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3.1

Mori Dream Spaces

On graded rings and quasi-torus actions

Assume from now on that X_Σ has no $(\mathbb{C}^*)$ factors, i.e., Σ is not contained in a hyperplane of $N \otimes_{\mathbb{Z}} \mathbb{R}$. Let

$$\mathbb{Z}^{\Sigma(1)} := \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} \cdot D_\rho$$

denote the lattice of torus-invariant Weil divisors on X_Σ. If $v_\rho \in N$ denotes the primitive lattice point on the ray $\rho \in \Sigma(1)$, then the lattice map

$$\mathrm{div} : M \longrightarrow \mathbb{Z}^{\Sigma(1)} \; : \; m \longmapsto \sum_{\rho \in \Sigma(1)} \langle m, v_\rho \rangle D_\rho$$

is injective (our assumption on X_Σ implies that the dual map to div is surjective) and its cokernel is isomorphic to the Class group

$$\mathrm{Cl}(X_\Sigma) := \{ \mathcal{O}_X(D) \mid D \text{ is Weil divisor} \} /_{\sim \text{isom}}$$

where for $U \subset X_\Sigma$ open, we have

$$\Gamma(U, \mathcal{O}_X(D)) = \{ f \in \mathbb{C}(X)^* \mid (\mathrm{div}(f) + D)|_U \geq 0 \} \cup \{0\}$$

Therefore

$$0 \longrightarrow M \xrightarrow{\mathrm{div}} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\deg} \mathrm{Cl}(X_\Sigma) \longrightarrow 0$$

$$D \longmapsto \mathcal{O}_X(D)$$

is a short exact sequence of finitely generated abelian groups. Note that $\mathrm{Cl}(X_\Sigma)$ is isomorphic to the Picard group of line bundles on X_Σ when X_Σ is smooth.

3.2 **Prop'n** For $E \in Cl(X_\Sigma)$ we have

$$\Gamma(E) \cong \bigoplus_{D \in \mathbb{N}^{\Sigma(1)} \cap \deg^{-1}(E)} \mathbb{C} \cdot x^D = \bigoplus_{\substack{D \text{ effective} \\ \mathcal{O}_X(D) = E}} \mathbb{C} \cdot x^D$$

Corollary The Cox ring of X_Σ is the $Cl(X_\Sigma)$-graded ring

$$Cox(X_\Sigma) = \bigoplus_{E \in Cl(X_\Sigma)} \Gamma(E) \cong \mathbb{C}[\mathbb{N}^{\Sigma(1)}] \cong \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$$

where a monomial $x^D \in Cox(X_\Sigma)$ has degree $\mathcal{O}_X(D) \in Cl(X_\Sigma)$.

Examples

(i) $\mathbb{P}^2$

$$\begin{tikzcd} M \arrow[d, phantom, "\cong" rotate=90] & & \mathrm{Pic}(\mathbb{P}^2) \arrow[d, phantom, "\cong" rotate=90] \\ 0 \arrow[r] & \mathbb{Z}^2 \arrow[r, "\begin{pmatrix} -1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}"] & \mathbb{Z}^3 \arrow[r, "{(1,1,1)}"] & \mathbb{Z} \end{tikzcd}$$

and $\mathbb{C}[z_0, z_1, z_2]_d \cong \Gamma(\mathcal{O}_{\mathbb{P}^n}(d))$ spanned by monomials of deg d.

(ii) $\mathbb{F}_1$

$$\begin{tikzcd} M \arrow[d, phantom, "\cong" rotate=90] & & & \mathrm{Pic}(\mathbb{F}_1) \arrow[d, phantom, "\cong" rotate=90] \\ 0 \arrow[r] & \mathbb{Z}^2 \arrow[r, "\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ -1 & -1 \\ 1 & 0 \end{pmatrix}"] & \mathbb{Z}^4 \arrow[r, "\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}"] & \mathbb{Z}^2 \arrow[r] & 0 \end{tikzcd}$$

and $\Gamma(\mathcal{O}_{\mathbb{F}_1}(1,1)) \cong \mathbb{C}^5$ has basis $x_1 x_4, x_3 x_4, x_1^2 x_2, x_1 x_2 x_3, x_2 x_3^2$

(iii) Y

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^3 \arrow[r, "\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}"] \arrow[d, phantom, "\cong" rotate=90] & \mathbb{Z}^5 \arrow[r, "\begin{pmatrix} 1 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 \end{pmatrix}"] & \mathbb{Z}^2 \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\ & M & & \mathrm{Pic}(Y) \end{tikzcd}$$

and $\Gamma(\mathcal{O}_Y(-1,2))$ is gen'd as $\mathbb{C}[\mathbb{N}^5 \cap \deg^{-1}(0)]$-module by 4 elements.

3.3 <u>Def'n.</u> An <u>affine algebraic group</u> G is an affine variety together with morphisms $\nu : G\times G \to G$, $e : \mathrm{Spec}\,\mathbb{C} \to G$ and $\iota : G \longrightarrow G$ satisfying a compatibility condition for each defining axiom of a group,

e.g., associativity becomes
$$\begin{tikzcd} G\times G\times G \arrow[r, "\nu\times id"] \arrow[d, "id\times\nu"'] \arrow[dr, phantom, "\square"] & G\times G \arrow[d, "\nu"] \\ G\times G \arrow[r, "\nu"] & G \end{tikzcd}$$

An affine algebraic group G is a <u>quasi-torus</u> (a.k.a. diagonalisable algebraic group) if the character group $G^\vee := \mathrm{Hom}(G, \mathbb{C}^*)$ provides a $\mathbb{C}$-vector space basis for the $\mathbb{C}$-algebra $\Gamma(\mathcal{O}_G)$.

<u>Example</u> $G := (\mathbb{C}^\times)^n$ has $G^\vee \cong \mathbb{Z}^n$ and
$\Gamma(\mathcal{O}_G) := \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \cong \bigoplus_{\lambda\in\mathbb{Z}^n} \mathbb{C}\cdot t^\lambda =: \mathbb{C}[\mathbb{Z}^n]$

<u>Remarks</u> In fact every quasi-torus is isomorphic to $(\mathbb{C}^*)^n \times \prod_{i=1}^k \mu_{r_i}$.

<u>Theorem</u>
There is a contravariant equivalence of categories
$$\begin{tikzcd} \{\text{finitely-generated abelian groups}\} \arrow[r, bend left, "\Lambda \mapsto \mathrm{Spec}\,\Bbbk[\Lambda]"] & \{\text{quasi-tori}\} \arrow[l, bend left, "G^\vee = \mathrm{Hom}(G,\mathbb{C}^*) \mapsfrom G"] \end{tikzcd}$$

<u>Sketch</u> as ν on $\mathrm{Spec}\,\Bbbk[\Lambda]$ via $\Bbbk[\Lambda] \longrightarrow \Bbbk[\Lambda]\otimes\Bbbk[\Lambda] : t^\lambda \mapsto t^\lambda \otimes t^\lambda$.
The construction in both directions is clear. The defn of quasi-torus ensures that $\mathrm{Spec}\,\Bbbk[G^\vee] \cong G$. Given Λ, choose presentation $\mathbb{Z}^d \twoheadrightarrow \Lambda$ induces closed immersion $\mathrm{Spec}\,\Bbbk[\Lambda] \hookrightarrow (\mathbb{C}^*)^d$ and exact sequence with cokernel $(\mathbb{C}^*)^k$, so $\mathrm{Spec}\,\Bbbk[\Lambda] \cong (\mathbb{C}^*)^{d-k} \times \prod_{i=1}^k \mu_{r_i}$ whose character group is isomorphic to Λ.

3.4 <u>Def'n</u> An affine variety X admits an <u>algebraic action</u> of G if $\exists$ morphism $\mu : G \times X \longrightarrow X$ satisfying a compatibility condition for each defining axiom of a group action, e.g.,

$g\cdot(h\cdot x) = (gh)\cdot x$ becomes

$$\begin{tikzcd} G\times G\times X \arrow[r, "\mathrm{id}\times\mu"] \arrow[d, "\nu\times\mathrm{id}"'] & G\times X \arrow[d, "\mu"] \\ G\times X \arrow[r, "\mu"] & X \end{tikzcd}$$

The dual action of G on the ring $\Gamma(\mathcal{O}_X)$ of regular functions is given by $(g\cdot f)(p) = f(g^{-1}\cdot p)$ for $f\in\Gamma(\mathcal{O}_X)$, $g\in G$, $p\in X$.

<u>Def'n</u> Given $\lambda\in G^\vee$, we say $f\in\Gamma(\mathcal{O}_X)$ is <u>homogeneous of weight λ</u> (and we write $f\in\Gamma(\mathcal{O}_X)_\lambda$) iff $f(g\cdot p) = \lambda(g)f(p)$ for $g\in G$ and $p\in X$.

<u>Theorem</u>

Let Λ be a finitely generated abelian group. There is a contravariant equivalence of categories

$$\begin{tikzcd}[column sep=huge] \{\text{graded. fin. gen'd } \Bbbk\text{-algebras}\} \arrow[r, bend left, "R=\bigoplus_{\lambda\in\Lambda}R_\lambda \;\mapsto\; \operatorname{Spec}\Bbbk[\Lambda]\text{ action on }\operatorname{Spec}(R)"] & \{\text{affine varieties with quasitorus actions}\} \arrow[l, bend left, "\Gamma(\mathcal{O}_X)=\bigoplus_{\lambda\in G^\vee}\Gamma(\mathcal{O}_X)_\lambda \;\mapsfrom\; X\text{ with }G\text{-action}"] \end{tikzcd}$$

<u>Sketch</u>

From left to right, the action of $\operatorname{Spec}\Bbbk[\Lambda]$ on $\operatorname{Spec} R$ is obtained by pulling back along $\mu^* : R \longrightarrow \Bbbk[\Lambda]\otimes_{\Bbbk} R$ given by

$$\mu^*(f) = t^\lambda\otimes f \qquad \text{for } f\in R_\lambda$$

From right to left, the dual action of G on $\Gamma(\mathcal{O}_X)$ comes from a rational representation which splits as the given direct sum. $\square$.

3.5 **Examples**

① Let X_Σ be a toric variety. The polynomial ring $\mathrm{Cox}(X_\Sigma) = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ is graded by the finitely generated abelian group $\mathrm{Cl}(X_\Sigma)$, so we obtain an action of the quasitorus

$$G = \operatorname{Spec} \mathbb{C}[\mathrm{Cl}(X_\Sigma)] \quad \text{on the affine space } \mathbb{A}^{\Sigma(1)} = \operatorname{Spec} \mathrm{Cox}(X_\Sigma)$$

Explicitly:

(i) $\mathbb{P}^2$ has $G = \operatorname{Spec} \mathbb{C}[\mathbb{Z}] \cong \mathbb{C}^\times$ acting on $\mathbb{C}^3 = \operatorname{Spec} \mathbb{C}[z_0, z_1, z_2]$ where z_0, z_1, z_2 are homogeneous of weight $1 \in \mathbb{Z}$, i.e.

$$g(p_0, p_1, p_2) = (gp_0, gp_1, gp_2)$$

(ii) $\mathbb{F}_1$ has $G \cong (\mathbb{C}^\times)^2$ acting on $\mathbb{A}^4 = \operatorname{Spec} \mathbb{C}[x_1, x_2, x_3, x_4]$ where

$$(g_1, g_2) \cdot (p_1, p_2, p_3, p_4) = (g_1 p_1, g_1^{-1} g_2 p_2, g_1 p_3, g_2 p_4).$$

② Consider a finitely-generated integral $\mathbb{C}$-algebra with a $\mathbb{Z}$-grading

$$R = \bigoplus_{\lambda \in \mathbb{Z}} R_\lambda. \qquad \text{s.t. } R_\lambda = 0 \text{ for } \lambda < 0.$$

Then

$$\operatorname{Proj}(R) := \left\{ \begin{array}{l} \mathbb{Z}\text{-homogeneous prime ideals in } R \\ \text{that do not contain } \bigoplus_{\lambda > 0} R_\lambda \end{array} \right\}$$

More geometrically, choose homogeneous generators $f_0, f_1, \ldots, f_n$ with $f_i \in R_{a_i}$ and write

$$\psi : \mathbb{k}[x_0, x_1, \ldots, x_n] \twoheadrightarrow R$$

satisfy $\psi(x_i) = f_i$ and set $\deg(x_i) = a_i$, so ψ is a homomorphism of graded rings. Then $\ker(\psi) \subset \mathbb{k}[x_0, \ldots, x_n]$ is a $\mathbb{Z}$-homogeneous ideal that encodes relations

3.6 between the f_i, and

$$\operatorname{Proj}(R) \cong \mathbb{V}(\ker\psi) \subset \mathbb{P}^n_{R_0}(a_0, a_1, .., a_n);$$

the ambient space here is weighted projective space, ie the quotient of $\mathbb{C}^{n+1} - \{0\}$ mod $\mathbb{C}^\times$ acting with weights $a_0, .., a_n$ [when $\mathbb{C} = R_0$; otherwise its a quotient of $\mathbb{A}^{n+1}_{R_0} - \{0\}$]

More geometrically still, ψ induces a closed immersion $\psi^* : \operatorname{Spec} R \hookrightarrow \mathbb{C}^{n+1}$ sending a point p to $(f_0(p), \ldots, f_n(p))$. The $\mathbb{Z}$-grading $\mathbb{C}^\times = \operatorname{Spec} \mathbb{C}[\mathbb{Z}]$-actions on both $\operatorname{Spec} R$ and $\mathbb{C}^{n+1}$ that make ψ^* into a $\mathbb{C}^\times$-equivariant morphism. If we remove from $X = \operatorname{Spec} R$ the common zero-locus of $f_0, f_1, .., f_n$, then the restriction of ψ^* descends to a morphism on $\mathbb{C}^\times$-orbits

assume $R_0 = \mathbb{C}$

$$\begin{tikzcd}
X \setminus \mathbb{V}(f_i) \arrow[d, phantom, "\shortparallel"] & & \\
\operatorname{Spec} R[f_i^{-1}] \arrow[r, hook, "\text{open}"'] \arrow[d] & X \setminus \mathbb{V}(f_0, f_1, .., f_n) \arrow[r, hook] \arrow[d, "\pi"] & \mathbb{C}^{n+1} \setminus \{0\} \arrow[d] \\
\operatorname{Spec} R[f_i^{-1}]^{\mathbb{C}^\times} \arrow[r, hook, "\text{open}"'] & \operatorname{Proj}(R) \arrow[r, hook] & \mathbb{P}^n_{\mathbb{C}}(a_0, .. a_n)
\end{tikzcd}$$

where the vertical maps identify points in the same $\mathbb{C}^\times$-orbit. The locus $X \setminus \mathbb{V}(f_0, f_1, .., f_n)$ on which the quotient map is well-defined is covered by the principal affine open subsets

$$X \setminus \mathbb{V}(f_i) = \operatorname{Spec} R[f_i^{-1}],$$

so $\operatorname{Proj}(R)$ is covered by local charts $\operatorname{Spec} R[f_i^{-1}]^{\mathbb{C}^\times}$ where

$$R[f_i^{-1}]^{\mathbb{C}^\times} = \left\{ \frac{f}{f_i^j} \;\middle|\; f \in R_{ja_i} \quad j \geq 0 \right\}$$