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9.1 # Mori Dream Spaces

Goal: X $\mathbb{Q}$-factorial, projective $\mathrm{Pic}(X)\otimes\mathbb{R} \simeq N^1(X)$

X MDS $\iff$ $\mathrm{Cox}(X)$ is finitely generated $\Bbbk$-alg.

§1 Zariski's theorem

$f: X \longrightarrow Y$ $\mathcal{F}$ on X, $\mathcal{G}$ vector bundle on Y, projection formula gives $f_*(\mathcal{F}\otimes f^*(\mathcal{G})) \simeq f_*(\mathcal{F}) \otimes \mathcal{G}$.

Th'm [Zariski].
L semi-ample, then $R(X,L) := \bigoplus_{j\geq 0} \Gamma(L^{\otimes j})$ is fin. gen'd / $\Bbbk$.
Pf.
1. A ample, $\mathcal{F}$ coherent on X, $\exists\, m_0 \geq 0$ s.t. $\forall\, a,b \geq m_0$

$$\Gamma(\mathcal{F}\otimes A^{\otimes a}) \otimes_{\Bbbk} \Gamma(A^{\otimes b}) \twoheadrightarrow \Gamma(\mathcal{F}\otimes A^{\otimes(a+b)}) \qquad (*)$$

Sketch: $0 \to I_\Delta \to \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta \to 0$ $\otimes\, \mathcal{G}_{a,b} := (\mathcal{F}\otimes A^{\otimes a}) \boxtimes A^{\otimes b}$
and take long exact sequence in cohomology

$$\begin{array}{ccccc} H^0(X\times X, \mathcal{G}_{ab}) & \to & H^0(X\times X, \mathcal{G}_{ab}|_\Delta) & \to & H^1(X\times X, I_\Delta \otimes \mathcal{G}_{ab}) \\ \text{Künneth } \wr\wr & & \wr\wr & & \downarrow \\ \Gamma(\mathcal{F}\otimes A^a)\otimes\Gamma(A^b) & \to & \Gamma(\mathcal{F}\otimes A^{\otimes(a+b)}) & & 0 \end{array}$$

after taking resol'n of $I_\Delta \otimes (\mathcal{F}\boxtimes\mathcal{O})$ and apply Serre van.

Case $\mathcal{F} = \mathcal{O}_X$ gives $R(X,A)$ gen'd in degree $\leq 2m_0$

2. L semi-ample $\therefore$ $f: X \twoheadrightarrow Y \subseteq |L^{\otimes m}|$ s.t.
$f^*(A) = L^{\otimes m}$ and $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$. Then

$$\Gamma(X, L^{jm}) \cong \Gamma(X, f^*(A^j)) \cong \Gamma(Y, f_*f^*(A^j)) \simeq \Gamma(Y, A^j\otimes f_*\mathcal{O}_X) \simeq \Gamma(Y, A^j)$$

9.2 $\therefore R(X, L^m)$ is fin. gen'd / $\Bbbk$ by Step 1.

3. For $G = L^i \quad (1 \leq i \leq m-1)$

$$\Gamma(X, G \otimes L^{am}) \simeq \Gamma(Y, f_*(G \otimes L^{am})) \cong \Gamma(Y, A^{\otimes a} \otimes \underset{\substack{\shortparallel \\ \mathcal{F}}}{f_*(G)})$$

and sub into (*) to get $\qquad$ coh sheaf as f proper

$$\Gamma(X, L^{i+am}) \otimes_{\Bbbk} \Gamma(X, L^{bm}) \twoheadrightarrow \Gamma(X, L^{i+(a+b)m})$$

for $a, b \geq m(i)$ and $1 \leq i \leq m-1$, so $R(X, L)$ fin gen'd. $\square$

<u>Corollary</u>
$L_1, \dots, L_r$ semi-ample, $R(X, L_1, \dots, L_r) := \bigoplus_{(m_1, \dots, m_r)} \Gamma(L_1^{m_1} \otimes \dots \otimes L_r^{m_r})$
finitely generated $\Bbbk$-algebra.

<u>Pf</u>

$$\mathbb{P} := \mathbb{P}(L_1 \oplus \dots \oplus L_r) \xrightarrow{\pi} X$$

$$\shortparallel$$

$$\mathrm{Proj}_{\mathcal{O}_X} \mathrm{Sym}^{\cdot}(L_1 \oplus \dots \oplus L_r)$$

$$\pi_* \mathcal{O}_{\mathbb{P}}(d) = \mathrm{Sym}^d(L_1 \oplus \dots \oplus L_r) = \bigoplus_{\Sigma m_i = d} L_1^{m_1} \otimes \dots \otimes L_r^{m_r}$$

$$\therefore \Gamma(\mathcal{O}_{\mathbb{P}}(d)) = \Gamma(X, \oplus_{\Sigma m_i = d} L_1^{m_1} \otimes \dots \otimes L_r^{m_r})$$

$$\therefore R(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \cong R(X, L_1, \dots, L_r)$$

$\uparrow$ semiample $\qquad \therefore$ fin. gen'd / $\Bbbk$. $\square$

§2 <u>Mori Chambers</u>

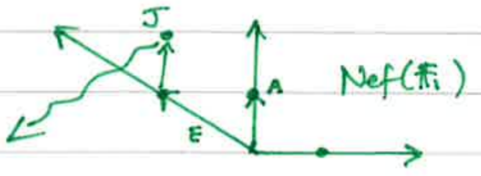
X MDS, $\exists\ Y_1, \dots, Y_l$ MDS's and birational contractions

9.3

$$\begin{tikzcd} & W_i \arrow[dl, "p_i"'] \arrow[dr, "q_i"] & \\ X \arrow[rr, dashed, "g_i"] & & Y_i = \mathrm{Proj}(X, J_i) \end{tikzcd} \qquad J_i = g_i^*(\underbrace{A_i}_{\text{ample}}) \otimes \mathcal{O}_X(\underbrace{E_i}_{\text{'fixed'}})$$

where $(q_i)_*\left(\mathcal{O}_{W_i}(\widetilde{E_i} + \mathrm{excep}(p_i))\right) \cong \mathcal{O}_{Y_i}$

e.g. $\begin{array}{ccc} \mathbb{F}_1 & \xrightarrow{g} & \mathbb{P}^2 \\ \cup & & \cup \\ E & \longrightarrow & pt \end{array}$



$$J = g^*(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes \mathcal{O}_{\mathbb{F}_1}(E)$$

Recall

$$\mathrm{Eff}(X) = \bigcup_{1 \le i \le l} \underbrace{(g_i)^*(\mathrm{Nef}(Y_i)) \times E_i}_{\text{Mori chamber } C_i}$$

$D \sim D' \iff \varphi_{|mD|} = \varphi_{|mD'|} \quad m \gg 0$

$$\cup\!\!\mid$$

$$\mathrm{Mov}(X) = \bigcup_{1 \le i \le k} (g_i)^*(\mathrm{Nef}(Y_i)) \qquad k \le l.$$

<u>Lemma</u>

For line bundle L on Y_i s.t. $g_i^*(L) := (p_i)_*(q_i^* L)$ is line bundle on X, we have

$$\Gamma(Y_i, L) \cong \Gamma(X, g_i^* L).$$

<u>Proof</u>

Use additive notation D for $L = \mathcal{O}_{Y_i}(D)$. Since $g_i^* D = p_{i*} q_i^* D$,

$p_i^* g_i^* D = p_i^* p_{i*} q_i^* D$ so

$$p_i^* g_i^* D - q_i^* D \quad \text{is} \quad p_i\text{-exceptional}$$

ie, $\exists$ effective, exceptional E_1, E_2 s.t

$$p_i^* g_i^* D + E_1 = q_i^* D + E_2$$

Then

$$(p_i)_*(p_i^* g_i^* D + E_1) \cong g_i^* D + (p_i)_*(E_1) \cong g_i^* D$$
$$(q_i)_*((q_i)^* D + E_2) \cong D + (q_i)_*(E_2) \cong D,$$

so

9.4 $$\Gamma(X, g_i^* D) \cong \Gamma(W_i, p_i^* g_i^* D + E_1) \cong \Gamma(W_i, (g_i)^* D + E_2) \cong \Gamma(Y_i, D)$$ $\square$.

§3 X MDS $\Rightarrow$ $\mathrm{Cox}(X)$ finitely gen'd.

For Mori chamber $C \subset \mathrm{Eff}(X)$, $R_C := \bigoplus_{D \in C} \Gamma(\mathcal{O}_X(D))$.

Each homog $f \in R = \mathrm{Cox}(X)$ is $f \in R_C$ some C and $\exists$ finitely many mori chambers; RTS R_C fin. gen'd.

Choose $J_1, .., J_d \in C$ Hilbert basis for semigroup $C \cap \mathrm{Pic}(X)$ ($\exists$ fin. many generators by Gordan's lemma). Since C top-dim'l, expressing each J_i in terms of given basis for $\mathrm{Pic}(X)$ gives

$$R(X, J_1, ..., J_d) \twoheadrightarrow R_C$$

and

$$J_i = g_i^*(A_i) \otimes \mathcal{O}_{Y_i}(F_i) \qquad A_i \text{ semiample}, F_i \text{ fixed}.$$

Then

$$\Gamma(X, J_1^{m_1} \otimes \cdots \otimes J_d^{m_d}) \cong \Gamma(X, g_i^*(A_1^{m_1} \otimes \cdots \otimes A_d^{m_d}) \otimes \mathcal{O}_{Y_i}(F))$$

$$\cong \Gamma(Y, (g_i)_*(g_i^*(A_1^{m_1} \otimes \cdots \otimes A_d^{m_d}) \otimes \mathcal{O}(F))$$

$$\cong \Gamma(Y, A_1^{m_1} \otimes \cdots \otimes A_d^{m_d})$$

as $\mathcal{O}(F) \cong \mathcal{O}(F_1^{m_1} \otimes \cdots \otimes F_d^{m_d})$ is fixed. Thus summing over all $(m_1, .., m_d)$ gives

$$R(X, J_1, .., J_d) \cong R(Y, A_1, .., A_d)$$

with each A_i semi-ample on Y, so it's finitely gen'd by the Corollary to Zariski's theorem. $\square$.

9.5 §4 GIT cone to pseudo-effective cone

$R := \mathrm{Cox}(X)$ finitely generated as $\Bbbk$-algebra; it's $\Lambda := \mathrm{Pic}(X)$-graded (in fact $\mathrm{Cl}(X)$-graded). Then $G := \mathrm{Spec}\,\Bbbk[\Lambda]$ acts algebraically on ~~on~~ $V = \mathrm{Spec}\,R$.

Fact 1: R is a UFD. It follows that V normal, $\mathrm{Cl}(V) = 0$.

Fix $A \in \mathrm{Pic}(X)$ ample. Then

$$V/\!\!/_A G = \mathrm{Proj} \bigoplus_{j \geq 0} R_{jA} \cong \mathrm{Proj} \bigoplus_{j \geq 0} \Gamma(X, A^{\otimes j}) \cong X.$$

Fact 2: $V/\!\!/_A G$ is good geometric quotient of $U := V^s_A$ by G, giving $\pi : U \longrightarrow V/\!\!/_A G$; also $\mathrm{codim}(U) \geq 2$.

$\forall \lambda \in \Lambda$, obtain G-linearised line bundle $L_\lambda \in \mathrm{Pic}^G(V)$, i.e., by Fact 1 this must be $\mathcal{O}_V$ with action of G determined by λ:

$$\begin{array}{ccccc} \Lambda \otimes \mathbb{Q} & \longrightarrow & \mathrm{Pic}^G(U) & \xleftarrow[\cong]{\pi^*} & \mathrm{Pic}(V/\!\!/_A G) \\ \lambda & \longmapsto & L_\lambda|_U & & \text{by Kempf's descent.} \end{array}$$

Lemma $\ell : \Lambda \otimes \mathbb{Q} \longrightarrow \mathrm{Pic}(X)$ is isom.

Pf

For $E \in \mathrm{Pic}^G(U)$, underlying bundle on U extends uniquely to bundle on V by Fact 2; must be $\mathcal{O}_V$ by Fact 1, so G-linearisation determined by some $\lambda \in \Lambda$ giving ℓ^{-1}. □

Fix $A \in \mathrm{Pic}(X)$, then for $\lambda \in \Lambda$

$$\Gamma(V, L_\lambda^{j})^G = R_{j\lambda} \cong \Gamma(X, \ell(j\lambda)), \qquad (*)$$

9.6 $\therefore$ ℓ identifies cone of effective G-linearisation

$$\begin{aligned}\Lambda^+ &= \{\lambda \in \Lambda \mid V/\!\!/_\lambda G \neq \phi\} \\ &= \{\lambda \in \Lambda \mid R_{j\lambda} \neq 0 \text{ for } j\} \\ &= \{\ell(\lambda) \in \mathrm{Pic}(X) \mid \Gamma(X, \ell(j\lambda)) \neq 0\} \\ &= \mathrm{Eff}(X)\end{aligned}$$

with the (pseudo)-effective cone from birational geometry.

§5 '<u>$\mathrm{Cox}(X)$ finitely gen'd $\Rightarrow$ X MDS</u>' by variation of GIT.

The above (*) also implies for $\lambda \in \Lambda^+$

$$\begin{tikzcd} & X = V/\!\!/_A G \arrow[dl, dashed, "f_\lambda"'] \arrow[dr, dashed, "\varphi_{|\ell(j\lambda)|}"] & \\ V/\!\!/_\lambda G = \mathrm{Proj} \bigoplus_{j \geq 0} R_{j\lambda} \arrow[rr, leftrightarrow, "\cong"] & & \mathrm{Proj}\, R(X, \ell(\lambda)) \end{tikzcd}$$

Compare equivalence relations:

GIT $\quad \lambda \sim \lambda' \iff V_\lambda^{ss} = V_{\lambda'}^{ss}$

Mori $\quad \ell(\lambda) \sim \ell(\lambda') \iff \varphi_{|\ell(j\lambda)|} = \varphi_{|\ell(j\lambda')|} \quad j \gg 0$

$\therefore \quad \lambda \sim \lambda' \Rightarrow \ell(\lambda) \sim \ell(\lambda')$

i.e.

GIT-equivalence $\Rightarrow$ Mori-equivalence.

<u>Fact 3</u> [Thaddeus / Dolgachev-Hu] GIT decompositions are necessarily finite polyhedral

$\Longrightarrow$ $\Lambda^+ \cong \mathrm{Eff}(X)$ is finite union of polyhedral chambers.

9.7 Fact 2 implies that Mori-equivalence $\Rightarrow$ GIT-equivalence, and further, that for

$$\lambda \in C \quad , \quad f_\lambda : X \dashrightarrow V/\!\!/_\lambda G \text{ satisfies}$$

$$\underline{\ell(\lambda) = f_\lambda^*(\mathcal{O}(1)) \otimes \mathcal{O}(E)} \qquad E \text{ fixed.}$$

$$\therefore \quad C = \overline{f_\lambda^*(\mathrm{Amp}(V/\!\!/_\lambda G)) \times E}.$$

Remains to show: for any such $Y = V/\!\!/_\lambda G$ with $\mathrm{Eff}(Y) = \mathrm{Eff}(X)$, $\mathrm{nef}(Y)$ gen'd by finitely many semi-ample bundles. Each chamber finite. polyhedral, so is each class semi-ample?

For $A \in C = \overline{\mathrm{Amp}(X)}$, pick $\lambda \in \partial C$;

$$U := V_A^s \subseteq V_\lambda^{ss} \quad \therefore \quad \begin{array}{rcl} f_\lambda : X = V/\!\!/_A G & \longrightarrow & V/\!\!/_\lambda G \\ p & \longmapsto & [\overline{G\cdot\tilde{p}}] \end{array}$$

and

$$\ell(\lambda) = f_A^*(\text{ample on } V/\!\!/_\lambda G).$$