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# The Cox ring of a flag variety

Plan
- What is a flag variety $X$?
- $\mathrm{Cox}(X)$ f.g. $\therefore$ MDS
- $\mathrm{Cox}(X) = \mathbb{C}[V]/I$ ← explicit quadratic generators (Plücker rel^ns)

[More examples than proofs]

## 1. Flag varieties $(K = \mathbb{C})$

$G$ cmplx simply connected semisimple algebraic group (no covers) (eg $SL(n,\mathbb{C})$)

$B \le G$ Borel if max. connected solvable (eg upper $\Delta$'ar)

$P \le G$ parabolic if $B \le P$ some Borel (eg block upper $\Delta$'er)

Flag variety is $G/P$ some parabolic $P$

[so $G$-homog variety with parabolic stabilisers]

Why important?

FACT These exhaust complete ~~fin~~ $G$-homog. varieties by Borel Fixed Pt Thm: If connected solvable $B$ acts on complete $X$ then $B$ has a fixed pt....

Examples $G = SL(n,\mathbb{C})$

$$B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} = \mathrm{stab}(\mathbb{C} \le \mathbb{C}^2 \le \dots \le \mathbb{C}^n)$$

$$G/B \cong \{ V_1 \le V_2 \le \dots \le V_n = \mathbb{C}^n \mid \dim V_k = k \} \quad \text{flag manifold}$$

$$P = \left\{ \begin{array}{c|c|c} & k & n-k \\ k & * & * \\ & 0 & * \end{array} \right\} = \mathrm{stab}(\mathbb{C}^k \le \mathbb{C}^n)$$

$$G/P \cong \{ V \le \mathbb{C}^n \mid \dim V = k \} = G_k(\mathbb{C}^n) \quad \text{Grassmannian}$$

In general $\{ V_1 \le \dots \le V_r \le \mathbb{C}^n \mid \dim V_k = d_k \} = G_{\underline{d}}(\mathbb{C}^n)$ for fixed $1 \le d_1 < d_2 < \dots < d_r \le n$

These are projective

$$G_k(\mathbb{C}^n) \hookrightarrow \mathbb{P}(\Lambda^k \mathbb{C}^n) \quad \text{Plücker embedding}$$
$$V \longmapsto \Lambda^k V$$

$$\begin{array}{ccccc} G_{\underline{d}}(\mathbb{C}^n) & \to & G_{d_1}(\mathbb{C}^n)\times\cdots\times G_{d_r}(\mathbb{C}^n) & \xrightarrow{\text{Plücker}} & \mathbb{P}(\Lambda^{d_1}\mathbb{C}^n)\times\cdots\times\mathbb{P}(\Lambda^{d_r}\mathbb{C}^n) \\ (V_1\le V_2\le\dots) & \mapsto & (V_1, V_2, \dots) & & \downarrow \text{Segre} \\ & & & & \mathbb{P}(\Lambda^{d_1}\mathbb{C}^n\otimes\cdots\otimes\Lambda^{d_r}\mathbb{C}^n) \end{array}$$

## 2. Homogeneous line bundles & characters

$L \to X$ line bundle over flag variety is homogeneous if $G$-action on $X$ lifts to $G$-action on $L$ by bundle morphisms:

$$\begin{array}{ccc} L & \xrightarrow{g} & L \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & X \end{array}$$

Then:

- $V := \Gamma L$ is $G$-module:
  $$(g\cdot\sigma)(x) := g(\sigma(g^{-1}x))$$
- Base locus $G$-stable so either
  - $V = \{0\}$
  - $G$-morphism $f: X \to \mathbb{P}(V^*)$ with $f^*\mathcal{O}(1) = L$
    $$f(x) \overset{G\text{-mo}}{=} [\sigma \mapsto \sigma(v_x)^*] \quad v_x \in L_x$$
- $P := \mathrm{Stab}(x)$ has rep$^n$ $P \to GL(L_x) = \mathbb{C}^\times$
  $\Leftrightarrow$ char of $P/P'$ ← commutator subgrp.

always a torus

Converse: $\chi \in \mathrm{Char}(P/P') \rightsquigarrow L = G\times_\chi \mathbb{C}$ homog. line bundle.

assoc to principal $G \to X$, $g \mapsto gx$

Example Tautological rk $k$ bdle $\mathbb{V} \le G_k(\mathbb{C}^n)\times\mathbb{C}^n$

$$\mathbb{V}_V = V$$

$$L = (\Lambda^k \mathbb{V})^* = f^*\mathcal{O}(1) \quad f \text{ Plucker embedding.}$$

[need $G = SL(n,\mathbb{C})$ not $PSL(n,\mathbb{C})$ for this]

# 3. Schubert cells

$$P \rhd U = \exp(\mathfrak{p}^{\perp})$$

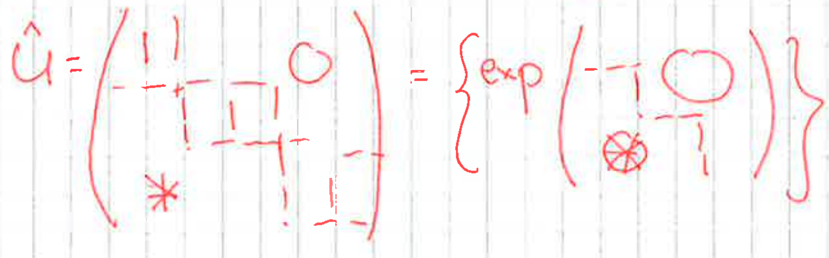
$U$: unipotent radical; $\mathfrak{p}$: Lie alg; $\mathfrak{p}^{\perp}$: nilpt Lie alg, polar wrt Killing

Example


Opposite or complementary parabolic is $\hat{P}$ with unipotent radical $\hat{U}$ s.t.

$$\hat{U} \cap P = \{1\}$$
$$\hat{U} P \text{ open in } G$$

eg




FACTS

- $$\begin{aligned} \hat{\mathfrak{p}}^{\perp} &\longrightarrow G/P \\ \xi &\longmapsto \exp\xi\, P \end{aligned}$$ (polyn$^l$) regular isom onto dense open affine $\Omega \subsetneq G/P$

- $G/P \setminus \Omega$ union of divisors of homog. line bundles $L_1, \dots, L_r$

Examples
- $P = \mathrm{Stab}(\mathbb{C}^k \leq \mathbb{C}^n)$
  $\hat{P} = \mathrm{Stab}(W \leq \mathbb{C}^n)$ $\quad \mathbb{C}^k \oplus W = \mathbb{C}^n$

$$\Omega = \{V \in G_k(\mathbb{C}^n) \mid V \oplus W = \mathbb{C}^n\} \cong \mathrm{Hom}(\mathbb{C}^k, W) \cong \hat{\mathfrak{p}}^{\perp}$$
$$\mathrm{graph}(T) \longleftarrow T$$

$$G_k(\mathbb{C}^n) \setminus \Omega = \{V \mid \Lambda^k V \wedge \Lambda^{n-k} W = 0\} \longleftarrow \text{hyperplane section of } f^*\mathcal{O}(1)$$

- $B = \mathrm{Stab}(\mathbb{C} \leq \dots \leq \mathbb{C}^n)$

$\hat{B} = \mathrm{Stab}(W_1 \leq W_2 \leq \dots \leq \mathbb{C}^n)$ s.t. $W_k \cap \mathbb{C}^{n-k} = \{0\}$ $\forall k$

$\Omega = \{V_1 \leq V_2 \dots \leq V_n = \mathbb{C}^n \mid V_{n-k} \oplus W_k = \mathbb{C}^n\}$

$G/_B \setminus \Omega = \bigcup_{i=1}^{n-1} D_i$

$D_i = \{V_1 \leq \dots \leq V_n \mid V_i \cap W_{n-i} = \mathbb{C}^n\}$

In fact:
- $\Omega$ is $\hat{B}$-orbit for $\hat{B} \leq \hat{P}$
- $G/P$ is union of finite # of $\hat{B}$ orbits $\therefore$ spherical

Application

$\mathrm{Pic}(G/_P)$ freely generated by $L_1, \dots, L_r$

Proof $\mathrm{Pic}(\Omega) = \{0\}$ since $\Omega$ affine.

1/ $D$ any divisor

$D \cap U = \mathrm{div}(f)$ $f \in K(\Omega) = K(G/_P)$

$\therefore$ $D - \mathrm{div}(f) = \sum_{i=1}^{r} a_i D_i$

2/ $\sum_{i=1}^{r} a_i [D_i] = 0$ iff $\exists f \in K(G/_P)$ with

$\sum a_i D_i = \mathrm{div}(f)$

Then $f|_U$ regular & never zero, ~~$\Rightarrow$ nonconstant~~ $\times$

unless $a_i = 0$ $\forall i$.

$\square$

## 4. Facts from rep$^n$ Thy

$G$ complex semisimple, ~~$V$ finite dim$^l$ $G$-module~~

- (complete reducibility) $V$ finite-dim$^l$ $G$-module

$V = V_1 \oplus \dots \oplus V_k$ $V_i$ irreducible $G$-mod

- (Thm of highest wt) $V$ irred (finite dim) $G$-module

$\exists$ unique 1-dim$^l$ $W \leq V$ fixed by $B$ | unique from partial order on wts...

$W$: highest wt space

$\therefore$ char $\chi : B/_{B'} \longrightarrow \mathbb{C}^\times = \mathrm{End}(W)$

$B/_{B'}$: a torus

## 4. Facts from rep$^n$ thy

$\mathfrak{g}$ cmplx semisimple  $V$ finite dim$^l$ $\mathfrak{g}$-module

- (complete reducibility)

  $V \cong V_1 \oplus \cdots \oplus V_k$  $V_i$ irred. $\mathfrak{g}$-mod.

- (Thm of highest wt) $V$ irred, $B \trianglerighteq B' =: N \leftarrow$ unipot. rad.

  $\uparrow$ Borel

  $\dim V^N = 1$ — highest wt space

  $V^N := \{v : nv = v \ \ \forall n \in N\}$

  $\sigma$ ~~$B$~~ $V^N$ $B$-stable so acts by char $\chi : B/N \to \mathbb{C}^\times = \mathrm{End}(V^N)$

  ($B/N$ ← torus)

Put two together: $V \neq \{0\}$ irred iff $\dim V^N = 1$

Application (key point) $L \to X$ homog then

$\Gamma L$ irred.

Pf Have $\underbrace{P \geq B \geq N \geq U}_{\text{unipotent rad}}$

s.t. $\Omega = U \cdot x$ open in $X$

Let $\sigma \in \Gamma L^N$ then for $u \in U$

$u \cdot \sigma = \sigma$  i.e.  $\sigma(ux) = u \cdot \sigma(x)$

$\therefore$ $\sigma|_\Omega$ determined by $\sigma(x)$ but $\sigma|_\Omega$ determins $\sigma$ $\therefore$ $\dim \Gamma L^N = 1$ unless $\Gamma L = \{0\}$ $\boxtimes$

- $V \mapsto \chi$  $\{\text{irred } \mathfrak{g}\text{-mod}\} \to \mathrm{Char}(B/N)$ injects

  Image: $\Gamma L_1, \ldots, \Gamma L_n$  $L_1, \ldots, L_n$ generators of $\mathrm{Pic}(G/B)$

  $G/B = G/\hat{B}$ ← all Borels conjugate (fixed pt thm again)

N.B. $\chi \in \mathrm{Char}(B/N) \rightsquigarrow L = G \times_\chi \mathbb{C} \to G/B$

$\rightsquigarrow \Gamma L$ ~~wh~~ with char $\chi^{-1}$ ← !

Punchline (Borel-Weil)

- each nonzero $\Gamma L$ is irrep
- all irreps arise this way. [Bott computes $H^k(X, \mathcal{O}(L))$ ...]

## 5. Cox ring

Take $X = G/B$ henceforth (most complicated case)

$$\mathrm{Cox}(X) = \bigoplus_{(m_1,\dots,m_n)\in\mathbb{Z}^n_{\geq 0}} \Gamma(L_1^{m_1}\otimes\dots\otimes L_n^{m_n}) \leftarrow G\text{-irrep}$$

Observe: $\Gamma L, \Gamma L' \neq \{0\}$ then

$$\Gamma L \otimes \Gamma L' \longrightarrow \Gamma(L\otimes L')$$ surjects [nonzero image by Schur $\therefore$ onto]

$\therefore$ $\mathrm{Cox}(X)$ f.g. by induction.

More precisely: $V_i := \Gamma L_i$ $\quad 1 \leq i \leq n$

[ N.B. $V_i^* \cong V_j$ [Nakayama]

Example: $G = SL(n+1, \mathbb{C})$

$V_k = \Lambda^k \mathbb{C}^{n+1}$ $\quad V_k^* \cong V_{n+1-k}$ $\perp$

~~Induct~~

$$S^{m_1}V_1\otimes\dots\otimes S^{m_n}V_n \longrightarrow \Gamma(L_1^{m_1}\otimes\dots\otimes L_n^{m_n}) \quad \text{onto}$$

$$\therefore \quad R := \mathbb{C}[\underbrace{V_1^*\oplus\dots\oplus V_n^*}_{\substack{\shortparallel \\ V_1\oplus\dots\oplus V_n}}] \longrightarrow \mathrm{Cox}(X) \quad \text{surjects}$$

- graded by degree $\underline{m} = (m_1, \dots, m_n)$
- generated in total degree $|\underline{m}| = (m_1+\dots+m_n)$ $\underline{1}$

Rel'ns: Decompositions of $G$-mod:

$$S^2 V_i = \Gamma(L_i^2) \oplus Q_i$$

$$V_i \otimes V_j = \Gamma(L_i \otimes L_j) \oplus Q_{ij}$$

$$I = \langle Q_i, Q_{ij} \mid 1 \leq i < j \leq n \rangle$$

(Kostant, essentially)

**Thm** $\mathrm{Cox}(X) \cong R/I$

**Sketch proof** Work with rep$^{\underline{n}}$ of $\mathfrak{g} = \mathrm{Lie}(G)$

- Casimir: Killing form $\in \mathfrak{g}^* \otimes \mathfrak{g}^* \rightsquigarrow \mathfrak{g} \otimes \mathfrak{g} \hookrightarrow C \in Z(U(\mathfrak{g}))$

  $C = \sum a_i b_i$ $\quad a_i, b_i$ Killing dual bases.

  $C|_V = c(V) 1_V$ $\quad V$ irrep.

  $\uparrow$ can compute from $\chi \in \mathrm{Char}\, B/N$.

- In $R_{\underline{m}}$ $\quad Cv = c(V_{\underline{m}}) v$ iff $v \in V_{\underline{m}}$

- In $(R/I)_{\underline{m}}$ $\quad C = c(V_{\underline{m}}) 1$
  - true by def$^{\underline{n}}$ in $|\underline{m}| = 2$
  - Leibniz rule using $|\underline{m}| = 2$ for $|\underline{m}| > 2$

$\square$