s(H) = j^s(H^*)$, for $s \geq 0$ (see [7] in the normal toric case).*

We have a partition $H^* = \sqcup_{\nu \in \overset{\circ}{\sigma} \cap N} H_\nu^*$ where $H_\nu^* := \{h \in H^* / \nu_h = \nu\}$. It follows that $j^s(H) = j^s(H^*) = \cup_{\nu \in \overset{\circ}{\sigma} \cap N} j^s(H_\nu^*)$, but this union is non-finite and non-disjoint since different arcs $h \in H_\nu^*$ and $h' \in H_{\nu'}^*$ may have the same s -jet. We follow the strategy of Lejeune and Reguera [4]: we show that $j^s(H_\nu^*)$ is locally closed in H^s and we compute the class $[j^s(H_\nu^*)]$; then we exhibit a finite subset $\Xi(s)$ of $\overset{\circ}{\sigma} \cap N$ determining a partition $j^s(H) = \sqcup_{\nu \in \Xi(s)} j^s(H_\nu^*)$. This description is given in terms of a sequence of monomial ideals of the local ring $\mathbf{C}[[\sigma^\vee \cap M]]$. In the case of Lejeune and Reguera these ideals are the *maximal ideal* and the *logarithmic jacobian ideal*, which determines the *Nash modification* of S , see [4].

Let us define the following subsets of Γ , for $k = 1, \dots, d$:

$$(1) \quad \mathcal{J}_k := \{e_{i_1} + \dots + e_{i_k} / e_{i_1}, \dots, e_{i_k} \text{ linearly independent} \}_{1 \leq i_1 < \dots < i_k \leq d}.$$

If $\mathcal{J} \subset \sigma^\vee \cap M$ we denote also by \mathcal{J} the corresponding monomial ideal of $\mathbf{C}[[\sigma^\vee \cap M]]$. The *Newton polyhedron* $\mathcal{N}(\mathcal{J})$ of the monomial ideal \mathcal{J} is the convex hull of $\mathcal{J} + \sigma^\vee$. We denote by $\text{ord}_{\mathcal{J}}$ the support function of the polyhedron $\mathcal{N}(\mathcal{J})$, defined by: $\text{ord}_{\mathcal{J}} : \sigma \rightarrow \mathbf{R}, \nu \mapsto \inf_{\omega \in \mathcal{N}(\mathcal{J})} \langle \nu, \omega \rangle$. We use the following notations: $\phi_1 := \text{ord}_{\mathcal{J}_1}, \phi_2 := \text{ord}_{\mathcal{J}_2} - \text{ord}_{\mathcal{J}_1}, \dots, \phi_d := \text{ord}_{\mathcal{J}_d} - \text{ord}_{\mathcal{J}_{d-1}}$. We have that $\phi_1 \leq \phi_2 \leq \dots \leq \phi_d$ on σ (point-wise). For any $s \geq 0$, we have a partition of $\overset{\circ}{\sigma} \cap N$:

$$\begin{aligned} \rho_0(s) &:= \{ \mu \in \overset{\circ}{\sigma} \cap N \quad / \quad s < \phi_1(\mu) \} \\ \rho_1(s) &:= \{ \mu \in \overset{\circ}{\sigma} \cap N \quad / \quad \phi_1(\mu) \leq s < \phi_2(\mu) \} \\ \dots & \quad \dots \quad \quad \quad \dots \quad \quad \dots \\ \rho_{d-1}(s) &:= \{ \mu \in \overset{\circ}{\sigma} \cap N \quad / \quad \phi_{d-1}(\mu) \leq s < \phi_d(\mu) \} \\ \rho_d(s) &:= \{ \mu \in \overset{\circ}{\sigma} \cap N \quad / \quad \phi_d(\mu) \leq s \}. \end{aligned}$$

Theorem 1. *If $s \geq 0$ and $\nu \in \overset{\circ}{\sigma} \cap N$, let k be the unique integer such that $\nu \in \rho_k(s)$ then we have that if $k = 0$ the jet space $j^s(H_\nu^*)$ is equal to $\{0\}$ otherwise it is isomorphic to $(\mathbf{C}^*)^k \times \mathbf{A}_{\mathbf{C}}^{sk - \text{ord}_{\mathcal{J}_k}(\nu)}$.*

Remark. The ideal \mathcal{J}_d gives the Nash modification of a normal $(S, 0)$, see [4].

We define an equivalence relation \sim in the set $\rho_k(\sigma)$ for any $s > 0$ and $1 \leq k \leq d$:

$$\nu \sim \nu' \in \rho_k(s) \Leftrightarrow \left\{ \begin{array}{l} \nu \text{ and } \nu' \text{ define the same face of } \mathcal{N}(\mathcal{J}_j) \\ \text{and } \text{ord}_{\mathcal{J}_j}(\nu) = \text{ord}_{\mathcal{J}_j}(\nu'). \end{array} \right\} \text{ for } 1 \leq j \leq k.$$

We denote by $\bar{\nu}$ the class of ν . The quotient set $\rho_k(s)/\sim$ is finite.

Corollary 2. If S is locally unibranch along its singular locus the class of $j^s(H)$ in the Grothendieck ring is of the form:

$$[j^s(H)] = 1 + \sum_{k=1}^d \sum_{\bar{\nu} \in \rho_k(s)/\sim} (\mathbf{L} - 1)^k \mathbf{L}^{ks - \text{ord}_{\mathcal{J}_k}(\nu)}.$$

2. QUASI-ORDINARY HYPERSURFACE CASE

An equidimensional germ $(S, 0)$ is *quasi-ordinary* (QO) if there exists a finite projection $\pi : (S, 0) \rightarrow (\mathbf{C}^d, 0)$ which is a local isomorphism outside a normal crossing divisor. The class of QO-singularities contains curve singularities and simplicial toric singularities; it is important in Jung's approach to resolution of singularities, see [5]. The normalization of a QO-singularity is a toric singularity, see [8]. A *QO-hypersurface* $(S, 0)$ has an equation $f = 0$, where $f \in \mathbf{C}[[x_1, \dots, x_d]][y]$ is the minimal polynomial over $\mathbf{C}[[x_1, \dots, x_d]]$ of a especial type of fractional power series $\zeta \in \mathbf{C}[[x_1^{1/m}, \dots, x_d^{1/m}]]$ possessing a finite set $\lambda_1, \dots, \lambda_g \in \mathbf{Q}^d$ of *characteristic exponents*, see [1] and [5]. This series generalizes Newton-Puiseux expansions. The normalization is the toric singularity $Z^{\sigma^\vee \cap M}$ where $\sigma^\vee := \mathbf{R}_{\geq 0}^d$ and $M := \sum_{i=1}^{d+g} \mathbf{Z}e_i$, for $x_i = \chi^{e_i}$, $i = 1, \dots, d$; and $e_{d+j} := \lambda_j$, $j = 1, \dots, g$, see [3]. The strategy is similar to the toric case, though proofs are more involved. We denote by H^* the set of arcs $h \in H$ such that $\pi \circ h$ has its generic point in the torus, we set H_ν^* analogously for $\nu \in \overset{\circ}{\sigma} \cap N$. We define the sets for $k = 1, \dots, d$:

$$(2) \quad \mathcal{J}_k := \left\{ \sum_{r=1}^k e_{i_r}/e_{i_1}, \dots, e_{i_k} \text{ linearly independent} \right\}_{\substack{i_{k-1} < i_k \leq d+g \\ 1 \leq i_1 < \dots < i_{k-1} \leq d}},$$

Theorem 3. With the notations and hypothesis analogous to those of section 1, the statement of Theorem 1 and Corollary 2 holds with respect to the ideals (2).

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Thom polynomials of singularities

ANDRÁS SZENES

(joint work with Gergely Bérczi)

In joint work with Gergely Bérczi, we compute the so-called *Thom polynomials* [6] of a family of Boardman singularities. Broadly speaking, this is an expression describing the fundamental class of a singularity in the space of jets of functions. In more detail, let N, n, k be three positive integers; we will assume that $n \leq k$. We can model the jet space of holomorphic maps $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^k, 0)$ up to degree N as the direct sum

$$\mathcal{J}^{(N)} = \bigoplus_{l=1}^N \text{Hom}(\text{Sym}^l \mathbb{C}^n, \mathbb{C}^k),$$

or equivalently, as the set

$$\mathcal{J}^{(N)} = \{(P_1, \dots, P_k); P_i \in \mathbb{C}[x_1, \dots, x_n], P_i(0) = 0, \deg P_i \leq N\}$$

This is a linear space with the action of the substitution group $\mathcal{D} = \text{Diff}(\mathbb{C}^n, 0) \times \text{Diff}(\mathbb{C}^k, 0)$; we will be interested in the subgroup of linear substitutions $G = \text{GL}_n \times \text{GL}_k$.

Now, let \mathcal{A} be an Artinian ring, and consider the subspace

$$\mathcal{J}_{\mathcal{A}}^{(N)} = \{(P_1, \dots, P_k) \in \mathcal{J}^{(N)}; \mathbb{C}[x_1, \dots, x_n]/\langle P_1, \dots, P_k \rangle \cong \mathcal{A}\},$$

where $\langle \dots \rangle$ means “the ideal generated by”. This subset is \mathcal{D} -invariant, and thus it is G -invariant. Thus one can consider its equivariant Poincaré dual, a homogeneous polynomial $eP_{\mathcal{A}}$ in $n+k$ variables $\lambda_1, \dots, \lambda_n, \theta_1, \dots, \theta_k$, which is symmetric in the λ s and θ s separately. The degree of $eP_{\mathcal{A}}$ equals the codimension of $\mathcal{J}_{\mathcal{A}}^{(N)}$ in $\mathcal{J}^{(N)}$. This polynomial stabilizes for large N , and thus this stable version only depends on \mathcal{A}, n and k .

In this talk, we consider the case $\mathcal{A} = \mathbb{C}[t]/t^{d+1}$. For this algebra, one can choose $N = d$. This singularity is known as the A_d singularity, or the Boardman singularity of index 1_d . There has been considerable amount of work done on this case [2, 5, 4, 3, 1]. The known cases are the cases of $d = 1, d = 2$, with a conjectural formula for $d = 3$, and results for any d in the case $n = k$. We, motivated by localization theory, give a unified approach for arbitrary n, k and d , with the dependence on n and k completely separated, and thus the computation is reduced to an object depending on d only. This object turns out to be trivial for $d = 1, 2$ and 3 , may be computed by hand for $d = 4, 5$, and can be computed by computer for $d = 6, 7$.

It is also compatible with the following result of Thom:

Proposition 1. Consider the infinite sequence of elements c_i , $\deg c_i = i$, defined by the generating series

$$c(z) = 1 + c_1 z + c_2 z^2 + \dots = \frac{\prod_{m=1}^k (1 - \theta_m z)}{\prod_{i=1}^n (1 - \nu_i z)}.$$

Then the equivariant Poincaré dual $eP_{\mathcal{A}}$ is a polynomial in the classes c_i .

This statement holds in greater generality, not just for $\mathcal{A} = \mathbb{C}[t]/t^{d+1}$. Now we can introduce the *Thom polynomial* via

$$\text{Th}_{\mathcal{A}}(c_1, c_2, \dots) = eP_{\mathcal{A}}(\lambda_1, \dots, \lambda_n, \theta_1, \dots, \theta_k)$$

The final result may be described as follows.

Theorem 2. Let B_d be the space group of d -by- d upper-triangular matrices, with $T_d \subset B_d$ the torus of diagonal matrices. Then there exists a representation V_d of B_d , and a vector $v_1 \in V_d$ such that

$$\text{Th}_{\mathcal{A}=\mathbb{C}[t]/t^{d+1}} = \text{Res}_{z_d, z_1, z_2, \dots, z_{d-1}} \frac{Q_d \prod_{i < j} (z_i - z_j)}{\prod_{i \leq j, i+j \leq d} (z_i + z_j - z_l)} \prod_{i=1}^d \frac{c(z_i) dz_i}{z_i^{k-n+1}},$$

where Q_d the equivariant Poincaré dual of the orbit of the point v_1 on the representation V_d , the z_i s are the weights of the action of T_d , and the iterated residues are taken at infinity.

The representation V_d and the vector v_1 may be described explicitly, and are related to the symmetric square of the fundamental representation.

The proof is based on a careful study of the model of Porteous and Ronga. This characterizes the elements of $\mathcal{J}_{\mathcal{A}}^{(N)}$, as the (jets of) those maps $\psi : \mathbb{C}^n \rightarrow \mathbb{C}^k$ for which there exists a test curve C in \mathbb{C}^n such that the first d derivatives of $\psi(C)$ vanish at the origin. This allows one to fiber $\mathcal{J}_{\mathcal{A}}^{(N)}$ over a partial flag variety. The fibers themselves have the structure of a linear fibration which suggest a natural compactification. Finally, the theorem is obtained after applying two localization principles, a variant of the Berline-Vergne equivariant localization theorem and a global residue principle.

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Homological methods for hypergeometric families

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(joint work with Laura Felicia Matusevich and Uli Walther)

This abstract is taken nearly verbatim from the introduction to [MMW05].

In the late 1980s, Gelfand, Graev, and Zelevinsky introduced a class of systems of linear partial differential equations that are closely related to toric varieties [GGZ87]. These systems, now called *GKZ systems*, or *A-hypergeometric systems*, are constructed from a $d \times n$ integer matrix A of rank d and a complex parameter vector $\beta \in \mathbb{C}^d$, and are denoted by $H_A(\beta)$. We assume that the columns of A lie in a single open halfspace. A -hypergeometric systems arise in various instances in algebraic geometry. For example, solutions of A -hypergeometric systems appear as toric residues [CDS01], and special cases are mirror transforms of generating functions for intersection numbers on moduli spaces of curves [CK99], the A -hypergeometric systems there being Picard–Fuchs equations governing the variation of Hodge structures for Calabi–Yau toric hypersurfaces.

The first fundamental results about the systems $H_A(\beta)$ were proved by Gelfand, Graev, Kapranov, and Zelevinsky. These results concerned the case where the semigroup $\mathbb{N}A$ generated by the columns of A gives rise to a semigroup ring $\mathbb{C}[\mathbb{N}A]$ that is Cohen–Macaulay and graded in the standard \mathbb{Z} -grading [GGZ87, GKZ89]. In geometric terms, the associated projective toric variety X_A is arithmetically Cohen–Macaulay. The above authors showed that, in this case, the system $H_A(\beta)$ gives a holonomic module over the ring D of polynomial \mathbb{C} -linear differential operators in n variables, and hence $H_A(\beta)$ has finite *rank*; that is, the dimension of its holomorphic solution space is finite. Furthermore, they showed that this dimension can be expressed combinatorially, as the integer $\text{vol}(A)$ that is $d!$ times the Euclidean volume of the convex hull of the columns of A and the origin $0 \in \mathbb{Z}^d$. The remarkable fact is that their rank formula holds independently of the parameter β .

Even if $\mathbb{C}[\mathbb{N}A]$ is not Cohen–Macaulay or \mathbb{Z} -graded, Adolphson showed that $H_A(\beta)$ is always a holonomic ideal [Ado94]. He further proved that, for all parameters outside of a closed locally finite arrangement of countably many ‘semi-resonant’ affine hyperplanes, the characterization of rank as volume still holds.

It came as quite a surprise when in [ST98] an example was given showing that if $\mathbb{C}[\mathbb{N}A]$ is not Cohen–Macaulay then not all parameters β have to give the same rank. Thus the set \mathcal{E}_A of *exceptional parameters* $\beta \in \mathbb{C}^d$, for which the rank does not take the expected value, can be nonempty. Nearly at the same time, the case of projective toric curves was discussed completely in [CDD99]: the set \mathcal{E}_A of exceptional parameters is finite in this case, and empty precisely when $\mathbb{C}[\mathbb{N}A]$ is Cohen–Macaulay; moreover, at each $\beta \in \mathcal{E}_A$ the rank exceeds the volume by exactly 1.

It was shown soon after in [SST00] that the rank can never be smaller than the volume as long as $\mathbb{C}[\mathbb{N}A]$ is \mathbb{Z} -graded, and it was established in the same book that \mathcal{E}_A is in fact contained in a finite affine subspace arrangement. More recently, the

much stronger fact emerged that \mathcal{E}_A is a finite union of Zariski locally closed sets by means of Gröbner basis techniques [Mat03]. While rank-jumps can be arbitrarily large [MW04], the absence of rank-jumping parameters is equivalent to the Cohen–Macaulay property for \mathbb{Z} -graded $\mathbb{C}[\mathbb{N}A]$ when either $\mathbb{C}[\mathbb{N}A]$ has codimension two [Mat01], or if the convex hull of A is a simplex [Sai02], or if $\mathbb{C}[\mathbb{N}A]$ is a polynomial ring modulo a generic toric ideal [Mat03].

Encouraged by these results, which suggest an algebraic structure on the set of exceptional parameters, it was conjectured in [MM05] that the obstructions to the Cohen–Macaulayness of $\mathbb{C}[\mathbb{N}A]$ and the set of exceptional parameters are identified in an explicit manner. To be precise, let $H_{\mathfrak{m}}^{<d}(\mathbb{C}[\mathbb{N}A])$ be the direct sum of all the local cohomology modules supported at the maximal homogeneous ideal \mathfrak{m} of $\mathbb{C}[\mathbb{N}A]$ in cohomological degrees less than d . Then define the set E_A of *exceptional quasidegrees* to be the Zariski closure in \mathbb{C}^d of the set of \mathbb{Z}^d -graded degrees α such that $H_{\mathfrak{m}}^{<d}(\mathbb{C}[\mathbb{N}A])$ has a nonzero element in degree $-\alpha$. With this notation, our motivating result is the following.

Theorem. *For any rank d matrix $A \in \mathbb{Z}^{d \times n}$ as above, the set \mathcal{E}_A of exceptional (that is, rank-jumping) parameters equals the set E_A of exceptional quasidegrees.*

We note that there is no assumption on $\mathbb{C}[\mathbb{N}A]$ being \mathbb{Z} -graded. The \mathbb{Z} -graded simplicial case of this result was proved in [MM05] using results of [Sai02].

Methods and results. The first step in our proof of the Theorem is to construct a homological theory to systematically detect rank-jumps. To this end, we study rank variation in any family of holonomic modules over any base B , and not just A -hypergeometric families over $B = \mathbb{C}^d$. The idea is that under a suitable coherence assumption, holonomic ranks behave like fiber dimensions in families of algebraic varieties. In particular, we prove that rank is constant almost everywhere and can only increase on closed subsets of B . We develop a computational tool to check for rank-jumps at a smooth point $\beta \in B$: since the rank-jump occurs through a failure of flatness at β , ordinary Koszul homology detects it.

The second step toward the Theorem is to construct a homological theory for D -modules that reproduces the set E_A of exceptional quasi-degrees, which a priori arises from the commutative notion of local cohomology. Our main observation along these lines is that the *Euler–Koszul complex*, which was already known to Gelfand, Kapranov, and Zelevinsky for Cohen–Macaulay \mathbb{Z} -graded semigroup rings [GKZ89], generalizes to fill this need. Adolphson [Ado99] recognized that when the semigroup is not Cohen–Macaulay, certain conditions guarantee that this complex has zero homology. Here, we develop *Euler–Koszul homology* for the class of *toric modules*, which are slight generalizations of \mathbb{Z}^d -graded modules over the semigroup ring $\mathbb{C}[\mathbb{N}A]$. For any toric module M , we show that the set of parameters β for which the Euler–Koszul complex has nonzero higher homology is precisely the analogue for M of the exceptional quasi-degree set E_A defined above for $M = \mathbb{C}[\mathbb{N}A]$.

Having now two cohomology theories, one being a D -module theory to recover local cohomology quasi-degrees for hypergeometric families, and the other being

a geometric theory to detect rank-jumping parameters for general holonomic families, we link them in our central result: for toric modules, these two theories coincide. Consequently, we obtain our motivating Theorem as the special case $M = \mathbb{C}[\mathbb{N}A]$ of a result that holds for arbitrary toric modules M . From there, we deduce the equivalence of the Cohen–Macaulay condition on $\mathbb{C}[\mathbb{N}A]$ with the absence of rank-jumps in the GKZ hypergeometric system $H_A(\beta)$.

As a final comment, let us note that we avoid the explicit computation of solutions to hypergeometric systems. This contrasts with the previously cited constructions of exceptional parameters, which rely on combinatorial techniques to produce solutions. It is for this reason that these constructions contained the assumption that the semigroup ring $\mathbb{C}[\mathbb{N}A]$ is graded in the usual \mathbb{Z} -grading, for this implies that the corresponding hypergeometric systems are *regular holonomic* and thus have solutions expressible as power series with logarithms, with all the combinatorial control this provides. Our use of homological techniques makes our results valid in both the regular and non-regular cases.

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Syzygies of toric varieties

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(joint work with Hal Schenck, Greg Smith)

Understanding the equations defining an algebraic variety in projective space is a classical question in algebraic geometry. In the 1980's, Green realized that classical results by Castelnuovo, Noether, Petri and Fujita on the equations defining an algebraic curve generalize to higher syzygies and he and Lazarsfeld uncovered a beautiful connection between syzygies and geometry in the form of property N_p .

Let L be a globally generated line bundle on a normal projective variety X , let $R = \bigoplus H^0(X, L^m)$ be the section ring associated to L , and let $S = \text{Sym}^\bullet H^0(X, L)$. Then R is a finitely generated S -module and it admits a minimal free graded resolution over S , $E_\bullet \rightarrow R$. We say that L satisfies N_0 if $E_0 \cong S$ and that L satisfies N_p if $E_0 \cong S$ and $E_i \cong \bigoplus S(-i-1)$ for $1 \leq i \leq p$. Hence L satisfies N_0 if and only if it gives rise to a projectively normal embedding, and L satisfies N_1 , it and only if the ideal of this embedding is generated by quadratic equations. Moreover, L also satisfies N_2 if and only if the first syzygies of these equations are linear.

Green [8] showed that sufficient powers of ample line bundles satisfy N_p , but exact bounds are not even known in the case of Veronese embeddings or curves. In a paper with H. Schenck and G. Smith [11] we obtain sufficient criteria for ample line bundles on toric varieties by exploiting the connection of the regularity of a line bundle with N_p .

Let L be a globally generated line bundle on a toric variety X corresponding to a lattice polytope P . Then L satisfies N_0 if and only if P is normal, i.e., if for every m , every lattice point in mP can be written as a sum of m lattice points in P . P gives rise to a polytopal semigroup S_P and the associated semigroup ring $k[S_P]$ is naturally isomorphic with the section ring R associated to L . A k -algebra R is called *Koszul*, if $\text{Tor}_i^R(k, k)_m = 0$ for $i \neq m$, i.e., k admits a linear resolution over R . Then if R is Koszul, L satisfies N_1 . The converse is not true (see Sturmfels [21]), but evidence suggests that when a line bundle L has a natural reason to satisfy N_1 , then the associated section ring is Koszul.

Theorem 1 ([11, Corollary 1.2.]). *Let L be an ample line bundle on a toric variety of dimension n . Then L^{n-1+p} satisfies N_p for $p \geq 0$.*

The case $p = 0$ is the well known fact that for a polytope of dimension n , $(n-1)P$ is normal (cf. [4], [13] and [1]) and Bruns, Gubeladze and Trung [1] show that $k[S_{nP}]$ is Koszul which implies the case $p = 1$. Recently Ogata [14] proved that when $n \geq 3$, L^{n-2+p} satisfies N_p for $p \geq 1$.

Using information of the Hilbert polynomial, we obtain the following generalization.

Theorem 2 ([11, Corollary 1.3.]). *Let L be an ample line bundle on a toric variety of dimension n . If r denotes the number of integer roots of the Hilbert polynomial of L , then $L^{n-r-1+p}$ satisfies N_p for $p \geq 1$, unless $L \cong \mathcal{O}_{\mathbb{P}^n}(1)$.*

In fact, if L is a globally generated line bundle on a toric variety corresponding to a lattice polytope P , then the number of integer roots r of the Hilbert polynomial of L coincides with the largest integer r such that rP does not contain any lattice points in its interior. This motivates the following theorem.

Theorem 3 ([11, Corollary 1.4.], [10, Corollary IV.28]). *Let P be a lattice polytope of dimension n , and let r be the largest integer such that rP does not contain any interior lattice points. Then*

- (i) $(n-r)P$ is normal, and
- (ii) $K[S_{(n-r)P}]$ is Koszul.

A natural question generalizing Green's result on curves and extending Fujita's conjectures is whether adjoint line bundles of the form $K_X \otimes A^{n+2+p}$ for an ample line bundle A satisfy N_p . Ein and Lazarsfeld [3] have shown that $K_X \otimes A^{n+p}$ satisfies N_p for a very ample line bundle A on a smooth projective variety $X \not\cong \mathbb{P}^n$. We have a similar result for Gorenstein toric varieties.

Theorem 4 ([11, Corollary 1.6.]). *Let X be a Gorenstein projective toric variety of dimension n , and let B_1, \dots, B_r be the minimal generators of the nef cone of X . Let A be an ample line bundle such that for all i , $A \otimes B_i^{-1}$ is globally generated and assume that $A \not\cong \mathcal{O}_{\mathbb{P}^n}(1)$. Then $K_X \otimes A^{n+p}$ satisfies N_p for $p \geq 1$.*

When X is a smooth toric surface, a formula due to Schenck [20] and Gallego and Purnaprajna [6] implies the following precise criterion.

Theorem 5 ([10, Corollary IV.23]). *Let L be an ample line bundle on a smooth toric surface X corresponding to a lattice polygon P . Then $L(K_X)$ satisfies N_p if and only if*

$$|\partial P \cap M| + |\{\text{vertices of } P\}| \geq p + 15.$$

Our methods also apply to Segre-Veronese embeddings. Green [9] proved that the Veronese embedding $\mathcal{O}_{\mathbb{P}^n}(d)$ satisfies N_d . Ottaviani and Paoletti [15] conjecture that for $d \geq 3$ and $n \geq 2$, $\mathcal{O}_{\mathbb{P}^n}(d)$ satisfies N_{3d-3} , and they show the necessity of this condition. This conjecture is known for $n = 2$. We have the following generalization of Green's theorem to Segre-Veronese embeddings.

Corollary 6 ([11, Corollary 1.5.]). *Let $X = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$. Then $\mathcal{O}_X(d_1, \dots, d_r)$ satisfies $N_{\min\{d_1, \dots, d_r\}}$.*

In fact, the Segre-embedding $\mathcal{O}_X(1, \dots, 1)$ satisfies N_3 ; see Lascoux [12] and Pragacz-Weyman [17] for $r \leq 2$ and Rubei [18, 19] for $r \geq 3$.

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