MORI DREAM K3 SURFACES

Today we want to answer the question:
which K3 surfaces are Mori Dream spaces?

First of all, let's recall from James' talk the definition of MDS:

A normal, $\mathbb{Q}$-factorial, projective variety X is a MDS if

i) $\mathrm{Pic}(X)_{\mathbb{R}} := \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} \cong N^1(X)$;

ii) $\mathrm{Nef}(X)$ is generated by the classes of finitely many semiample classes;

iii) $\exists$ finite collection of small $\mathbb{Q}$-factorial modifications $f_i : X \dashrightarrow X_i$ such that X_i satisfies i) and ii) above and

$$\overline{\mathrm{Mov}}(X) = \bigcup_{i=1}^{r} f_i^*(\mathrm{Nef}(X_i)).$$

Rmk: ① for a projective surface S, the point iii) in the definition is empty. One way to see it is to notice that for any small $\mathbb{Q}$-factorial modification $f : X \dashrightarrow Y$, $f^*(\overline{\mathrm{Mov}}(Y)) = \overline{\mathrm{Mov}}(X)$ and that $\overline{\mathrm{Mov}}(X) = \mathrm{Nef}(X)$ by a theorem of Boucksom-Demailly-Paun-~~Peternell~~ Peternell in 2004.

② ii) says in particular that $\mathrm{Nef}(X)$ is rational polyhedral.

We start recalling ~~from~~ some facts from the theory of K3 surfaces. In this setting the nef cone is better understood via its dual, i.e. the Mori cone (aka closure of the cone of effective curves).
We will present the result of Kovacs about the Mori cone of a projective K3 surface, which will make clear that no all K3 surfaces can be Mori Dream spaces.
Finally we will sketch the proof of the main result, i.e. a projective K3 surface X is a MDS $\Longleftrightarrow$ $\mathrm{Aut}(X)$ is finite.

§1. K3 surfaces

__Def__: a compact connected complex manifold X of dimension 2 is called __K3 surface__ if $\omega_X := \Omega^2_X \simeq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

__Rmk__: By a theorem of Siu, K3 surfaces are always Kähler.

Since we are interested in projective varieties, we suppose from now on that all our K3 surfaces are projective.

__Rmk__: Projective K3 surfaces are the algebraic ones. We decided to work over $\mathbb{C}$, but most of the results hold true for any algebraically closed field.

Since X is smooth by hypothesis, we have that

$$\mathrm{Pic}(X) \cong \mathrm{Cl}(X) \cong \mathrm{WCl}(X)$$

Cartier divisors　　　Weil divisors

Some facts:

i) $Pic(X)$ is torsion-free;

ii) From the exponential sequence (using $H^1(X, \mathcal{O}_X) = 0$) we get

$$Pic(X)_{\mathbb{R}} \cong NS(X)_{\mathbb{R}} \cong N^1(X)$$

Néron-Severi group

Rmk: ① this is exactly i) in the definition of a Mori Dream Space

② we only used that $H^1(X, \mathcal{O}_X) = 0$, so this claim holds for a much wider class of surfaces (varieties actually).

iii) For any divisor D on X, the Riemann-Roch formula gives

$$\chi(D) = 2 + \frac{1}{2} D.D$$

intersection product

In particular, if C is a (integral) curve on X and $p_a(C) = 1 - \chi(C)$ is the arithmetic genus

$$\Rightarrow \quad 2p_a(C) - 2 = C.C$$

This implies that $C.C \geq -2$ is always even and moreover $C.C = -2$ iff $C \cong \mathbb{P}^1$.

iv) Hodge diamond of X

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Examples:

1) smooth quartic surfaces in $\mathbb{P}^3$

2) smooth complete intersections of type $(d_1, \ldots, d_k)$ in $\mathbb{P}^{n+2}$ with $\sum_1^k d_k = n+3$

3) Double cover of $\mathbb{P}^2$ ramified along a smooth sextic curve

4) Kummer surfaces, i.e. smooth resolution of singularities of the quotient of an abelian surface (or complex torus) by the natural involution.

Ample & Nef cones

Recall that the positive cone $\mathcal{C}_X$ is the connected component of the cone $\{\alpha \in NS(X)_{\mathbb{R}} \mid \alpha.\alpha > 0\}$ containing the ample cone.

The Nakai-Moishezon-Kleiman theorem and fact iii) above imply then

$$\mathrm{Amp}(X) = \{\alpha \in \mathcal{C}_X \mid \alpha.C > 0 \quad \forall\, C \simeq \mathbb{P}^1 \subseteq X\}$$

and

$$\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)} = \{\alpha \in \overline{\mathcal{C}_X} \mid \alpha.C \geq 0 \quad \forall\, C \simeq \mathbb{P}^1 \subseteq X\}$$

§2. Mori cone

The cone of effective curves is

$$NE(X) = \left\{ \sum a_i [C_i] \;\middle|\; a_i \in \mathbb{R}_{\geq 0} \text{ and } C_i \subseteq X \text{ integral curve} \right\} \subseteq N_1(X)$$

Rmk: ① Since X is a surface, a divisor and a curve are the same object. Nevertheless we prefer to keep notations separate to avoid confusion

② $NE(X)$ is neither open nor closed in general.

The Mori cone is $\overline{NE}(X)$ /

By definition, $\overline{NE}(X)^{*} = \mathrm{Nef}(X)$ (dual cone)

Since $C.C \geq -2$ for any integral curve, $NE(X) \subseteq \mathcal{C}_X + \sum_{C \cong \mathbb{P}^1} \mathbb{R}_{\geq 0} \cdot [C]$

Lemma: $\overline{NE}(X) = \overline{\mathcal{C}_X + \sum_{C \cong \mathbb{P}^1} \mathbb{R}_{\geq 0} \cdot [C]}$.

Proof: if D is a divisor s.t. $D.D \geq 0$, by R-R it (or its dual) is effective. Identifying curves and divisors, this implies that

$$\mathcal{C}_X \cap NS(X) \subseteq NE(X).$$

□

As a consequence, extremal rays are generated by curves with self-intersection equal to 0 or -2.

Lemma: Let $\alpha \in \mathrm{Amp}(X)$ be a (real) ample class and $N \in \mathbb{N}$ fixed

$\Rightarrow$ there are only finitely many curves $\mathbb{P}^1 \cong C \subseteq X$ with $C.\alpha \leq N$.

Sketch of proof: if $\alpha = H$ is a (integral) ample divisor, since $C.C = -2$ $\forall C \cong \mathbb{P}^1$, fixing the intersection product $C.H = N$ is equivalent to fixing the Hilbert polynomial

$\Rightarrow$ smooth rational curves on X are classified by a (zero-dimensional) Hilbert scheme, which is projective and hence consists of a finite number of points. $\square$

Corollary: outside $\overline{\mathcal{C}}_X$, the Mori cone $\overline{NE}(X)$ is locally polyhedral.

Before stating Kovács' theorem, recall that the rank of the Néron-Severi group is called Picard rank and is denoted by ρ_X. Moreover, for a (projective) K3 surface, $1 \leq \rho_X \leq 20$ and examples of K3 surfaces with given ρ_X can be found using lattice-theoretic facts and the global Torelli theorem.

<u>Theorem</u> (Kovács, 1994)

Let X be a smooth projective K3 surface

Denote with $\mathcal{E}(X)$ the set of irreducible rational curves E s.t. $E.E=0$

Then one of the following statements holds:

i) $\rho_X = 1$ and $\overline{NE}(X) = \mathbb{R}_{\geq 0} h$ where h is ample;

ii) $\rho_X = 2$ and $\overline{NE}(X) = \mathbb{R}_{\geq 0}[E] + \mathbb{R}_{\geq 0}[C]$ where $E \in \mathcal{E}(X)$ and $C \simeq \mathbb{P}^1$;

iii) $2 \leq \rho_X \leq 4$, $\overline{NE}(X) = \overline{\mathcal{C}}_X$ and $\partial \overline{NE}(X)$ does not contain any effective class (i.e. X does not contain any smooth rational or smooth elliptic curve);

iv) $2 \leq \rho_X \leq 11$, $\overline{NE}(X) = \overline{\mathcal{C}}_X = \overline{\sum_{E \in \mathcal{E}(X)} \mathbb{R}_{\geq 0}[E]}$

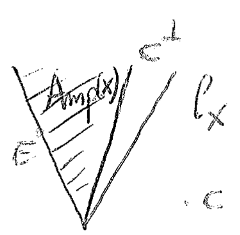
(in particular, X does not contain any smooth rational curve);

v) $2 \leq \rho_X \leq 20$ and $\overline{NE}(X) = \overline{\sum_{C \simeq \mathbb{P}^1} \mathbb{R}_{\geq 0}[C]}$.

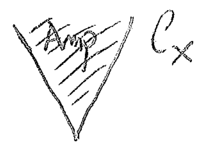
Moreover, all of these cases do occur for every value of ρ_X.

<u>Example</u>: we describe the shape of the nef cone when $\rho_X = 2$ (labels reflect the numbering in the theorem)

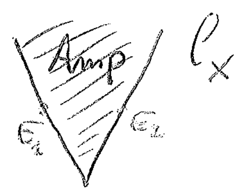
ii) $\mathrm{Amp}(X) \subsetneq \mathcal{C}_X$ and $\partial \mathrm{Nef}(X)$ is composed by $\mathbb{R}_{\geq 0}[E]$ for E smooth elliptic and by $[C]^\perp$ where C is smooth rational.



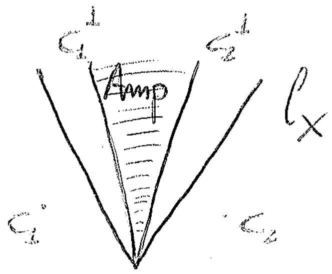
iii) $\mathrm{Amp}(X) = \mathcal{C}_X$ and $\partial \mathcal{C}_X \cap NS(X) = \{0\}$



iv) $\mathrm{Amp}(X) = \mathcal{C}_X$ and $\exists$ smooth elliptic curves E_1, E_2 such that $\partial \mathrm{Nef}(X)$ is the two rays spanned by $[E_1]$ and $[E_2]$



v) $\mathrm{Amp}(X) \subsetneq \mathcal{C}_X$ and $\exists$ smooth rational curves C_1, C_2 such that $\partial \mathrm{Nef}(X)$ is the union of the two rays spanned by the orthogonals to $[C_1]$ and $[C_2]$



<u>Rmk</u>: from the example it follows that $\mathrm{Nef}(X)$ (and hence $\overline{NE}(X)$) is polyhedral in each case and it fails to be rational polyhedral only in case iii).

<u>Recall</u>: If $\mathcal{C}$ is a cone and $U \subseteq \partial \mathcal{C}$ is open, we say that $\mathbb{R}_{\geq 0} U$ is a <u>circular part</u> of $\mathcal{C}$ if $\mathcal{C}$ is not locally finitely generated at any point of U.

We say that $\mathcal{C}$ is <u>circular</u> if $\partial \mathcal{C}$ is a circular part.

Corollary Let X be a smooth projective K3 surface

$\Rightarrow$ $\overline{NE}(X)$ is either circular or has no circular parts at all!

Rmk: from here we see that not all K3 surfaces can be MDS but that there must ~~extist~~ exist a subset of them which are.

§ 3. Main result

Theorem

A smooth projective K3 surface X is MDS $\Leftrightarrow$ $Aut(X)$ is finite.

The first remark is the following result.

Proposition

semi-ample cone

For a smooth projective K3 surface X, $SAmp(X) = Nef(X)$.

Rmk: this already implies that X is MDS iff $Nef(X)$ is rational polyhedral.

$\boxed{\text{Proof}}$ First of all note that $SAmp(X) \subseteq Nef(X)$ always.

Let hence $D \neq 0$ be a nef divisor. We have two cases:

·) $D.D > 0$ $\Rightarrow$ D is semiample by Kawamata's base-point free thm

OK.

·) $D.D = 0$

In this case, by R-R, $h^0(D) = \dim H^0(X, D) \geq 2$

$\Rightarrow$ decompose D in its fixed and moving parts

i.e. $D = F + M$

where $F \in \mathrm{Eff}(X)$, $M \in \mathrm{Nef}(X)$ $\quad h^0(F) = 1$ and $h^0(M) \cong h^0(D) \geq 2$

Now, $0 \leq D.M \leq D.(M+F) = D.D = 0 \Rightarrow D.M = 0 = D.F$

$\Rightarrow M.M = -F.M$ but $F.M \geq 0$ and $M.M \geq 0$ (because M is nef)

$\Rightarrow M.M = 0 = F.M$

$\Rightarrow F.F = 0$.

Finally, if $F \not\equiv 0 \Rightarrow$ by R-R again $h^0(F) \geq 2$ contradicting the hypothesis that $h^0(F) = 1 \Rightarrow F = 0$ and $D = M$

In particular, the rational map associated to D could be non regular in dimension 0, i.e. $|D|$ could have only isolated fixed points.

On the other hand, if $|M|$ has isolated fixed points $\Rightarrow M.M > 0$

$\Rightarrow D = M$ is semi-ample. $\square$

The final result is the following corollary of the cone ~~theorem~~ conjecture which was proved by Kawamata for Calabi-Yau manifolds in 1997}

<u>Proposition</u>

Let X be a smooth and projective K3 surface, then

$$\mathrm{Nef}(X) \text{ is rational polyhedral} \iff \mathrm{Aut}(X) \text{ is finite.}$$

$\boxed{\text{Proof}}$ This is only a sketch of proof.

Define the effective nef cone $\mathrm{Nef}^e(X)$ as the real convex hull of $\mathrm{Nef}(X) \cap NS(X)$.

Then $\mathrm{Nef}(X)$ is rational polyhedral iff $\mathrm{Nef}^e(X)$ is rational polyhedral.

The cone ~~theorem~~ conjecture ~~states~~ claims that there exists a rational polyhedral subcone $\mathcal{C} \subseteq \mathrm{Nef}^e(X)$ such that

$$\mathrm{Nef}^e(X) = \bigcup_{g \in \mathrm{Aut}(X)} g^* \mathcal{C} \quad \text{and} \quad g^*\mathcal{C} \cap h^*\mathcal{C} \text{ has no inner points } \forall\, g \neq h \in \mathrm{Aut}(X)$$

(such a subcone is called a rational polyhedral fundamental domain).
Since this conjecture holds for K3 surfaces, the thesis follows. □

The proof of the theorem follows.

<u>Example</u> if $\rho_X = 2$, we can see that X is a MDS iff there exists a class $\alpha \in NS(X)$ s.t. either $\alpha.\alpha = 0$ or $\alpha.\alpha = -2$.