6.1 Mori Dream Spaces

Recall that a quasi-torus $G = \operatorname{Spec} \Bbbk[\Lambda]$ acts on affine space $X = \operatorname{Spec} \Bbbk[z_1, \dots, z_r]$ with weights determined by the map $\mathbb{Z}^r \twoheadrightarrow \Lambda$ satisfying $D_\rho \longmapsto w_\rho$. Then

$$\operatorname{Cone}(\underline{w}) = \Big\{ \sum_\rho \alpha_\rho w_\rho \in \Lambda_{\mathbb{R}} \;\Big|\; \alpha_\rho \geq 0 \;\; \forall \rho \in \Sigma(1) \Big\}$$

decomposes into the cones of the <u>secondary fan</u>, and we say $\lambda \in \operatorname{Cone}(\underline{w})$ is <u>generic</u> if λ lies in the interior of a top-dimensional cone in this fan.

<u>Theorem</u>

If $\lambda \in \operatorname{Cone}(\underline{w})$ is generic, then $X^s_\lambda = X^{ss}_\lambda$ and the polyhedron P_λ is simple.

<u>Cor</u>

For λ generic, $X /\!\!/_\lambda G$ is a $\mathbb{Q}$-factorial toric variety of dim'n $r - \dim G$ that's projective over $X/G = \operatorname{Spec} \Bbbk[z_1, \dots, z_r]^G$, and it's a good <u>geometric</u> quotient of X^s_λ by G.

<u>Proof of Th'm.</u>

<u>Step 1.</u> λ generic $\implies \lambda \notin \partial \operatorname{Cone}(\underline{w})$ [illegible]

$\implies \dim X /\!\!/_\lambda G = \dim P_\lambda = r - \dim G.$

Thus, for each vertex $b \in P_\lambda$, $\exists$ at least $r - \dim G$ components b_ρ of b that vanish (since P_λ is intersection of positive orthant with $\deg^{-1}(\lambda)$), and if $s < \dim G$ nonvanishing then write $b = \sum_{\rho \in S} b_\rho D_\rho$ with $s = |S|$ to see that $\lambda \in \operatorname{Cone}(\underline{w}')$ for $\underline{w}' = \{ w_\rho \mid \rho \in S \}$, ie λ not generic, #.

It follows that P_λ is simple, i.e., Σ_λ is simplicial.

6.2 <u>Step 2.</u> λ-unstable ($:=$ not λ-semistable) locus is $\mathbb{V}(B_\lambda) \subset X$ for

$$
\begin{aligned}
B_\lambda &= \mathrm{rad}\left(z^u \in k[z_1, \ldots, z_r] \mid u \in \mathbb{N}^r \cap \deg^{-1}(\lambda) \right) \\
&= \left(\textstyle\prod_{b_\rho \neq 0} z_\rho \mid b \in P_\lambda \text{ is vertex} \right) \\
&= \left(\textstyle\prod_{\rho \notin \sigma(1)} z_\rho \mid \sigma \in \Sigma_\lambda(n) \right).
\end{aligned}
$$

Let

$$X_\lambda^{ss}(\sigma) := \{ (p_1, \ldots, p_r) \in X = \mathbb{C}^r \mid p_\rho \neq 0 \text{ for } \rho \notin \sigma(1) \}.$$

Then

$$X_\lambda^{ss} = \bigcup_{\sigma \in \Sigma_\lambda(n)} X_\lambda^{ss}(\sigma).$$

<u>Step 3.</u> For $p \in X_\lambda^{ss}(\sigma)$, we show orbit $G \cdot p \subset X_\lambda^{ss}(\sigma)$ closed and stabiliser G_p is finite, hence $p \in X_\lambda^{s}$ as required. Suffices to show for $G^\circ \subseteq G$ connected comp of identity since it has finite index. Now

(i) For $\bar{p} \in \overline{G^\circ \cdot p} \subset X_\lambda^{ss}(\sigma)$ is point in affine toric var. where torus G° acts transitively on each orbit. Then

$$\bar{p} = \lim_{t \to 0} v(t) \cdot g \cdot p$$

for $g \in G^\circ$ and $v : \mathbb{C}^* \longrightarrow G^\circ \subseteq (\mathbb{C}^*)^r$; here $v(t) = (t^{a_\rho})$ for $a_\rho \in \mathbb{Z}$ satisfying $\sum_{\rho \in \Sigma(1)} a_\rho v_\rho = 0$. Componentwise

$$\bar{p}_\rho = \lim_{t \to 0} t^{a_\rho} g_\rho \cdot p_\rho$$

so

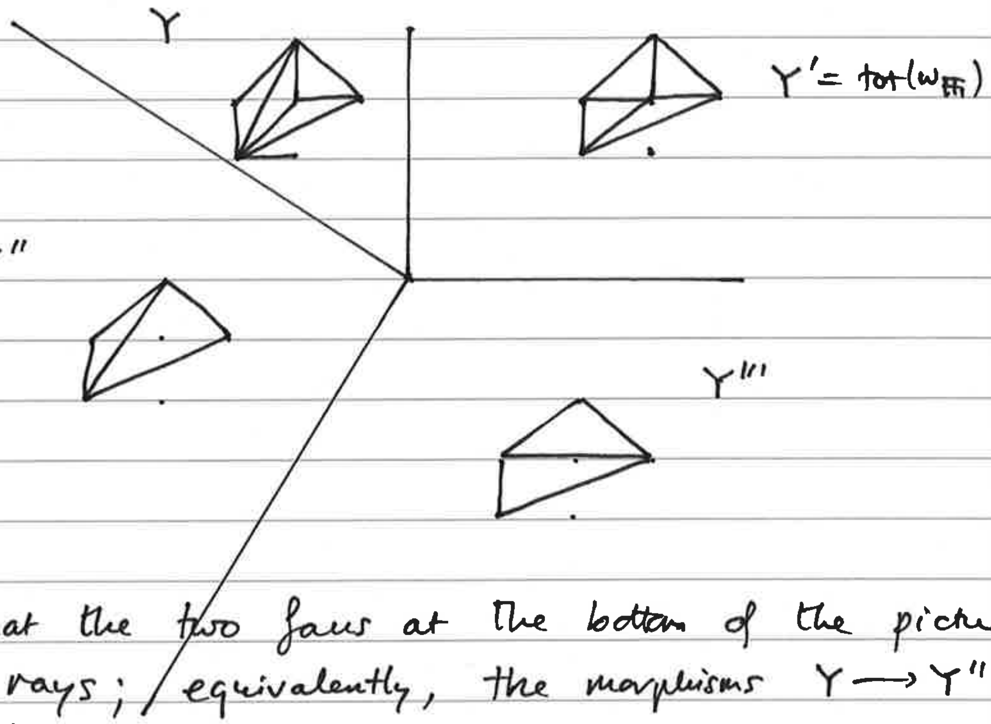
- $\rho \notin \sigma(1)$, $p_\rho, \bar{p}_\rho \neq 0$ and $g_\rho \neq 0$ so $a_\rho = 0$.

Therefore $\sum_{\rho \in \sigma(1)} a_\rho v_\rho = 0$, but P_λ simple shows $\{v_\rho \mid \rho \in \sigma(1)\}$ linearly indep, so $a_\rho = 0$ $\forall \rho$ giving $v = 0$, ie $\bar{p} \in G^\circ \cdot p$.

(ii) Write $G = \{ (t_\rho) \in (\mathbb{C}^*)^r \mid \prod_\rho t_\rho^{\langle m, v_\rho \rangle} = 1 \ \forall m \in M \}$. If $g \cdot p = p$, ie, $t_\rho \cdot p_\rho = p_\rho$ $\forall \rho \in \Sigma(1)$, then for $\rho \notin \sigma(1)$ $p_\rho \neq 0$ so $t_\rho = 1$; thus $\prod_{\rho \in \sigma(1)} t_\rho^{\langle m, v_\rho \rangle} = 1$. If $\{v_\rho \mid \rho \in \sigma(1)\}$'s were $\mathbb{Z}$-basis, dual basis m's forces $t_\rho = 1$ $\forall \rho$, so G_p trivial. In fact $\{v_\rho\}$ generate finite index sublattice, so G_p finite. $\square$.

6.3 Example

Consider again $\mathbb{Z}^5 \xrightarrow{\begin{pmatrix} 1 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 \end{pmatrix}} \mathbb{Z}^2$ for which the secondary fan has four chambers defining simplicial (= $\mathbb{Q}$-factorial) toric varieties with fans as shown, two of which we've seen before:



Notice that the two fans at the bottom of the picture have only 4 rays; equivalently, the morphisms $Y \longrightarrow Y''$ and $Y' \longrightarrow Y'''$ each contract a divisor.

We now give the geometric interpretation. For any normal variety X, recall

$$\mathrm{Cl}(X) = \mathbb{Z}\langle \text{Weil divisors on } X \rangle / \sim_{\text{lin}}$$

where

$$D \sim_{\text{lin}} D' \iff D - D' = \mathrm{div}(f) \quad f \in \Bbbk(X) \iff \mathcal{O}_X(D) \cong \mathcal{O}_X(D')$$

which contains the group of line bundles

$$\mathrm{Pic}(X) = \mathbb{Z}\langle \text{Cartier divisors on } X \rangle / \sim_{\text{lin}}.$$

Compare

$$N^1(X) = \mathbb{Z}\langle \text{Cartier divisors on } X \rangle / \equiv$$

where

$$D \equiv D' \iff D.C = D'.C \quad \forall \text{ proper irred curves } C \subset X.$$

6.4 **Lemma 1**

For toric variety X, Cartier divisor $D \sim_{\text{lin}} 0 \iff D \equiv 0$. Thus

$$N^1(X) \cong \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} =: \Lambda_{\mathbb{R}}.$$

Proof

If $D \sim_{\text{lin}} 0$ then $D.C = \deg \mathcal{O}_X(D)|_C = \deg \mathcal{O}_X|_C = 0$, so $D \equiv 0$.

If $D \equiv 0$, describe D via local data $\{(U_\sigma, m_\sigma)\}_{\sigma \in \Sigma(n)}$ where $D|_{U_\sigma} = \mathrm{div}(-m_\sigma)|_{U_\sigma}$ [if $\mathcal{A} = \mathcal{O}_X(D)$ is glob. gen'd, these m_σ are the vertices of polyhedron $P_{\mathcal{A}}$]. Then one can show

$$0 = D.C = \langle m_\sigma - m_{\sigma'}, v \rangle \quad (*)$$

where $C = V(\tau)$ for $\tau = \sigma \cap \sigma'$ where $\sigma, \sigma' \in \Sigma(n)$, where $v \in \sigma'$ has image in $N(\tau)$ primitive gen of $\overline{\sigma'}$, and $m_\sigma - m_{\sigma'} \in \tau^\perp$.

Since $v \notin \sigma$, $v \notin \tau$ so $(*)$ forces $m_\sigma = m_{\sigma'}$, and this holds $\forall \sigma \in \Sigma(n)$, so $D = \mathrm{div}(-m_\sigma) \sim_{\text{lin}} 0$. □.

Proposition 2

For toric variety X, Cartier divisor D is nef (ie $D.C \geq 0$ $\forall$ proper irred curves $C \subset X$) iff $\mathcal{O}_X(D)$ is basepoint-free.

Sketch.

($\Leftarrow$) The morphism $\varphi_{|D|} : X \longrightarrow \mathbb{P}(\Gamma(\mathcal{O}_X(D)))$ gives

$$\begin{aligned} D.C &= \varphi_{|D|}^*(H).C \\ &= \varphi_{|D|*}([C]).H \\ &\geq 0 \end{aligned}$$

(doesn't need toric hypothesis)

($\Rightarrow$) For any $C = V(\tau)$ for $\tau \in \Sigma(n-1)$, write $\tau = \sigma \cap \sigma'$ and

$$\begin{aligned} 0 &\leq D.C \\ &= \langle m_\sigma - m_{\sigma'}, v \rangle \end{aligned}$$

as in the Lemma above. Thus $\langle m_\sigma, v \rangle \geq \langle m_{\sigma'}, v \rangle = \varphi_D(v)$ where φ_D is the 'support function' for divisor D. Since $v \notin \sigma$, one can then show φ_D is convex and hence $\mathcal{O}_X(D)$ is basepoint-free. □.

6.5 Def'n.

Let $D = \sum a_\rho D_\rho$ be an $\mathbb{R}$-Cartier divisor. Its class $[D] \in N^1(X)$ is

- effective if $\exists\ f \in \Gamma(\mathcal{O}_X(nD))$ for some $n > 0$
- movable if it has no fixed component, i.e. $\nexists$ effective nonzero D_0 s.t each $D' \in \Gamma(\mathcal{O}_X(D))$ is $D' = D_0 + E$ for some effective E.

The pseudoeffective cone $\overline{\mathrm{Eff}}(X)$ is the closure of the cone of effective classes, and the movable cone $\overline{\mathrm{Mov}}(X)$ is the closure of the cone of movable classes.

Theorem 3

For X a $\mathbb{Q}$-factorial toric variety, projective over an affine toric var and with $\mathrm{Cox}(X) = \mathbb{k}[z_1, .., z_r]$ graded by $\Lambda := \mathrm{Cl}(X)$. Under the isomorphism $N^1(X) \cong \mathrm{Pic}(X) \otimes \mathbb{R} = \Lambda_{\mathbb{R}}$, we have

① $\overline{\mathrm{Eff}}(X) = \mathrm{Cone}(\underline{w}) = \mathrm{Cone}(w_\rho \mid \rho \in \Sigma(1))$ is closed;

② $\overline{\mathrm{Mov}}(X) = \bigcap_{\rho \in \Sigma(1)} \mathrm{Cone}(w_1, \dots, \hat{w}_\rho, \dots, w_r)$ is closed;

③ $\mathrm{Nef}(X)$ is the unique top-dimensional cone in the secondary fan whose interior contains ample classes.

Corollary

$$\overline{\mathrm{Mov}}(X) = \bigcup_{\substack{\Sigma' \text{ simplicial} \\ \text{supported on} \\ \text{same cone as } \Sigma, \\ \Sigma'(1) = \Sigma(1)}} g^*(\mathrm{Nef}(X'))$$

where $g : X_\Sigma \dashrightarrow X_{\Sigma'}$ is the birational map