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1.1 # Mori Dream Spaces

Goal: to understand why every smooth Fano variety X is a Mori Dream Space; here, 'Fano' means that the sheaf $\omega_X^{-1} := (\Lambda^n \Omega_X^1)^{-1}$ is ample, i.e., $\exists\ j>0$ and a closed immersion $X \hookrightarrow \mathbb{P}(V)$ for $V = \Gamma(\omega_X^{-j})$.

Why might one care about Mori Dream Spaces?

Any Mori Dream Space (MDS) X has many nice properties, for example:

1. the closed cones
$$\mathrm{Nef}(X) \subseteq \overline{\mathrm{Mov}(X)} \subseteq \overline{\mathrm{Eff}(X)}$$
in the vector space $N^1(X)$ of real Cartier divisors up to numerical equivalence are all rational polyhedral cones, i.e., they're determined by finitely many supporting hyperplanes.
2. every nef divisor class is semi-ample: given $L \in \mathrm{Nef}(X)$ $\exists\ j>0$ such that $L^{\otimes j}$ is globally generated, i.e., $\exists$ morphism $X \longrightarrow \mathbb{P}(V)$ for $V = \Gamma(L^j)$.

More generally, and more succinctly, X behaves well with respect to the Mori programme in birational geometry.

The original definition of a MDS due to Hu-Keel in 2000 was phrased in birational geometric terms. For an alternative definition - let's call it a working definition for now - let X be a smooth projective variety such that the group $\mathrm{Pic}(X)$ is finitely generated and free. The vector space
$$\mathrm{Cox}(X) := \bigoplus_{L \in \mathrm{Pic}(X)} \Gamma(X, L)$$

1.2 can be made into a ring: every $L \in \mathrm{Pic}(X)$ is isomorphic (but not canonically!) to a sheaf of the form $\mathcal{O}_X(D)$ for some (Cartier) divisor D on X; these sheaves are subsheaves of the sheaf of rational functions on X, and therefore we can multiply

$$\begin{array}{ccc} \Gamma(X, \mathcal{O}_X(D)) \times \Gamma(X, \mathcal{O}_X(D')) & \longrightarrow & \Gamma(X, \mathcal{O}_X(D+D')) \\ (s, s') & \longmapsto & s \cdot s' \end{array}$$

using the multiplication in the sheaf of rational functions. To make this work, therefore, we choose a $\mathbb{Z}$-basis of $\mathrm{Pic}(X)$ of the form $\mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_\rho)$ and then

$$\mathrm{Cox}(X) = \bigoplus_{(m_1, \ldots, m_\rho) \in \mathbb{Z}^\rho} \Gamma(\mathcal{O}_X(m_1 D_1 + \ldots + m_\rho D_\rho))$$

becomes a ring. In particular it's a $\mathbb{k}$-algebra.

<u>Working Defn</u> [this is a special case of theorem of Hu Keel] A smooth projective variety X with a finitely generated and free Picard group is a MDS $\iff$ $\mathrm{Cox}(X)$ is finitely generated as a $\mathbb{k}$-algebra.

<u>Rmks</u>

① In general X need not be smooth. Birational geometry of varieties in dimension ≥ 3 forces one to work with mildly singular varieties even if one only cares about smooth varieties.

② In fact one has to assume that $N^1(X)$ is isomorphic to $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, i.e., the space of Cartier divisors up to numerical equivalence should be isomorphic to the

1.3 Space of Cartier divisors up to linear equivalence.
(We'll review these notions in the weeks ahead.)

Using the working definition, we can work towards a geometric construction of any MDS.

Choose a surjective $\mathbb{k}$-algebra homomorphism

$$\varphi : \mathbb{k}[x_1, .., x_n] \twoheadrightarrow \mathrm{Cox}(X)$$

Note that $\mathrm{Cox}(X)$ is a $\mathbb{Z}^\rho$-graded ring (here, note that $\mathbb{Z}^\rho = \mathrm{Pic}(X)$) and we define a $\mathbb{Z}^\rho$-grading on $\mathbb{k}[x_1, .., x_n]$ so that φ is a homomorphism of graded rings, ie $f \in \mathbb{k}[x_1, .., x_n]$ has $\mathbb{Z}^\rho$-degree equal to the $\mathbb{Z}^\rho$-degree of $\varphi(f) \in \mathrm{Cox}(X)$.

Having a $\mathbb{Z}^\rho$-graded ring R is equivalent to specifying an action of the algebraic torus $(\mathbb{k}^\times)^\rho = \mathrm{Hom}(\mathbb{Z}^\rho, \mathbb{k}^\times)$ on the affine scheme $\mathrm{Spec}\, R$. More generally, applying the functor $\mathrm{Spec}(-)$ to the graded ring homomorphism φ induces a closed immersion

$$\mathrm{Spec}\, \mathrm{Cox}(X) \hookrightarrow \mathbb{A}^n := \mathrm{Spec}\, \mathbb{k}[x_1, .., x_n] \qquad (*)$$

[with image $\mathbb{V}(\ker(\varphi)) = \{ p \in \mathbb{A}^n \mid f(p) = 0 \ \forall f \in \ker(\varphi) \}$] together with a compatible action of $\mathrm{Hom}(\mathbb{Z}^\rho, \mathbb{k}^\times)$.

We now take Geometric Invariant Theory (GIT) quotients: for this, any character of the acting group leads to a GIT quotient of both spaces in $(*)$. Here, the character group of the torus is $\mathbb{Z}^\rho = \mathrm{Pic}(X)$ and we choose an ample line bundle $L \in \mathrm{Pic}(X)$, 1.4 giving rise to a closed immersion

$$\operatorname{Spec} \operatorname{Cox}(X) /\!\!/_L (\Bbbk^*)^\rho \hookrightarrow \mathbb{A}^n /\!\!/_L (\Bbbk^*)^\rho \qquad (**)$$

We'll say more about GIT later, but for now we note that the GIT quotient on the left is <u>defined to be</u>

$$\operatorname{Spec} \operatorname{Cox}(X) /\!\!/_L (\Bbbk^*)^\rho = \operatorname{Proj}\Big(\bigoplus_{j\geq 0} \Gamma(X, L^{\otimes j}) \Big)$$
$$\cong X \qquad (***)$$

where the isomorphism here follows from the fact that L is ample on X.

A crucial point for us is that the ambient space in $(**)$ is a toric variety; indeed, just as $\mathbb{A}^n$ has a natural action of $(\Bbbk^*)^n$, the GIT quotient $\mathbb{A}^n /\!\!/_L (\Bbbk^*)^\rho$ has an action of $(\Bbbk^*)^{n-\rho}$. Moreover, the Picard group (in fact the Class group as this toric variety need not be smooth) is isomorphic to $\mathbb{Z}^\rho$. We'll see that the birational geometry of the MDS behaves nicely precisely because it inherits nice birational-geometric properties from an ambient toric variety as in $(**)$.

<u>Rmks</u>

① Projective toric varieties are MDS's – just take φ equal to the identity.

② A MDS does not inherit the birational geometry properties from <u>any</u> toric variety (otherwise an ambient $\mathbb{P}^n$ would do – very dull birational geometry), only those as in $(**)$.

1.5 ③ To put a bit more flesh on the phrase "inherit birational geometry", note that

$$\mathrm{Nef}(X) \subseteq N^1(X) = \mathbb{R}\langle [D] \mid D \text{ Cartier divisor} \rangle /_{\equiv_{num}}$$

is a closed, full-dimensional cone whose interior is the ample cone of X, denoted $\mathrm{Amp}(X)$. Each $L \in \mathrm{Amp}(X)$ encodes X via the isom $(\ast\ast\ast)$, whereas moving L beyond the ample cone induces a rational map

$$\underset{\text{here } L \in \mathrm{Amp}(X)}{X = \mathrm{Proj}\left(\bigoplus_{j\geq 0} \Gamma(L^{\otimes j})\right)} \dashrightarrow \underset{\text{here } L \in \mathrm{Eff}(X)}{\mathrm{Proj}\left(\bigoplus_{j\geq 0} \Gamma(L'^{\otimes j})\right) = X'}$$

We'll see that for a MDS X, these rational maps are induced by rational maps of the ambient toric varieties:

$$\begin{tikzcd} \mathbb{A}^n /\!\!/_L (\mathbb{k}^\times)^\rho \arrow[r, dashed] & \mathbb{A}^n /\!\!/_{L'} (\mathbb{k}^\times)^\rho \\ X \arrow[u, hook] \arrow[r, dashed] & X' \arrow[u, hook] \end{tikzcd}$$

In summary, the key properties of a MDS – namely the rational polyhedral cones in $N^1(X)$, the fact that nef divisor classes (= line bundles) are semi-ample, and more generally, the nice birational-geometric properties of X – are inherited from a particular type of ambient toric varieties. We'll start by learning about toric varieties.

2.1 Mori Dream Spaces

Introduction to toric varieties

We'll introduce toric varieties by using the following diagram to highlight different topics:

$$\begin{array}{ccccc}
\mathbb{F}_1 & & \xrightarrow{\text{blow-up}} & & \mathbb{P}^2 \\
\text{closed immersion} \downarrow & & & & \downarrow \text{closed immersion} \\
\mathrm{tot}(\omega_{\mathbb{F}_1}) & \overset{\text{flop}}{\dashleftarrow\dashrightarrow} & Y & \underset{\text{open immersion}}{\hookrightarrow} & \mathrm{tot}(\omega_{\mathbb{P}^2}) \\
& \text{resolution} \searrow & & \swarrow \text{resolution} & \downarrow \text{resolution} \\
& & X & & \mathbb{C}^3/G
\end{array}$$

smooth proj. surfaces

smooth quasi-proj threefolds

singular affine threefolds

(i) The orbit-decomposition : $\mathbb{P}^2$

$$\mathbb{P}^2 = \bigsqcup_{0 \leq i \leq 2} U_i / \sim \quad \text{where} \quad U_i = \{ [z_0 : z_1 : z_2] \in \mathbb{P}^2 \mid z_i \neq 0 \}$$

The algebraic torus $(\Bbbk^\times)^2 = \{ [z_0 : z_1 : z_2] \in \mathbb{P}^2 \mid z_0 z_1 z_2 \neq 0 \}$ in U_i acts on $\mathbb{P}^2$ as

$$(\lambda_0 : \lambda_1 : \lambda_2) \cdot [z_0 : z_1 : z_2] = [\lambda_0 z_0 : \lambda_1 z_1 : \lambda_2 z_2],$$

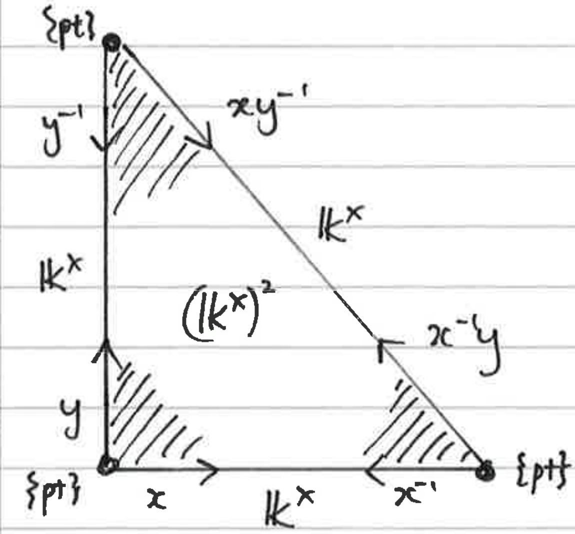
i.e., the action of $(\Bbbk^\times)^2$ on itself extends to an action of $(\Bbbk^\times)^2$ on $\mathbb{P}^2$.

Defn A (normal) toric variety is a variety containing $(\Bbbk^\times)^d$ as an open subvariety such that the action of

$(\Bbbk^\times)^d$ on itself extends to an action on the variety.

For $\mathbb{P}^2$, there exist seven $(\Bbbk^\times)^2$-orbits determined by vanishing of the coordinates z_0, z_1, z_2. To draw this decomposition, write $U_i = \operatorname{Spec} \mathbb{C}[t_1, t_2]$:

$$\mathbb{C}[U_0] = \mathbb{C}\left[\frac{z_1}{z_0}, \frac{z_2}{z_0}\right] \qquad x := z_1/z_0 \quad y := z_2/z_0$$
$$\mathbb{C}[U_1] = \mathbb{C}\left[z_0/z_1, z_2/z_1\right] \qquad x^{-1}, \quad x^{-1}y$$
$$\mathbb{C}[U_2] = \mathbb{C}\left[z_0/z_2, z_1/z_2\right] \qquad y^{-1}, \quad xy^{-1}$$



and ~~record~~ this polytope has inner normal fan