2/11/2016

5.1

Mori Dream Spaces

The secondary fan and geometric quotients

Sketch Proof

The locus X^{ss}_{λ} is covered by open affines $U_f := \operatorname{Spec} R[f^{-1}]$ for $f \in R_{d\lambda}$, $d > 0$, and good quotient is

$$U_f/G = \operatorname{Spec} R[f^{-1}]^G.$$

Compare with Proj of $S := \bigoplus_{j\geq 0} R_{j\lambda}$; it's covered by open affines $\operatorname{Spec} S[f^{-1}]^{\mathbb{C}^\times}$ for $f \in R_{d\lambda}$, $d>0$, where

$$S[f^{-1}]^{\mathbb{C}^\times} = \left\{ \frac{h}{f^l} \;\middle|\; h \in R_{ld\lambda},\ l > 0 \right\} = R[f^{-1}]^G$$

The covers agree if we pick the same set of f's; we may take a finite subcover since varieties are Noetherian. □.

Application

X toric variety, $\Lambda = \operatorname{Cl}(X)$, $R = \operatorname{Cox}(X) = \bigoplus_{E\in\Lambda} \Gamma(E)$. Set $r := |\Sigma(1)|$. For $\lambda \in \Lambda$

$$\begin{aligned}\mathbb{C}^r /\!\!/_{\lambda} G &= \operatorname{Spec} \operatorname{Cox}(X) /\!\!/_{\lambda} G \\ &\cong \operatorname{Proj} \bigoplus_{j\geq 0} \Gamma(\lambda^{\otimes j}) \\ &= \operatorname{Proj} \bigoplus_{j\geq 0} \mathbb{C}\langle \mathbb{N}^r \cap \deg^{-1}(\lambda^{\otimes j})\rangle\end{aligned}$$

is the toric variety whose fan Σ_λ is the inner normal fan to the polyhedron

$$P_\lambda := \operatorname{conv}(\mathbb{N}^r \cap \deg^{-1}(\lambda)).$$

(may need $\lambda \mapsto \lambda^{\otimes j}$)

The ~~lineality space~~ recession cone of the polyhedron is the cone $P_0 = \operatorname{conv}(\mathbb{N}^r \cap \deg^{-1}(0))$, and $\mathbb{C}^r /\!\!/_{\lambda} G$ is projective over $\operatorname{Spec} \mathbb{C}[\mathbb{N}^r \cap \deg^{-1}(0)]$.

5.2 The λ-unstable (:= not λ-semistable) locus is $\mathbb{V}(B_\lambda) \subset \mathbb{C}^n$ cut out by the ideal

$$B_\lambda := \mathrm{rad}\left(z^v \in \mathbb{C}[z_1,..,z_n] \mid v \in \mathbb{N}^n \cap \deg^{-1}(\lambda) \right)$$
$$= \left(\prod_{b \notin H_\rho} z_\rho \;\Big|\; b \text{ is vertex of } P_\lambda \right)$$

where H_ρ is coordinate hyperplane in $\mathbb{R}^\rho$. $\Rightarrow$ λ-unstable locus is union of coordinate subspaces.

Examples

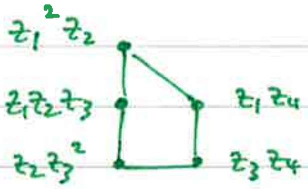
1. $X = \mathbb{P}^2$, then $\mathbb{C}^3 /\!/_\lambda \mathbb{C}^\times$ is either $\emptyset$, $\mathrm{Spec}\,\mathbb{C}$, $\mathbb{P}^2$ according to whether $\lambda = \mathcal{O}_{\mathbb{P}^2}(d)$ for $d<0$, $d=0$, $d>0$ resp. For $d>0$, $B_\lambda = (z_0, z_1, z_2)$ so $\mathbb{V}(B_\lambda) = 0 \in \mathbb{C}^3$.

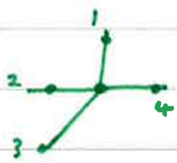
2. $X = \mathbb{F}_1$, then $\mathbb{C}^4 /\!/_\lambda (\mathbb{C}^\times)^2$ is either $\emptyset$, $\mathrm{Spec}\,\mathbb{C}$, $\mathbb{P}^1$, $\mathbb{P}^2$ or $\mathbb{F}_1$ according to where $\lambda = \mathcal{O}_{\mathbb{F}_1}(a,b)$ lies in $\mathrm{Pic}(\mathbb{F}_1)$:

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}} \mathrm{Pic}(\mathbb{F}_1) \longrightarrow 0$$

e.g. $\mathcal{O}(1,1) = \lambda$

P_λ :

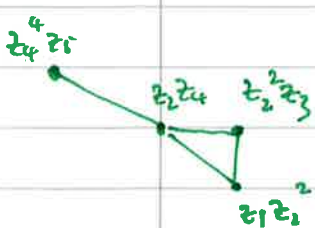




: Σ_λ

$$B_\lambda = \left(z_1 z_4,\; z_3 z_4,\; z_1 z_2,\; z_2 z_3 \right)$$

3. $X = Y$, then



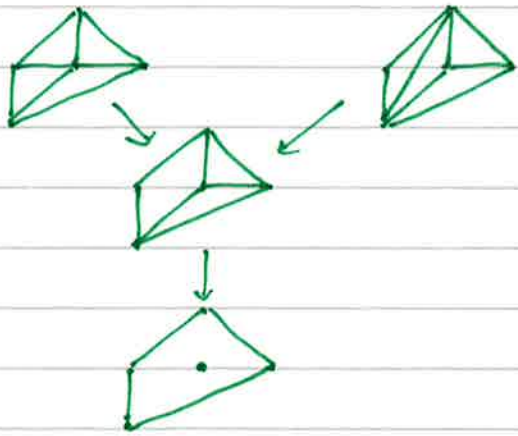
- $\lambda = \mathcal{O}(-1,2)$ then P_λ is polyhedron with vertices $\{ z_4^4 z_5,\ z_2 z_4,\ z_1 z_2^2,\ z_2^2 z_3 \}$ and recession cone P_0 generated by $z_3 z_4^2 z_5$, $z_1 z_4^2 z_5$, $z_1^2 z_2^2 z_5$, $z_2^2 z_3^3 z_5$, e.g.
$$P_\lambda \ni z_1 z_2 z_4^3 z_5 = (z_1 z_4^2 z_5).\, z_2 z_4.$$

5.3 • This choice of λ gives $\mathbb{C}^5 /\!\!/_\lambda (\mathbb{C}^\times)^2 \cong Y$

- $\lambda = \vartheta(1,1)$, then P_λ is polyhedron with vertices $\{z_1z_4,\ z_3z_4,\ z_1^2z_2,\ z_2z_3^2\}$ and recession cone P_0 as for $\lambda = \vartheta(-1,2)$. This choice of λ gives $\mathbb{C}^5 /\!\!/_\lambda (\mathbb{C}^\times)^2 \cong \mathrm{tot}(\omega_{\mathbb{F}_1})$
- $\lambda = \vartheta(0,1)$ determines a variety $\mathbb{C}^5 /\!\!/_\lambda (\mathbb{C}^\times)^2 \cong Y'$ which is badly singular; both Y and $\mathrm{tot}(\omega_{\mathbb{F}_1})$ admit a morphism to Y' which itself admits a morphism to $X := \mathrm{Spec}\ \mathbb{C}[z_3z_4^2z_5,\ z_1z_4^2z_5,\ z_1^2z_2^2z_5,\ z_2^2z_3^3z_5]$

We can summarize the situation with the flop diagram of fans:

$$\begin{tikzcd}
\mathrm{tot}(\omega_{\mathbb{F}_1}) \arrow[dr] \arrow[rr, dashed, leftrightarrow] & & Y \arrow[dl] \\
& Y' \arrow[d] & \\
& X &
\end{tikzcd}$$



To explain this phenomenon we need the secondary fan. First we define two cones. For $\Lambda = G^\vee$, we have

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^r \xrightarrow{\deg} \Lambda \longrightarrow 0.$$

For $1 \le \rho \le r$, set $w_\rho := \deg(D_\rho) = \vartheta_X(D_\rho)$; assume $w_\rho \neq 0$ for all ρ. Define

$$\mathrm{cone}(\underline{w}) := \mathrm{cone}(w_1, \dots, w_\rho) \subseteq \Lambda_{\mathbb{R}}$$

Also

$$0 \longleftarrow N \longleftarrow (\mathbb{Z}^r)^\vee \longleftarrow \Lambda^\vee \longleftarrow 0$$

5.4 where the dual basis in $(\mathbb{Z}^r)^\vee$ is mapped down to the primitive ~~vect~~ lattice points $v_\rho \in N$ on the rays $\rho \in \Sigma(1)$. Define

$$\operatorname{cone}(\underline{v}) = \operatorname{cone}(v_1, \ldots, v_\rho) \subset N_{\mathbb{R}}.$$

Remarks.

1. [Gale duality] $\operatorname{cone}(\underline{w}) = \Lambda_{\mathbb{R}} \iff \operatorname{cone}(\underline{v})$ is strongly convex
   and hence $\operatorname{cone}(\underline{v}) = N_{\mathbb{R}} \iff \operatorname{cone}(\underline{w})$ is strongly convex.
2. The $\operatorname{cone}(\underline{v})$ is the support of the fan Σ; this is the inner normal cone to the cone $P_0 = \operatorname{cone}(\mathbb{N}^r \cap \deg^{-1}(0))$ and defines the affine toric variety $\operatorname{Spec} \mathbb{C}[P_0 \cap M]$.
3. The $\operatorname{cone}(\underline{w})$ satisfies
   $$\lambda \in \operatorname{cone}(\underline{w}) \iff P_\lambda \neq \emptyset \iff \mathbb{C}^r /\!/_\lambda G \neq \emptyset \iff (\mathbb{C}^r)^{ss}_\lambda \neq 0$$
   and
   $$\lambda \in \operatorname{interior}(\operatorname{cone}(\underline{w})) \iff \dim \mathbb{C}^r /\!/_\lambda G = r - \dim G.$$

To explain the geometric significance of this decomposition, we need another standard definition from GIT. Again, let $G = \operatorname{Spec} \Bbbk[\Lambda]$ be a quasitorus that acts on $X = \operatorname{Spec} R \cong \mathbb{C}^r$

__Defn.__ $\lambda \in G^\vee$ is __generic__ if $\lambda \in \operatorname{cone}(\underline{w})$ and $\lambda \notin \operatorname{cone}(\underline{w}')$ for all subsets $\underline{w}' \subset \{w_1, \ldots, w_\rho\}$ with $\dim \operatorname{Cone}(\underline{w}')$ less than $\dim \operatorname{Cone}(\underline{w}) = \dim G$.

Terminology

The subcones of the form $\operatorname{cone}(\underline{w}')$ subdivide $\operatorname{cone}(\underline{w})$, and the resulting collection of cones is a fan in $\Lambda_{\mathbb{R}}$ called the __secondary fan__ (also the GKZ fan). A __chamber__ is the interior of a top-dimensional cone in this fan.

5.5 To understand the link between this combinatorics and geometry, we first return to the situation where $G = \operatorname{Spec} \Bbbk[\Lambda]$ acts on an affine variety $X = \operatorname{Spec} R$.

<u>Defn.</u> For $\lambda \in G^\vee$, a λ-semistable point $x \in X$ is <u>λ-stable</u> if the stabiliser G_x is finite and all G-orbits in $X - \mathbb{V}(f)$ are closed. Let $X^s_\lambda \subset X$ denote the locus of λ-stable points.

<u>Theorem</u>
For $\lambda \in \operatorname{Cone}(\underline{w})$, we have λ generic $\iff (\mathbb{C}^r)^s_\lambda = (\mathbb{C}^r)^{ss}_\lambda$.
When this is the case, we have:
- the polyhedron P_λ is simple (equiv, Σ_λ is simplicial), so $X_{\Sigma_\lambda} = \mathbb{C}^r /\!\!/_\lambda G$ is $\mathbb{Q}$-factorial;
- the GIT quotient $\mathbb{C}^r /\!\!/_\lambda G$ is a good <u>geometric</u> quotient of $(\mathbb{C}^r)^s_\lambda$ by G, ie fibres of $\pi| : (\mathbb{C}^r)^s_\lambda \longrightarrow \mathbb{C}^r /\!\!/_\lambda G$ are G-orbits.