4.1

Mori Dream Spaces

Geometric Invariant Theory

Let $R = \bigoplus_{\lambda \in \mathbb{N}} R_\lambda$ be a $\mathbb{Z}$-graded (with $R_\lambda = 0$ for $\lambda < 0$) finitely generated integral $\mathbb{k}$-algebra, and let $f_0, f_1, \dots, f_n \in R$ be $\mathbb{k}$-algebra generators with $f_i \in R_{a_i}$. The graded $\mathbb{k}$-algebra homomorphism

$$\psi : \mathbb{k}[z_0, z_1, \dots, z_n] \longrightarrow R$$

sending z_i (where $\deg(z_i) = \deg(f_i) = a_i$) to f_i induces

$$\psi^* : X = \operatorname{Spec} R \hookrightarrow \mathbb{C}^{n+1}$$

sending x to $(f_0(x), \dots, f_n(x))$, and $\exists$ compatible action of $\mathbb{C}^\times = \operatorname{Spec} \mathbb{k}[\mathbb{Z}]$. Consider the diagram

$$\begin{tikzcd}
X \setminus \mathbb{V}(f_i) = \operatorname{Spec} R[f_i^{-1}] \arrow[r, hook, "\text{open}"'] \arrow[d] & X \setminus \mathbb{V}(f_0, \dots, f_n) \arrow[r, hook, "\psi^*", "\text{closed}"'] \arrow[d] & \mathbb{C}^{n+1} \setminus \{0\} \arrow[d] \\
\operatorname{Spec} R[f_i^{-1}]^{\mathbb{C}^\times} \arrow[r, hook, "\text{open}"'] & \operatorname{Proj}(R) \arrow[r, hook, "\text{closed}"'] & \mathbb{P}^n_{R_0}(a_0, \dots, a_n)
\end{tikzcd}$$

where the vertical maps identify points in the same $\mathbb{C}^\times$-orbit; here the affine open subsets $\operatorname{Spec} R[f_i^{-1}]^{\mathbb{C}^\times}$ cover $\operatorname{Proj}(R)$, where

$$R[f_i^{-1}]^{\mathbb{C}^\times} = \left\{ \frac{f}{f_i^j} \;\middle|\; j \geq 0,\ f \in R_{ja_i} \right\}$$

Therefore, locally, taking Proj corresponds to taking Spec of $\mathbb{C}^\times$-invariant subalgebras.

4.2 Let G be an algebraic group acting algebraically on a variety X. We needn't assume G or X is affine here.

<u>Defn.</u> A surjective morphism $\pi : X \longrightarrow Y$ that's constant on G-orbits as a <u>good quotient</u> if

1) $\forall\, U \subseteq Y$ open, $\Gamma(U, \mathcal{O}_Y) \xrightarrow{\pi^*} \Gamma(\pi^{-1}(U), \mathcal{O}_X)^G$ is an isomorphism;
2) $\forall\, W \subseteq X$ closed and G-invariant, $\pi(X) \subseteq Y$ is closed;
3) $\forall\, W_1, W_2 \subseteq X$ closed, G-inv. and disjoint, $\pi(W_1) \cap \pi(W_2) = \emptyset$.

<u>Propn.</u>
Let $\pi : X \longrightarrow Y$ be a good quotient. Then

(i) if $\phi : X \longrightarrow Z$ is constant on G-orbits, $\exists$ unique morphism $\bar{\phi} : Y \longrightarrow Z$ s.t. $\bar{\phi} \circ \pi = \phi$

(ii) $x, x' \in X$ satisfy $\pi(x) = \pi(x') \iff \overline{G \cdot x} \cap \overline{G \cdot x'} \neq \emptyset$

(iii) $\forall\, y \in Y$, fibre $\pi^{-1}(y)$ contains exactly one closed G-orbit.

<u>Remarks</u>
By (i), if a good quotient exists then it's unique. If so, we write $\pi : X \longrightarrow X/G$ for this <u>good 'categorical' quotient</u>.
By (iii), a good categorical quotient parametrize closed G-orbits.

<u>Sketch</u>
For (i):

- for $y \in Y$, $\phi(\pi^{-1}(y))$ is a single point $z \in Z$ else we contradict 3, so $\bar{\phi}$ sends $y \longmapsto z$ as map of sets.
- for $V \subseteq Z$ open, $\bar{\phi}^{-1}(V) \cong Y \setminus \pi(X \setminus \phi^{-1}(V))$ is open by 2, so $\bar{\phi}$ continuous.
- for $V \subseteq Z$ open, <u>ϕ constant on G-orbits</u> and 1 holds so
$$\phi^* : \Gamma(V, \mathcal{O}_Z) \cong \Gamma(\phi^{-1}(V), \mathcal{O}_X)^G \cong \Gamma(\bar{\phi}^{-1}(V), \mathcal{O}_Y)$$
so $\bar{\phi} : Y \longrightarrow Z$ is algebraic.

4.3 For (ii), the direction $\Rightarrow$ is immediate from ③; for the $\Leftarrow$ direction, having $\pi(x) \neq \pi(x')$ forces x, x' to lie in distinct fibres, so $\overline{G \cdot x}$ and $\overline{G \cdot x'}$ also lie in different fibres so they're disjoint.
For (iii), if two closed G-orbits lie in the same fibre they must intersect which is absurd. To show each fibre contains a closed orbit, one shows that an orbit of minimal dimension is closed. □

Rmk.
The notion of good quotient is local: if $U \subset X/G$ is open, then $\pi| : \pi^{-1}(U) \to U$ is a good quotient; conversely, if $\pi : X \to Y$ constant on G-orbits and $Y = \bigcup U_i$ is an open affine cover s.t. $\pi| : \pi^{-1}(U_i) \to U_i$ is a good quotient for each i, then so is $\pi : X \longrightarrow Y$.

<u>Theorem</u>
Let G be quasi-torus acting algebraically on $X = \operatorname{Spec} R$. Then R^G is a finitely generated integral $\mathbb{k}$-algebra, and the morphism

$$\pi : X = \operatorname{Spec} R \longrightarrow X/G := \operatorname{Spec} R^G$$

induced by $R^G \hookrightarrow R$ is a good categorical quotient.
<u>Sketch.</u>
The first statement is Hilbert's -Nagata theorem (this holds more generally for any linearly reductive G). The idea is to take a G-inv. subspace $V \subset R$ that contains $\mathbb{k}$-algebra generators; can take V finite dimensional, and hence get surjective

$$\operatorname{Sym}^*(V) \longrightarrow R$$

which, by linear reductivity, induces a surjective map

$$\operatorname{Sym}^*(V)^G \longrightarrow R^G$$

so we may assume that R is a polynomial ring. Now apply Hilbert's basis theorem to the ideal in R generated by homogeneous

4.4 polynomials of positive degree from R^G. Then any homogeneous $f \in R^G_d$ becomes an R-linear combination of these generators, i.e.

$$f = \sum_i h_i f_i \qquad \circledast$$

where $f_1, \dots, f_k \in R^G$. Since G is linearly reductive, the action of G on $\mathbb{A}^n = \operatorname{Spec} R$ is completely reducible hence too for the action on $R \cong \mathbb{k}[x_1, \dots x_n]$; then using a G-invariant subspace complementary to R^G, one can project away to get a linear homomorphism (not ring homo)

$$r : R \longrightarrow R^G$$

s.t $r(fh) = f \cdot r(h)$ for $f \in R^G$ and $h \in R$; this is the 'Reynolds operator'. Apply r to $\circledast$ gives

$$f = r(f) = \sum_i r(h_i) f_i$$

and now each $r(h_i) \in R^G$ so we can apply induction since they have degree smaller than the degree of f. $\square$.

Rmk: The proof of the second statement needs the fact that for any $W_1, W_2 \subset X$ G-inv. closed disjoint subsets, $\exists f \in R^G$ s.t $f(W_1) \equiv 0$ and $f(W_2) \equiv 1$.

Examples

1. For $\mathbb{C}^\times \curvearrowright \mathbb{C}^{n+1}$ with weights $(1, 1, \dots, 1)$,

$$\pi : \mathbb{C}^{n+1} \longrightarrow \operatorname{Spec} \mathbb{C}[z_0, \dots z_n]^{\mathbb{C}^\times} \cong \operatorname{Spec} \mathbb{C}$$

is a good quotient!

2. For $\mathbb{C}^\times \curvearrowright \mathbb{C}^{n+1} \setminus \{0\}$ with weights $(1, 1, \dots, 1)$, the open affines $U_i = \operatorname{Spec} \mathbb{C}[z_0, \dots, z_n][z_i^{-1}]$ satisfy

$$\pi_i : U_i \longrightarrow U_i/G \cong \operatorname{Spec} \mathbb{C}[z_0, \dots, z_n, z_i^{-1}]^{\mathbb{C}^\times}$$
$$\cong \operatorname{Spec} \mathbb{C}[z_0/z_i, \dots, z_n/z_i] \cong \mathbb{C}^n$$

is a good quotient; these patch to give good quotient $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$.

4.5 We now go from local to global as in the last example. Let $G = \operatorname{Spec} \mathbb{k}[\Lambda]$ be a quasi-torus that acts algebraically on $X = \operatorname{Spec} R$, giving
$$R = \bigoplus_{\lambda \in \Lambda} R_\lambda.$$

Fix $\lambda \in \Lambda = G^\vee$ and lift the G-action to an action on $X \times \mathbb{C}$ where
$$g \cdot (x, p) = (g \cdot x, \lambda(g) \cdot p);$$

this is a 'linearisation' of the trivial bundle $\mathcal{O}_X$ w.r.t. G-action. The dual action of G on $R[t] = \Gamma(\mathcal{O}_{X \times \mathbb{C}})$ is determined by the weight λ of the action on t, and for $j \geq 0$
$$f \cdot t^j \in R[t] \text{ is } G\text{-invariant} \iff f \in R_{j\lambda}$$

$\therefore$ the ring $\bigoplus_{j \geq 0} R_{j\lambda} \cong R[t]^G$

so it's a finitely generated $\mathbb{k}$-algebra by Hilbert–Nagata.

__Defn.__ The __GIT quotient__ of X by G, linearised by $\lambda \in G^\vee$ is
$$X /\!\!/_\lambda G := \operatorname{Proj} \left(\bigoplus_{j \geq 0} R_{j\lambda} \right)$$

__Defn.__ For $\lambda \in G^\vee$, a point $x \in X$ is λ-semistable if $\exists j > 0$ and $f \in R_{j\lambda}$ s.t. $f(x) \neq 0$. Let $X^{ss}_\lambda \subset X$ denote the locus of λ-semi-stable points in X.

__Theorem__
The GIT quotient $X /\!\!/_\lambda G$ is the good categorical quotient of X^{ss}_λ by G. It admits a projective morphism to the 'affine quotient' $X/G := \operatorname{Spec} R^G$.