16/11/16

7.1 <u>Last Time</u>: For a $\mathbb{Q}$-factorial toric variety X that is projective over an affine, we listed some nice properties about the cones $\mathrm{Nef}(X)$, $\overline{\mathrm{Mov}(X)}$ and $\overline{\mathrm{Eff}(X)}$. In the toric case, these are the defining properties of a Mori Dream Space. Why are they important? Why are they called "dream" spaces?
Answer: The Minimal Model Programme (MMP).

Recall that a <u>rational map</u> of irreducible varieties $f: X \dashrightarrow Y$ is a morphism $f: U \to Y$ for some $U \subset X$ open. f is <u>birational</u> if there is an inverse rational map.
<u>GOAL</u>: Classify all (irreducible) algebraic varieties up to birational equivalence, and find a good representative of each class.
<u>Theorem (Hironaka, 1964)</u>: Over a field of characteristic 0, every variety is birational to a smooth projective variety. [Fields Medal!]
Hence it is enough to classify smooth projective varieties.

<u>Dimension 1</u>: There is a unique smooth projective curve in each birational equivalence class, and for each genus $g = \dim \Gamma(C, \omega_C) \geq 0$ there is a connected, irreducible moduli space parameterising the classes.

g	C	Moduli
≥ 2	general type	variety of dim $3g-3$
1	elliptic	$\mathbb{A}^1_{\mathbb{K}}$
0	rational	$\mathrm{Spec}(\mathbb{K})$

For higher dimensional varieties, it is useful to generalise the notion of genus.

7.2 <u>Def^n</u>: For X smooth and projective, define its <u>plurigenera</u> $P_d := \dim \Gamma(X, \omega_X^{\otimes d})$ for $d \in \mathbb{N}$. Then the <u>Kodaira dimension</u> of X is $\kappa(X) := \begin{cases} -\infty & \text{if } P_d = 0 \ \forall d > 0 \\ \text{minimum } k \text{ s.t } P_d/d^k \text{ is bounded} \end{cases}$

<u>Facts</u>:
- The plurigenera are birational invariants, hence so is $\kappa(X)$.
- $\kappa(X) \in \{-\infty\} \cup \{0, 1, .., n\}$, where $n = \dim X$.

<u>Example</u>: $X = \mathbb{P}^n$. $\omega_X = \mathcal{O}_X(-n-1)$ so $\dim \Gamma(\mathbb{P}^n, \mathcal{O}(-n-1)^{\otimes d}) = 0$ $\forall d > 0 \Rightarrow \kappa(\mathbb{P}^n) = -\infty$.

<u>Dimension 2</u>: There ~~may be~~ are many non-isomorphic smooth projective surfaces per birational equivalence class. For example, blowing up a point on a smooth surface produces a birational map to another smooth surface.

<u>Def^n</u>: A smooth rational curve C on a smooth projective surface S satisfying $C \cdot C = -1$ is called a (-1)-curve.

<u>Theorem (Castelnuovo)</u>: Let C be a (-1)-curve on a smooth surface $\bar{S}$. Then there exists a smooth surface S such that $\bar{S}$ is the blow-up of S at a point and the exceptional divisor is C.

The good representatives will be those surfaces containing no (-1)-curves, called <u>minimal surfaces</u>. Every surface is birational to a minimal surface: if it contains a (-1)-curve, blow it down using Castelnuovo's Theorem, and repeat the process. This process must terminate as blowing down decreases the <u>Picard number</u> ($\rho_S = \dim N^1(S) \geq 0$).

7.3 Enriques-Kodaira Classification:

$\kappa(S)$	S
2	general type
1	elliptic surface
0	K3, abelian, Enriques, hyperelliptic
$-\infty$	rational, ruled (birational to $\mathbb{P}^2$ / $\mathbb{P}^1\times C$ (curve) respectively)

Each class can be parameterised by a moduli space, most of which are well understood (general type, not so much). When $\kappa(S)\geq 0$ there is a unique minimal surface in each birational equivalence class, but not for $\kappa(S)<0$. Example:

$$\begin{tikzcd}
& Bl_{p,q}\mathbb{P}^2 \arrow[d] \arrow[ddl, "\substack{\text{blow down proper}\\ \text{transform of}\\ \text{line joining}\\ p,q}"'] \\
& Bl_p\mathbb{P}^2 \arrow[d] \\
\mathbb{P}^1\times\mathbb{P}^1 \arrow[r, dashed, leftrightarrow] & \mathbb{P}^2
\end{tikzcd}$$

__Higher Dimensions__: The definition of a (-1)-curve uses an ambient surface, so we will need a new notion of minimal.

__Proposition:__ A surface S that is minimal and satisfies $\kappa(S)\geq 0$ has nef canonical divisor ($K_S\cdot C\geq 0$ $\forall$ irreducible curves $C\subset S$).

For higher dimensional varieties X, we would like to take "K_X is nef" as our definition of minimal when $\kappa(X)\geq 0$ (we will see what happens when $\kappa(X)=-\infty$ later). Given smooth projective X, the natural first step would be to contract curves for which $K_X\cdot C<0$. When $\dim X\geq 3$ however, this may produce singularities. Hence we will need to expand our category from only allowing smooth varieties to allowing certain mild singularities.

7.4 When discussing nefness, we will also require our varieties to be $\mathbb{Q}$-factorial to ensure the intersection theory is well-defined.

<u>Def^n:</u> Let X be projective with K_X $\mathbb{Q}$-Cartier and let $f: Y \to X$ be a resolution of X (Y smooth, f a proper morphism, $f^{-1}(X \setminus X_{sing}) \cong X \setminus X_{sing}$) with E_i, $i=1,..,r$ the irreducible components divisors of the exceptional divisor ~~locus~~ of f ($\mathrm{Exc}(f)$ = inverse image of smallest closed set in X outside of which f is an isomorphism). Then $\exists a_i \in \mathbb{Q}$ s.t $K_Y = f^*K_X + \sum_{i=1}^{r} a_i E_i$. We say X has <u>terminal</u> / <u>canonical</u> / <u>log terminal</u> singularities if $a_i > 0$ / $a_i \geq 0$ / $a_i > -1$ $\forall i = 1,..,r$. for some (hence any) f.

<u>Def^n:</u> A morphism of projective varieties $f: X \to Y$ is a <u>contraction</u> if it is surjective and has connected fibres.

The <u>Contraction Theorem</u> is an analogue of Castelnuovo's Theorem to higher dimensions. Roughly speaking, it says that if X is projective with (at worst) log terminal singularities and C is an irreducible "extremal" curve with $K_X \cdot C < 0$, there is a unique contraction $f: X \to Y$ contracting all curves in the class of C ~~to a point~~. There are 3 cases:

1) $\dim Y < \dim X$. In this case X is <u>uniruled</u> ($\exists Y'$, $\dim Y' = n-1$ and a dominant (dense image) rational map $\mathbb{P}^1 \times Y \dashrightarrow X$). In fact, $\kappa(X) = -\infty$ and like the surface case X is not birationally equivalent to a variety with nef canonical divisor. So X admits no minimal model, but understanding X is reduced to lower dimensions. f is a <u>Fano fibration</u>: each fibre is a Fano variety ($-K_X$ ample) of positive dimension.

Remark: It is still only a conjecture that $\kappa(X) = -\infty \Rightarrow f$ a Fano fibration

7.5 2) $\dim Y = \dim X$ and there is an exceptional divisor E whose image under f has codimension ≥ 2. f is a <u>divisorial contraction</u>. If X has log terminal and $\mathbb{Q}$-factorial singularities, so does Y.

3) $\dim Y = \dim X$ and $\operatorname{codim}(\operatorname{Exc}(f)) \geq 2$. Then f is a <u>small contraction</u>. These are nasty; in particular K_Y will not be $\mathbb{Q}$-Cartier. Solution: flips.

<u>Def^n</u>: Let $f: X \to Y$ be a proper morphism. A $\mathbb{Q}$-Cartier divisor D on X is <u>f-ample</u> iff $(D^{\dim V} . V) > 0$ $\forall$ irreducible subvarities $V \subset X$ that ~~map to a point~~ are contracted by f.

<u>Def^n</u>: Let $f: X \to Y$ be a small contraction with $-K_X$ $\mathbb{Q}$-Cartier and f-ample. A <u>flip</u> is a variety X' with a proper birational morphism $f': X' \to Y$ such that $K_{X'}$ is $\mathbb{Q}$-Cartier and f'-ample and f' is a small contraction.

$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[dr, "f"', "{\substack{-K_X \\ f\text{-ample}}}"] & & X' \arrow[dl, "f'", "{\substack{K_{X'} \\ f'\text{-ample}}}"'] \\
& Y &
\end{tikzcd}$$

In particular there is a birational map $f'^{-1} \circ f : X \dashrightarrow X'$. This is sometimes called the flip.

Remark: If a flip exists, it is unique.

<u>Minimal Model Programme</u>

1) Start with a variety X, replace with a smooth projective.
2) If X is minimal (K_X nef, $\mathbb{Q}$-factorial, terminal singularities), stop.
3) Contraction theorem: produce morphism $f: X \to Y$ from an extremal curve.
4) If f is a Fano fibre space, stop.
5) If f is a divisorial contraction, replace X by Y, return to 2.
6) If f is a small contraction, replace X by a flip X', return to 2.

7.6 The programme will work when

i) flips exist.

ii) it terminates (no infinite sequence of flips).

This is a huge area of active research. Milestones:

1988, Mori: i and ii in dimension 3. [Fields Medal!]

2003, Shokurov: i in dimension 4.

2010, Birkar, Cascini, Hacon, M^c^Kernan: i in all dimensions.

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<u>Def^n</u>: A dominant rational map $f: X \dashrightarrow Y$ is <u>contracting</u> if f has a resolution

$$\begin{tikzcd} X' \arrow[d, "\varphi"'] \arrow[dr, "f'"] & \\ X \arrow[r, dashed, "f"] & Y \end{tikzcd}$$

(i.e. X' smooth projective, φ birational, f' morphism) such that all effective φ-exceptional divisors E satisfy $f'_*(\mathcal{O}_{X'}(E)) = \mathcal{O}_Y$. Note that the definition does not depend on choice of resolution, and that f regular $\Rightarrow$ f a contraction.

<u>Def^n</u> A <u>small $\mathbb{Q}$-factorial modification</u> (SQM) is a contracting birational map $f: X \dashrightarrow X'$ of projective varieties

SQM example: flip! → with X' $\mathbb{Q}$-factorial and f an isomorphism in codim 1.

<u>Proposition</u> An SQM induces an isomorphism $f^*: N^1(X) \longrightarrow N^1(X)$ and preserves the pseudo-effective and movable cones: $f^*(\mathrm{Eff}(X')) = \mathrm{Eff}(X)$, $f^*(\mathrm{Mov}(X')) = \mathrm{Mov}(X)$.

<u>Def^n</u> (Hu, Keel, 2000): A normal, $\mathbb{Q}$-factorial projective variety X is a <u>Mori Dream Space</u> (MDS) if:

1) $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} \cong N^1(X)$,

2) $\mathrm{Nef}(X)$ is generated by the classes of finitely many semiample divisors (D <u>semi-ample</u> if $\exists m > 0$ s.t. $\mathcal{O}_X(mD)$ globally generated).

3) there is a finite collection of SQMs $f_i: X \dashrightarrow X_i$, $i = 1, \dots, r$ such that X_i satisfies 1, 2 $\forall i = 1, \dots, r$ and we have $\mathrm{Mov}(X) = \bigcup_{i=1}^{r} f_i^*(\mathrm{Nef}(X_i))$.

Remark: Each X_i is also a MDS.

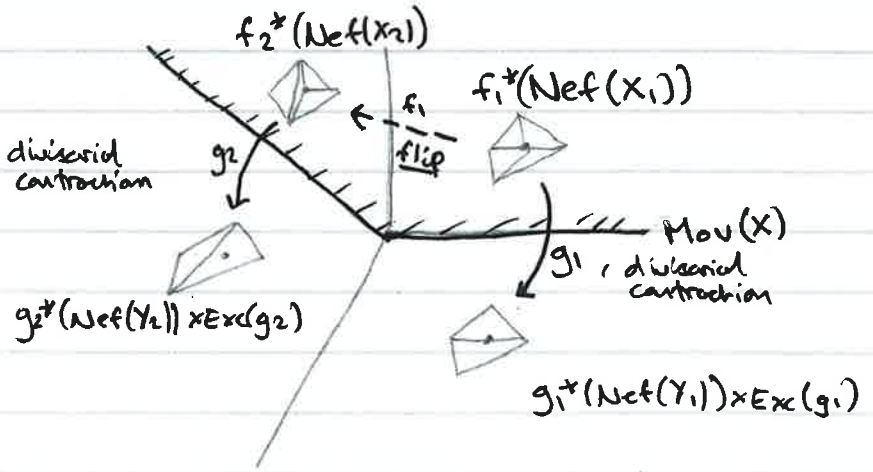
It was described above for the $\mathbb{Q}$-Cartier divisor K_X.

7.7 So why a "dream" space?

__Proposition (Hu, Keel):__ Let X be a MDS. Then Mori's programme (the MMP) can be carried out for any divisor on X. In addition, we get a complete birational description of X ~~that is very~~ with combinatorial elements:

- the cone of curves or effective cone $\overline{\mathrm{Eff}}(X) = \{\sum a_i [D_i] \mid D_i \subset X$ Cartier divisor, $a_i \in \mathbb{R}_{\geq 0}\} \subset N^1(X)$ has closure the affine hull of finitely many effective divisors. (In dim 2, $\overline{\mathrm{Eff}(X)} = \overline{NE(X)}$), i.e., it's a polyhedral cone.
- there are finitely many birational contractions $g_i : X \dashrightarrow Y$ with Y an MDS such that $\overline{\mathrm{Eff}(X)} = \bigcup_i g_i^*(\mathrm{Nef}(Y_i)) \times \mathrm{Exc}(g_i)$ is a decomposition into closed convex chambers with disjoint interiors – Mori chambers. These are one-to-one with birational contractions of X having $\mathbb{Q}$-factorial image
- the SQMs f_i from (3) are the only SQMs of X; X_i, X_j in adjacent chambers are related by a flip and these chambers with their faces form a fan with support $\mathrm{Mov}(X)$.

__Example:__ $\mathbb{Z}^5 \xrightarrow{\begin{pmatrix} 1 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 \end{pmatrix}} \mathbb{Z}^2$ from lecture 6:



Secondary fan in $N^1(X)$, here $N^1(X) = \overline{\mathrm{Eff}(X)} = \cancel{\overline{NE(X)}}$
$= \bigcup$ (4 chambers defining simplicial / $\mathbb{Q}$-factorial varieties)