12/10/2016

2.1 # Mori Dream Spaces

Introduction to toric varieties

We'll introduce toric varieties by using the following diagram to highlight different topics:

$$\begin{tikzcd}
\mathbb{F}_1 \arrow[rr, "\text{blow-up}"] \arrow[d, hook, "\text{closed immersion}"'] & & \mathbb{P}^2 \arrow[d, hook, "\text{closed immersion}"'] & \text{smooth proj. surfaces} \\
\mathrm{tot}(\omega_{\mathbb{F}_1}) \arrow[r, dashed, leftrightarrow, "\text{flop}"] \arrow[dr, "\text{resolution}"'] & Y \arrow[d, "\text{resolution}"] & \mathrm{tot}(\omega_{\mathbb{P}^2}) \arrow[l, hook', "\text{open immersion}"] \arrow[d, "\text{resolution}"] & \text{smooth quasi-proj threefolds} \\
 & X & \mathbb{C}^3/G & \text{singular affine threefolds}
\end{tikzcd}$$

(i) **The orbit-decomposition : $\mathbb{P}^2$**

$$\mathbb{P}^2 = \coprod_{0 \le i \le 2} U_i /\sim \qquad \text{where} \qquad U_i := \{ [z_0 : z_1 : z_2] \in \mathbb{P}^2 \mid z_i \neq 0 \}$$

The algebraic torus $(\mathbb{k}^\times)^2 = \{ [z_0 : z_1 : z_2] \in \mathbb{P}^2 \mid z_0 z_1 z_2 \neq 0 \}$ in U_i acts on $\mathbb{P}^2$ as

$$(\lambda_0 : \lambda_1 : \lambda_2) \cdot [z_0 : z_1 : z_2] = [\lambda_0 z_0 : \lambda_1 z_1 : \lambda_2 z_2],$$

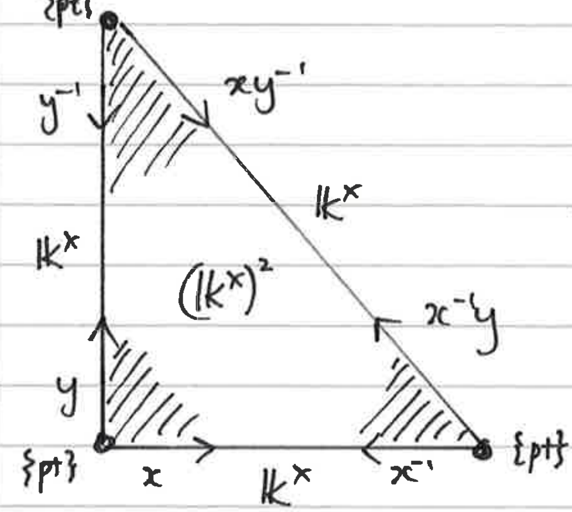
i.e., the action of $(\mathbb{k}^\times)^2$ on itself extends to an action of $(\mathbb{k}^\times)^2$ on $\mathbb{P}^2$.

Defn A (normal) toric variety is a variety containing $(\mathbb{k}^\times)^d$ as an open subvariety such that the action of

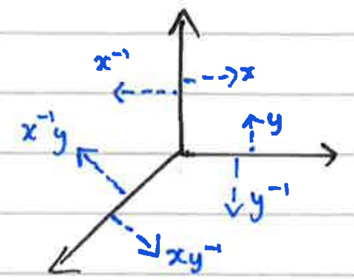
2.2 $(\Bbbk^\times)^d$ on itself extends to an action on the variety.

For $\mathbb{P}^2$, there exist seven $(\Bbbk^\times)^2$-orbits determined by vanishing of the coordinates z_0, z_1, z_2. To draw this decomposition, write $U_i = \operatorname{Spec} \mathbb{C}[t_1, t_2]$:

$$\mathbb{C}[U_0] = \mathbb{C}\left[\tfrac{z_1}{z_0}, \tfrac{z_2}{z_0}\right] \qquad x := z_1/z_0 \quad y := z_2/z_0$$
$$\mathbb{C}[U_1] = \mathbb{C}\left[z_0/z_1, z_2/z_1\right] \qquad x^{-1}, \quad x^{-1}y$$
$$\mathbb{C}[U_2] = \mathbb{C}\left[z_0/z_2, z_1/z_2\right] \qquad y^{-1}, \quad xy^{-1}$$

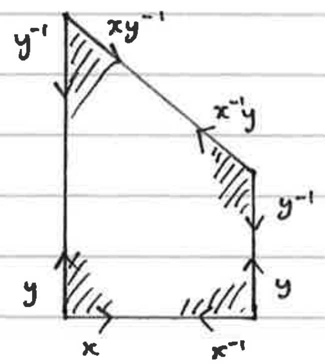


and ~~record~~ this polytope has inner normal fan

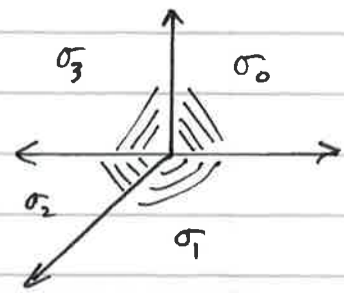


(ii) <u>Local charts and combinatorial description : $\mathbb{F}_1$</u>

Now let's blow-up the point $[0:1:0]$ in $\mathbb{P}^2$. At least topologically, the operation of blow-up deletes one point and replaces it by a copy of $\mathbb{P}^1$



which has inner normal fan



2.3 Motivated by the $\mathbb{P}^2$ example, we need four charts V_i $(0 \leq i \leq 3)$ where

$$\mathbb{C}[V_0] = \mathbb{C}[x,y] \quad \mathbb{C}[V_1] = \mathbb{C}[y^{-1}, xy^{-1}] \quad \mathbb{C}[V_2] = \mathbb{C}[x^{-1}y, y^{-1}] \quad \mathbb{C}[V_3] = \mathbb{C}[x^{-1}, y]$$

giving $V_i \cong \mathbb{C}^2$ for $0 \leq i \leq 3$. Convince yourself these charts cover

$$\mathbb{F}_1 := Bl_{[0:1:0]} \mathbb{P}^2 = \left\{ ([z_0 : z_1 : z_2], [s:t]) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid z_0 t = z_2 s \right\}$$

Each of these charts corresponds to one of the 2-dimensional cones $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ in the fan of $\mathbb{F}_1$. More generally:

$\tau := \sigma_0 \cap \sigma_1$ determines $V_0 \cap V_1 = \operatorname{Spec} \mathbb{C}[y, y^{-1}, x] \simeq \mathbb{C}^\times \times \mathbb{C}$

$\tau := \{0\}$ $\qquad V_0 \cap V_2 = \operatorname{Spec} \mathbb{C}[x, x^{-1}y, y^{-1}] \simeq (\mathbb{C}^\times)^2$

<u>Construction : toric variety from a fan Σ</u>

Fix
- lattice $N \simeq \mathbb{Z}^n$ — $n \times 1$ column vectors
- cone $\sigma \subseteq N \otimes_{\mathbb{Z}} \mathbb{R}$ — rational polyhedral w/ 0 max'l sub. ("strongly convex")

then
- dual lattice $M := N^\vee$ — $1 \times n$ row vectors
- dual cone $\sigma^\vee = \{ u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0 \ \ \forall v \in \sigma \}$

determine
- semigroup algebra
$$\mathbb{C}[\sigma^\vee \cap M] = \left\{ \sum c_m t^m \mid m \in \sigma^\vee \cap M \right\} \subseteq \mathbb{C}[M] = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

and hence
- affine variety $\quad U_\sigma := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$
  containing $\quad U_0 = \operatorname{Spec} \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \simeq (\mathbb{C}^\times)^n$.

Each face τ of σ is $\tau = \sigma \cap m^\perp$ for some $m \in M$, and

$$\mathbb{C}[\tau^\vee \cap M] = \mathbb{C}[\sigma^\vee \cap M]_{t^m} \quad \text{- localisation of the algebra}$$

2.4 therefore

$$U_\tau \xhookrightarrow{\text{open}} U_\sigma \qquad \text{embeds as the locus } t^m \neq 0$$

<u>Def</u> A <u>fan</u> $\Sigma \subset N \otimes_{\mathbb{Z}} \mathbb{R}$ is a collection of cones as above such that: for $\sigma, \sigma' \in \Sigma$, the cone $\sigma \cap \sigma'$ is a common face of both σ and σ'; and each face of $\sigma \in \Sigma$ also lies in Σ.

Given a fan $\Sigma \subset N \otimes_{\mathbb{Z}} \mathbb{R}$, define $X_\Sigma := \coprod_{\sigma \in \Sigma} U_\sigma \big/ \sim_{\text{glue}}$

<u>Rmks</u>

By this definition, toric varieties are necessarily normal (the semigroup $\sigma^\vee \cap M$ is saturated and hence $\mathbb{C}[\sigma^\vee \cap M]$ is integrally closed), and rational of dimension $n = \operatorname{rk} N$ (since each U_σ contains $U_0 \cong (\mathbb{C}^\times)^n$ as dense open subset).

(iii) <u>Singularities and the role of the lattice : $\mathbb{C}^3/G$</u>

Consider $\mathbb{C}^3 = \operatorname{Spec} \mathbb{C}[x,y,z] \cong \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap \overline{M}]$ where $\overline{N} = \mathbb{Z}^3$, $\sigma = \{ \sum \alpha_i e_i \mid \alpha_i \geq 0 \}$ is the positive octant and $\overline{M} = \overline{N}^\vee$. Now consider the overlattice

$$N := \overline{N} + \mathbb{Z} \cdot \tfrac{1}{3}(1,1,1) \supset \overline{N}$$

Then

$$M = \{ m \in \overline{M} \mid \langle m, n \rangle \in \mathbb{Z} \quad \forall n \in N \}$$

$$= \{ (m_1, m_2, m_3) \in \overline{M} \mid m_1 + m_2 + m_3 \in 3\mathbb{Z} \}$$

$$\therefore \quad \mathbb{C}[\sigma^\vee \cap M] = \left\{ \sum c_m x^{m_1} y^{m_2} z^{m_3} \in \mathbb{C}[x,y,z] \mid m_1 + m_2 + m_3 \in 3\mathbb{Z} \right\}$$

2.5 For $G = \mathbb{Z}/3\mathbb{Z} = \left\langle \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} \;\middle|\; \omega = e^{2\pi i/3} \right\rangle$ then $\mathbb{C}[\sigma^\vee \cap M] \cong \mathbb{C}[x,y,z]^G$ and hence

$$U_\sigma \cong \operatorname{Spec} \mathbb{C}[x,y,z]^G =: \mathbb{C}^3/G,$$

which is a singular affine variety of dimension three. Notice that the primitive cone generators of σ, namely e_1, e_2, e_3, do not form a $\mathbb{Z}$-basis of N : $\frac{1}{3}(1,1,1) \notin \mathbb{Z}^3 = \operatorname{span}(e_1, e_2, e_3)$
More generally, we have :

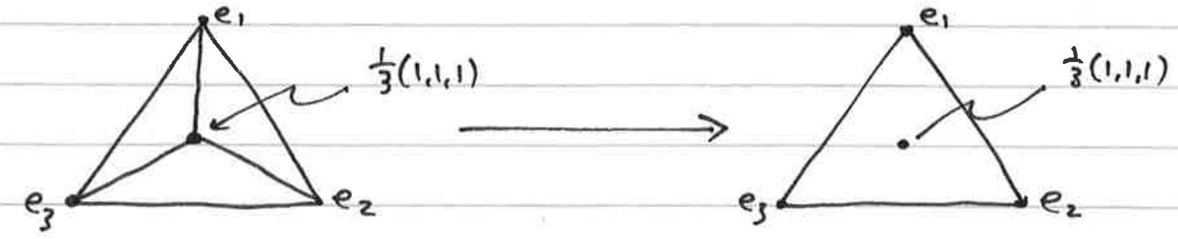
<u>Lemma</u>
U_σ is smooth $\iff$ the primitive cone generators of σ can be extended to a $\mathbb{Z}$-basis of the lattice N.
<u>Proof</u>
If $e_1, .., e_k$ are primitive cone generators then $U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}$.
For opposite direction, one can reduce to the case $\dim \sigma = n$, in which case the dimension of the Zariski tangent space at the torus-invariant point of U_σ equals the number of elements in a Hilbert basis of the semigroup $\sigma^\vee \cap M$. □

(iv) <u>Resolution of singularities : $\operatorname{tot}(\omega_{\mathbb{P}^2})$</u>

For a singular variety X, a resolution of singularities is a smooth variety Y and a proper birational morphism $\pi : Y \longrightarrow X$. Since $\operatorname{char}(\mathbb{C}) = 0$, Hironaka tells us that a resolution of singularities exists (and is highly non-unique). In toric geometry, computing a resolution is easy : the above lemma suggests that we should subdivide our cone σ to produce a collection of cones, and we repeat until each one has primitive cone generators

2.6 that can be extended to a $\mathbb{Z}$-basis of N.

In our case, $\exists$ natural choice of subdivision:



$$Y_0 := X_{\Sigma} \xrightarrow{\quad \tau \quad} U_\sigma \cong \mathbb{C}^3/G$$

(here, both fans contain cones of dimension three; the origin lies in a plane that is not this piece of paper).

<u>Claim</u>: In this example, $Y_0 = \mathrm{tot}(\omega_{\mathbb{P}^2})$ the total space of the canonical bundle on $\mathbb{P}^2$.

Here, $\mathrm{tot}(\omega_{\mathbb{P}^2}) = \underline{\mathrm{Spec}} \bigoplus_{j\geq 0} \omega_{\mathbb{P}^2}^{-j}$ is obtained by gluing

$$\mathrm{Spec} \bigoplus_{j\geq 0} \Gamma(U, \omega_{\mathbb{P}^2}^{-j}) \qquad \text{for } U \subseteq \mathbb{P}^2$$

providing an open cover.

<u>Fact</u>: $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1) \in \mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$

$\left(\frac{dz_0}{z_0} \wedge \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n} \overset{\text{descends to}}{\nearrow} \text{global holomorphic } n\text{-form on } \underset{\cup\!\!\mid}{\overset{\mathbb{P}^n}{}}(\mathbb{C}^\times)^n \right)$

so in our case $\omega_{\mathbb{P}^2}^{-1} \cong \mathcal{O}_{\mathbb{P}^2}(3)$. On $U_0 = \mathrm{Spec}\, \mathbb{C}[\frac{z_1}{z_0}, \frac{z_2}{z_0}]$ in $\mathbb{P}^2$, we have

$$\Gamma(U_0, \mathcal{O}_{\mathbb{P}^2}(3)) = \langle z_0^3 \rangle$$

generating as a $\underset{\mathcal{O}_{\mathbb{P}^2}(U_0)}{\underset{\shortparallel}{\Gamma(U_0, \mathcal{O}_{\mathbb{P}^2})}} = \mathbb{C}[\frac{z_1}{z_0}, \frac{z_2}{z_0}]$-module

2.7 More generally, $\Gamma(U_i, \mathcal{O}_{\mathbb{P}^2}(3j)) = \langle z_i^{3j} \rangle$ as an $\mathcal{O}_{\mathbb{P}^2}(U_i)$-module, so

$$\mathrm{tot}(\omega_{\mathbb{P}^2})|_{U_i} \cong U_i \times \mathbb{C} \cong \mathrm{Spec}\, \mathcal{O}_{U_i}[z_i^3]$$

and the gluing is: $\left(\frac{z_1}{z_0}\right)^3 \cdot z_0^3 = z_1^3$, i.e., $\mathrm{tot}(\omega_{\mathbb{P}^2})$ is covered by

$$V_0 \cong \mathrm{Spec}\, \mathbb{C}\left[\frac{z_1}{z_0}, \frac{z_2}{z_0}, z_0^3\right] \cong \mathbb{C}^3$$
$$V_1 \cong \mathrm{Spec}\, \mathbb{C}\left[z_0/z_1, z_2/z_1, z_1^3\right] \cong \mathbb{C}^3$$
$$V_2 \cong \mathrm{Spec}\, \mathbb{C}\left[z_0/z_2, z_1/z_2, z_2^3\right] \cong \mathbb{C}^3$$

Proof of the claim

We need only compute the local charts on $Y_0 = X_\Sigma$: for $N = \mathbb{Z}^3 + \mathbb{Z}\frac{1}{3}(1,1,1)$ and for

$$\sigma_0 = \mathrm{cone}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} \right)$$

the cone $\sigma_0^\vee = \mathrm{cone}(xz^{-1}, yz^{-1}, z^3)$ therefore

$$U_{\sigma_0} \cong \mathrm{Spec}\, \mathbb{C}\left[\frac{x}{z}, y/z, z^3\right]$$

and similarly for the other charts defined by cones of dim 3.
Comparing with the above, $Y_0 \cong \mathrm{tot}(\omega_{\mathbb{P}^2})$ $\square$.

Rmk.

Just as $Y_0 = X_\Sigma$ is a resolution of $\mathbb{C}^3/G$, also $\mathrm{tot}(\omega_{\mathbb{P}^2})$ can be contracted (contract the 0-section) to

$$\mathrm{Spec} \bigoplus_{j\geq 0} \Gamma(\omega_{\mathbb{P}^2}^{-j}) \cong \mathrm{Spec} \bigoplus_{j \geq 0} \mathbb{C}[x,y,z]_{3j} \cong \mathbb{C}^3/G.$$

2.8 (v) Closed toric strata : Y

To construct Y and study the embedding $\mathrm{tot}(\omega_{\mathbb{P}^2}) \hookrightarrow Y$ we study the closures of the torus orbits in a toric variety.

<u>Def</u> Let $\Sigma \subseteq N \otimes_{\mathbb{Z}} \mathbb{R}$ be a fan and $\tau \in \Sigma$. Define

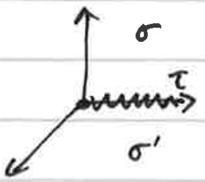
- a lattice $N(\tau) := N/(\mathrm{Span}(\tau) \cap N)$
- a fan $\mathrm{Star}(\tau) := \{ \mathrm{image}(\sigma) \subseteq N(\tau) \otimes \mathbb{R} \mid \tau \leq \sigma \in \Sigma \}$

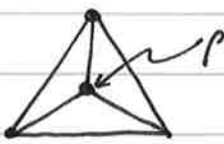
Let $V(\tau) := X_{\mathrm{Star}(\tau)}$ denote the toric variety defined by this fan; this is the (closed) toric (aka torus-invariant) stratum for τ.

<u>Rmks</u> Let $O(\tau) \simeq (\mathbb{C}^\times)^{n-\dim(\tau)}$ denote the torus orbit in X_Σ corresponding to $\tau \in \Sigma$. Then

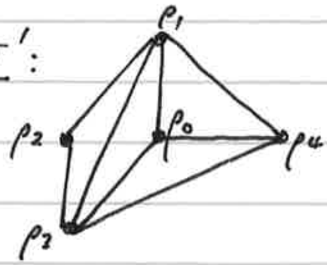
1. $V(\tau) = \overline{O(\tau)}$, so $\dim V(\tau) = n - \dim(\tau)$;
2. $V(\tau) = \bigcup_{\tau \leq \sigma} O(\sigma)$
3. Write $D_\rho := V(\rho)$ for the toric divisor (= codimension 1) in X_Σ defined by $\rho \in \Sigma(1) = \{\rho \in \Sigma \mid \dim \rho = 1\}$.

<u>Examples</u>

① $\mathbb{P}^2$  has $\mathrm{Star}(\tau)$  $\therefore D_\rho \cong \mathbb{P}^1$

② $\mathrm{tot}(\omega_{\mathbb{P}^2})$  has $\mathrm{Star}(\rho)$  $\therefore D_\rho \cong \mathbb{P}^2$

(this is the 0-section)

③ Consider Σ':  then $D_{\rho_2} \cong \mathbb{C}^2$ and

$X_{\Sigma'} = D_{\rho_2} \amalg \mathrm{tot}(\omega_{\mathbb{P}^2}) =: Y$