Sunpaths revisited

Introduction

The design of shading devices is influenced by a number of factors. These may range from the need to limit solar gain entering the building to the desire for a particular form of shade in order that façade takes on a particular appearance. Primarily, these notes will consider those factors that relate to the efficacy of shades in reducing or eliminating solar gain.

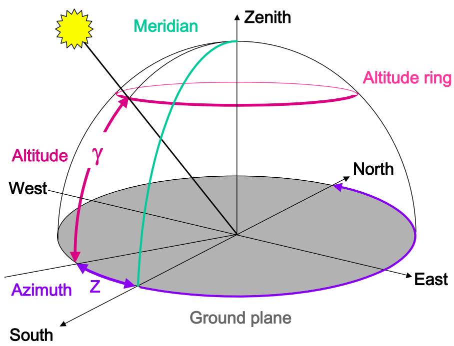

Figure 8.1 – Sun within sky vault

Sun position

The efficacy of various types of shades will depend upon the relative position of the sun to the façade. In order to establish this it is necessary first to determine whereabouts in the sky is the sun.

The position of the sun in the sky is given by two angles that are shown in Figure 8.1,

γ - the altitude of the sun above the ground or horizon plane
z – the compass direction of the sun on the ground plane.

The Azimuth may be measured either East or West of South, or clockwise from North. In general, the Azimuth is most often given in terms of the clockwise angle from North, but in these notes the angle will be given as an angle from East or West of South.

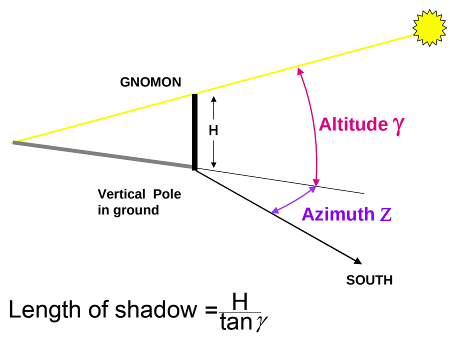

Figure 8.2 – Shadow cast by gnomon

Figure 8.2 shows a shadow cast by a vertical pole and how the length and position of the shadow are affected by the Altitude and Azimuth of the sun.

Where variables are defined as in Table 8.1, the sun's position may be determined from the following equations:

$$\sin\gamma = \cos D \times \cos L \times \cos H + \sin D \times \sin L,$$

$$\cos z = \frac{\cos D \times \cos H \times \sin L - \sin D \times \cos L}{\cos\gamma},$$

$$\sin z = \frac{\sin H \times \cos D}{\cos\gamma},$$

$$\tan z = \frac{\sin H}{\sin L \times \cos H - \cos L \times \tan D}.$$

These equations are derived in Appendix 8.

Symbol	Variable	Definition
D	Declination	The angle of the sun's rays to the equatorial plane, positive in summer.
L	Latitude	The angle from the equator to a position on Earth's surface.
H	Hour angle	The angle the Earth needs to rotate to bring the meridian to noon. Each hour of time is equivalent to 15 deg.
N	Day Number	The day number, January 1st is 1.

Table 8.1 – Definitions of variables

Application of the above formulae are not necessarily the best way to appreciate the various effects of the sun's position and a more graphical approach may usefully be adopted for the design of shading devices. However, once a design of shade has been developed, the above formulae may be usefully used to confirm dimensions.

Sunpath diagrams

There are many different ways of graphically displaying the relative position of the sun at different times of the day and year. These range from,

i) Sun dials, these are gnomic projections,
ii) Rectangular projections of the sky,
iii) Circular sunpath diagrams.

There is no one method that is preferable to all others, and each has its own advantages and disadvantages. For hand sketching diagrams, I believe that the Stereographic diagram has distinct advantages. The diagram allows the whole sky to be considered and this means there is no need to construct a new diagram for each different orientation of the façade. Also, all the sunpaths and time lines are arcs of circles and because of this they are reasonably easy to sketch, even the non artistic.

However its advantages are less pronounced with the advent of the computer, as now calculations and redrawing can be undertaken without effort. Never the less, it still is a most useful way of considering the sun's position. In computer applications, it does lend itself to serious design because it allows the consideration of a number of variables at one time, and this is not always possible with some of the other techniques of displaying sunpaths.

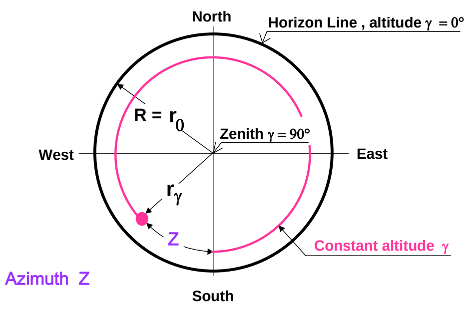

Figure 8.3 – Circular projections

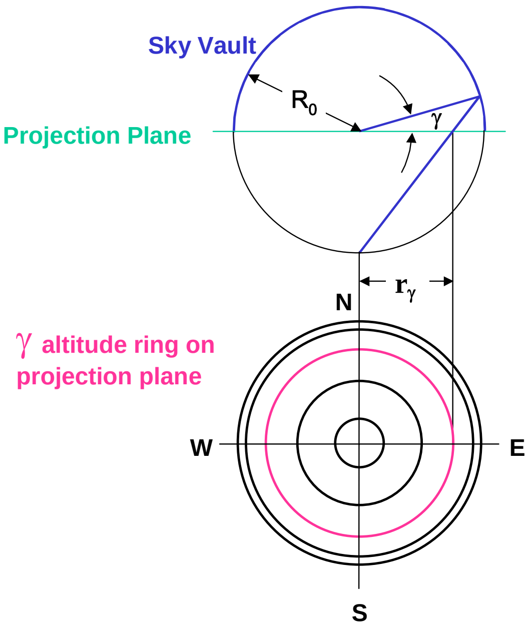

Figure 8.4 – Stereographic projection

The Stereographic projection

The basis of the circular projections is that a hemisphere of sky is projected down onto a horizontal plane.

This results in a diagram of the form shown in Figure 8.3, where the points of the compass are defined by the *direction* out from the centre of the diagram, and the altitude is defined by the *distance* out from the centre.

The construction of the Stereographic projection is shown in Figure 8.4 and results in the relation between radius and altitude,

$$r_\gamma = R_0 \tan\left(\frac{90^\circ - \gamma}{2}\right)$$

Sun's position in the sky

Sunpath loci are arcs of circles on the stereographic projection and the arc of a circle can be sketched quite easily if three positions on the arc are identified.

Three positions in the sky are used to sketch a Sunpath for a given day:

i) azimuth of the sun rise,
ii) altitude of the sun at solar noon,
iii) azimuth of the sun set.

The course of the sun across the sky during the day is caused by the Earth's rotation about its own axis, and as the declination changes only but a little during the course of a day, the sun rise and the sun set may be assumed to be symmetrically located on either side of south.

When *sketching* a sunpath for a particular location, it is really only necessary to consider four times in the year,

i) Winter Solstice,
ii) Summer Solstice,
iii) Vernal and Autumnal Equinoxes.

The two Equinoxes have the same sunpath loci and therefore it is rarely the case that more than three sun path loci need be plotted on the sun path diagram. Re-capping from earlier notes, the sun's altitude at noon can be seen from Figure 8.5 and will be given by,

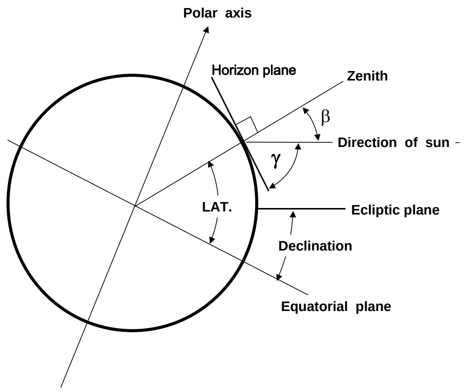

Figure 8.5 – Section of Earth at Noon

$$\gamma_{Noon} = 90° - L + D.$$

During the summer months, for locations above the arctic circle the sun will not set, and therefore the sun path locus will be a circle on a stereographic projection. All that is needed to draw a circle is its diameter. One end of the diameter will be given by the position of the sun a noon, and the other by the sun's position at midnight. If the altitude is measured from the southern direction it will be seen from Figure 8.6 that,

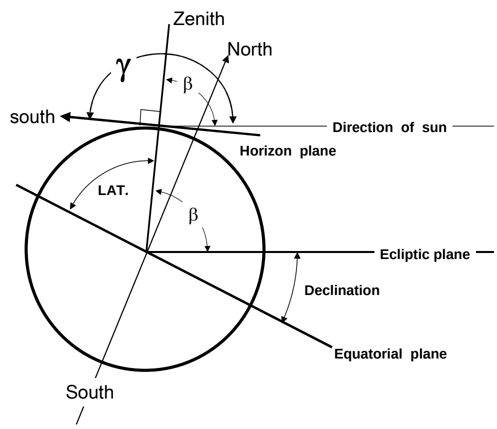

Figure 8.6 – Section of Earth at midnight

$$\gamma_{Midnight} = 270° - L - D$$

and if the altitude is measured from the northern direction it will be given by,

$$\gamma_{Midnight} = 180° - (270° - L - D)$$
$$= L + D - 90°$$

The azimuth of the sun at sunrise and sunset may be found using the relationship given below,

$$\cos z_{\gamma=0} = \frac{-\sin D}{\cos L} .$$

It is worthwhile noting that, where the sun does not cross the horizon, the above relation does not hold. Therefore, there will be no solution when the latitude is >66° for either the summer, or the winter solstice.

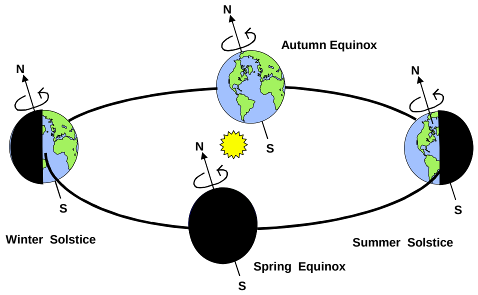

Figure 8.7 – Earth orbiting sun

Declination of the sun

The declination is one of the variables used in defining the sun's position. In order to appreciate the declination it is necessary to appreciate some aspects of the Earth's astronomical relation to the Sun.

The astronomical arrangement assumed is that shown in Figure 8.7. The Earth orbits the Sun once every year and the Earth rotates about its own axis about once every 24 hours. This axis of rotation is tilted from the ecliptic plane by an angle of 23.4°.

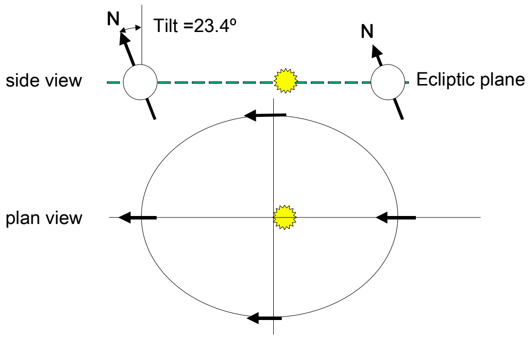

Figure 8.8 – Section of ecliptic

The tilt of the Earth's axis of rotation can be assumed to remain constant and its direction in space also stays the same. This is shown in Figure 8.8 , where it can be seen that the direction of tilt lies approximately in the direction of the major axis of the Earth's orbit.

It is the constant direction of the Earth's tilt that causes the sun's relative position to change with respect to the Earth's equator, as can be seen in Figure 8.7. It gives rise to the different seasons and results in the angle between the sun's rays and the equator changing. This angle is the declination and is shown diagrammatically in Figure 8.9.

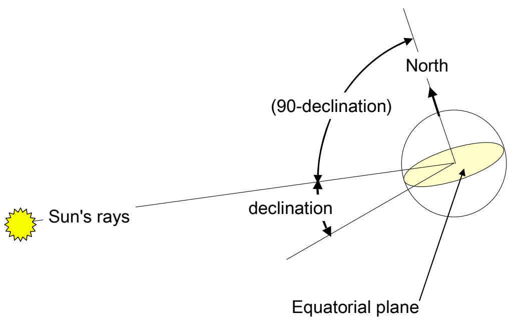

Figure 8.9 - Declination

There are 4 times in the year when the declination takes particular values that are especially significant and these are:

Astronomical occurance	D	Calendar date
Winter Solstice	-23.4	23$^{rd.}$ December
Vernal (Spring) Equinox	0	21$^{st.}$ March
Summer Solstice	+23.4	21$^{rt.}$ June
Autumnal Equinox	0	23$^{rd.}$ September

The Winter Solstice is that time of year when the declination is a minimum. Because the direction of Tilt is not exactly in line with major axis, it is not the case that this coincides with the shortest day of the year.

The Summer Solstice is that time of year when the declination is a maximum. Similarly to the other solstice, the longest day does not necessarily coincide with the summer solstice.

The Equinoxes are those times of year when the day and night are of equal time. Thus the sun will rise and set at 6am and 6pm respectively.

At other times in the year the Declination may be evaluated by the approximate equation:

$$Declination = 23.4 \times \sin\left(\frac{360 \times (284 + N)}{365}\right) \quad \text{degrees}$$

Where N is the day number of the date for which the declination is being calculated. January 1st. being day number 1.

A more accurate formulation for declination is given in Table 8.2, and this may be used for computer generated diagrams.

$$\text{Day angle} = \tau_d = \frac{2\pi(N-1)}{365} \quad \text{radians}$$

$$\text{Solar Declination} = \delta_s$$

$$\begin{aligned}\delta_s = {} & 0.006918 - 0.399912\cos\tau_d + 0.070257\sin\tau_d \\ & - 0.006758\cos 2\tau_d + 0.000907\sin 2\tau_d \\ & - 0.002697\cos 3\tau_d + 0.001480\sin 3\tau_d \quad \text{radians}\end{aligned}$$

Table 8.2 – Formula for Declination

Sketching a Sunpath

Sketching sunpaths for the City of Bath which is at a Latitude of 51·3° North.

Noting that at solar noon the sun is due South and at the maximum altitude given by the relation:

$$\gamma_{Noon} = 90° - L + D.$$

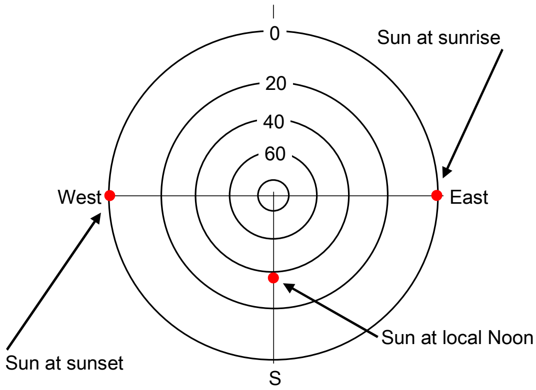

Figure 8.10 – Noon, sunrise, sunset

For the equinoxes, March 21st. and September 23rd., when the declination is 0°.

$$\begin{aligned}\gamma_{Noon} &= 90° - 51.3 \\ &= 38.7°\end{aligned}$$

Noting also that at the equinoxes, the sun rises due East and sets due West, there are three known positions on the Sunpath locus for the equinox, and these are shown on Figure 8.10. These may be used to sketch the first of the Sunpath loci, as is shown in Figure 8.11.

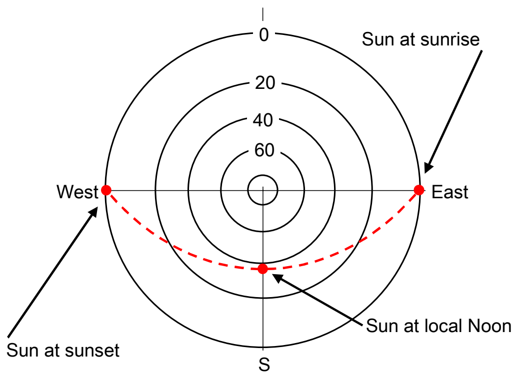

Figure 8.11 – Arc of circle

For the summer solstice, June 21st., when the declination is 23·4°,

$$\begin{aligned}\gamma_{Noon} &= 38.7° + 23.4 \\ &= 62.1°\end{aligned}$$

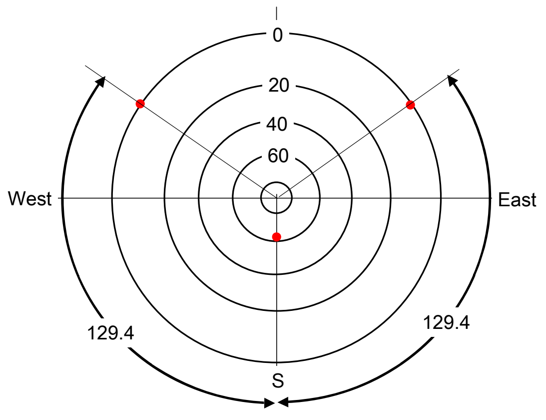

Figure 8.12 – Sun at Summer Solstice

Using the relation for the azimuth when the sun rises and sets,

$$\cos z_{\gamma=0} = \frac{-\sin D}{\cos L}$$

The declination is 23.4 and therefore,

$$\cos z_{\gamma=0} = \frac{-\sin 23.4}{\cos 51.3} = -\frac{0.397}{0.625} = -0.635$$

$$z_{\gamma=0} = \cos^{-1}(-0.635) = 129.4°$$

Therefore the azimuth of sunrise and sunset are respectively129·4° East and West of South, as is shown in Figure 8.12.

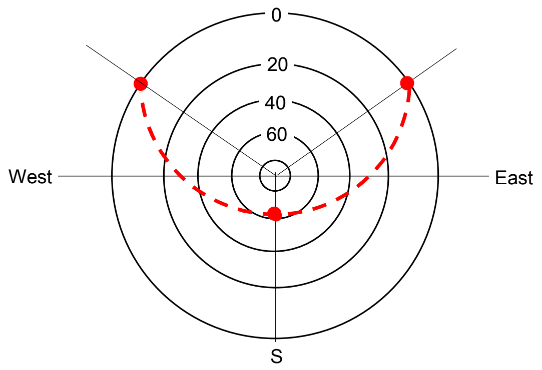

Figure 8.13 – Summer Solstice

There are then three positions of the sun that can be connected together by an arc of a circle that denotes its Sunpath, as shown in Figure 8.13.

For the winter solstice, December 23rd., when declination is –23·4°,

$$\gamma_{Noon} = 38.7° - 23.4$$
$$= 15.3°$$

The declination is –23·4° and therefore using the relation for azimuth at sunrise and sunset,

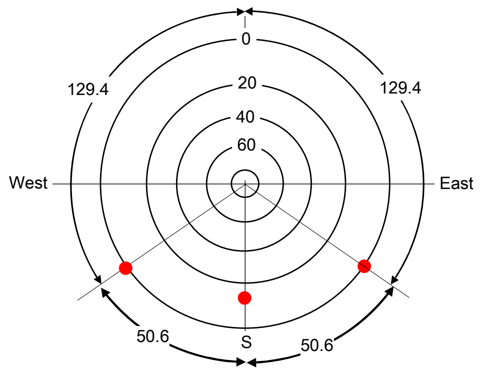

Figure 8.14 – Sun at Winter

$$\cos z_{\gamma=0} = \frac{-\sin(-23.4)}{\cos 51.3} = \frac{+\sin 23.4}{\cos 51.3} = \frac{0.397}{0.625} = 0.635$$

$$z_{\gamma=0} = \cos^{-1}(0.635) = 50.6°$$

It is worthwhile noting at this point that,

$$\cos(180° - z) = \cos 180° \cos z + \sin 180° \sin z$$

$$\cos(180° - z) = -\cos z$$

and this is confirmed by the observation that,

129·4°+50·6°=180°.

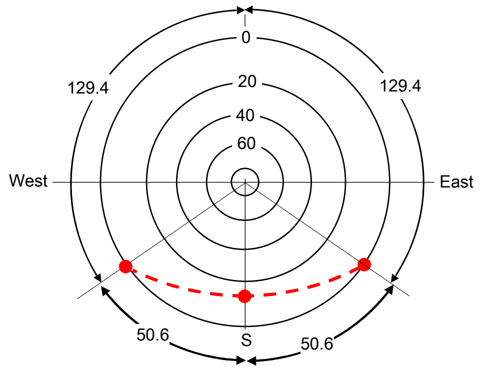

Figure 8.15 – Winter Solstice

Therefore, there is no need to go through the calculation of the azimuth twice. It is simpler to use the fact that the sunrise and sunset for the two Solstices are symmetrically positioned about the East-West axis. This is shown diagrammatically in Figure 8.14 where the known sun positions for the Winter Solstice are plotted. These are connected together in Figure 8.15.

The three Sunpath loci can then be collected together on the same diagram to give the range of sun positions throughout the year, as is shown in Figure 8.16.

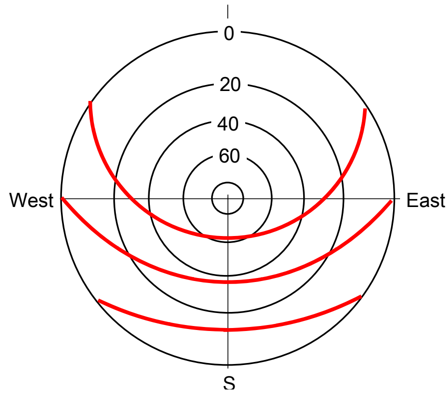

Figure 8.16 – Range of sunpaths

Sketching solar time lines

The hour lines on the stereographic projection are also arcs of circles. These hour lines always cross the sunpaths at 90° and this helps in constructing them.

The easiest hour line is that of noon, which is a straight line towards the South. At the Equinoxes the 6am and 6pm hour lines pass through the horizon line due East and due West respectively as shown in Figure 8.17.

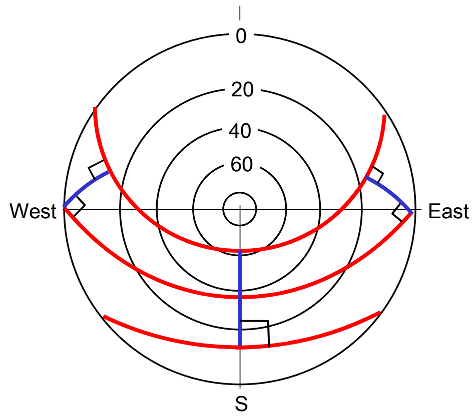

Figure 8.17 – Construction of hour

Although not exactly correct, for the purposes of sketching, the intermediate hour lines may be positioned on the basis of spacing them equally between the noon and 6 ' clock hour lines, as in Figure 8.18.

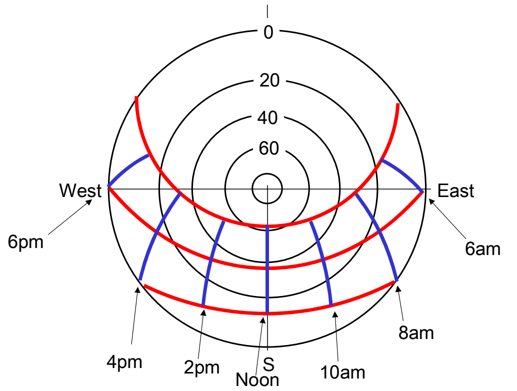

Figure 8.18 – Sunpath diagram for Bath

For Latitudes above the Arctic circle

At latitudes greater than 66·6°N, the sun will not rise above the horizon at the Winter Solstice. Therefore only two Sunpath loci are required to show the extreme ranges of the sun's position in the sky, that at the Equinoxes and that of the Summer Solstice.

At the Summer Solstice the sun will be above the horizon for the whole day and therefore its Sunpath is sketched using the position of the sun at noon and midnight. Considering the Figure 8.6 used to obtain the relation for the altitude of the sun at midnight, it should be noted that the sun will appear to be due North.

Considering the Sunpath for a Latitude of 70° North:

At the Equinoxes the Declination is zero and the maximum altitude at noon will be,

$$\gamma_{Noon} = 90° - 70° = 20° .$$

The sunrise and sunset are respectively due East and West.

At the Summer Solstice the Declination is +23.4° and therefore the maximum altitude at noon will be,

$$\gamma_{Noon} = 90° - 70° + 23.4° = 20° + 23.4° = 43.4° .$$

And the minimum altitude of the sun at midnight will be,

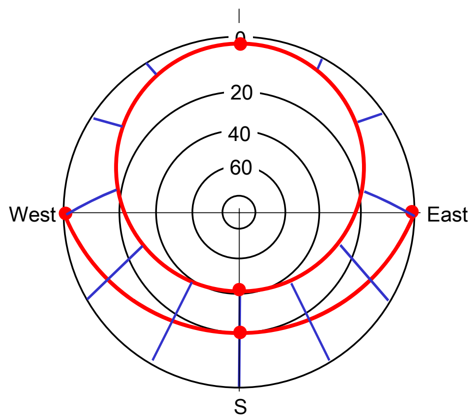

Figure 8.19 – Sunpaths for 70° North

$$\gamma_{Midnight} = 270° - L - D = 270 - 70 - 23.4 = 176.6°.$$

measured from South, and

$$180 - 176.6 = 3.4°$$

measured from North.

These are then used to plot the sunpaths as is shown in Figure 8.19 .

For Latitudes within the Tropics

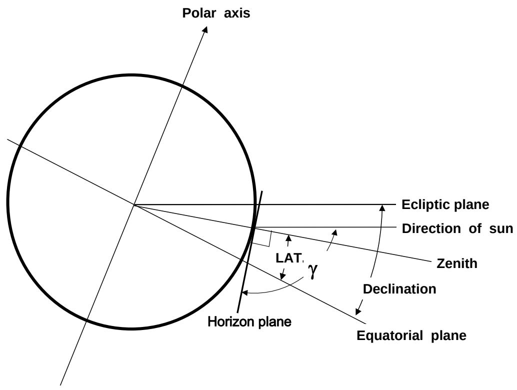

Figure 8.20 – Section of Earth at noon

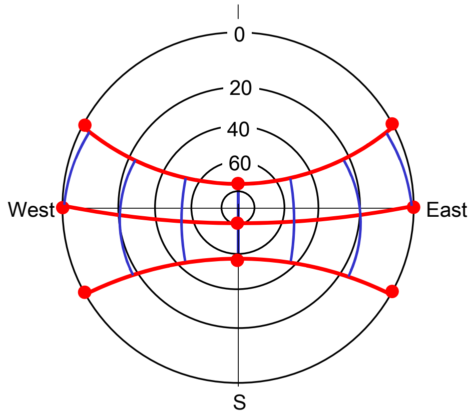

Figure 8.21 – Sunpaths for 10°

For Latitudes within the tropics the sun will pass overhead through the zenith at some time of year. Therefore, it is important to realise that at the summer solstice the sun may be to the North in northern latitudes and to the South in southern latitudes. This should be apparent from the cross section through the earth at noon shown in Figure 8.20.

As an example, sketching the sunpaths for the Latitude of 10° N:

At the Equinoxes, the Declination = 0°, and the max altitude is,

$$\gamma_{Noon} = 90° - L + D = 90 - 10 + 0 = 80°$$

At the Summer solstice, the declination is 23.4, and the altitude is,

$$\gamma_{Noon} = 90° - L + D = 90 - 10 + 23.4 = 103.4° \text{ to the south}$$
$$\equiv 180 - 103.4 = 76.6° \text{ to the North}$$

The azimuth at sunrise and sunset is,

$$\cos z = \frac{-\sin D}{\cos L} = \frac{-\sin 23.4}{\cos 10} = \frac{-0.397}{0.985} = -0.403$$
$$z_{\gamma=0} = \cos^{-1}(-0.403) = 113.8°$$

At The Winter Solstice, the declination is –23.4°, and the altitude is,

$$\gamma_{Noon} = 90° - L + D = 90 - 10 - 23.4 = 56.6° \text{ to the south}$$

and the azimuth of sunrise and sunset will be given by,

$Z_{\gamma=0}$ = 180° - 113.8° = 66.2°.

The whole Sunpath diagram for a latitude of 10° N is sketched in Figure 8.21.

For Southern Latitudes

In southern latitudes the sun will primarily be due north at solar noon. The sun will still rise in the East and set in the West. If there is doubt in your mind about where the sun is, then consider again a diagram of the cross section through the ecliptic plane as in Figure 8.8.

If a convention is adopted that the altitude of the sun at noon and midnight is always measured from the south, then by convention:

Northern latitudes are positive and Southern Latitudes are negative.

From Figure 8.22, it will be seen that the altitude of the sun at noon will be given by,

$$\gamma_{Noon} = 90 + \beta = 90 + (|L| + D)$$

but as the Latitude is to the South, by convention L is negative, and so,

$$\gamma_{Noon} = 90 + (-(-L) + D)$$
$$\gamma_{Noon} = 90 - L + D$$

The three latitudes previously considered for northern latitudes are reconsidered here as being southern latitudes, and the altitudes and azimuths needed for sketching the diagrams are listed in the table. The diagrams for southern latitudes are shown in the margin in Figures 8.23-8.25

	Equinox		June 21st.			December 23rd.		
Latitude	Altitude At Noon	Az. At Sunrise	Altitude At Noon	Altitude At midnight	Az. At Sunrise	Altitude At Noon	Altitude At midnight	Az. At Sunrise
- 51.3	141.3 S Or 38.7 N	90 East	164.7 S Or 15.3 N	NA	129.4	117.9 S Or 62.1 N	NA	50.6
- 70	160 S Or 20 N	90 East	183.4 S Or -3.4 N No sun	NA	NA	136.6 S Or 43.4 N	3.4 S Or 176.6 N	NA
- 10	100 S Or 80 N	90 East	123.4 S Or 56.6 N	NA	113.8	76.6 S Or 103.4 N	NA	66.2

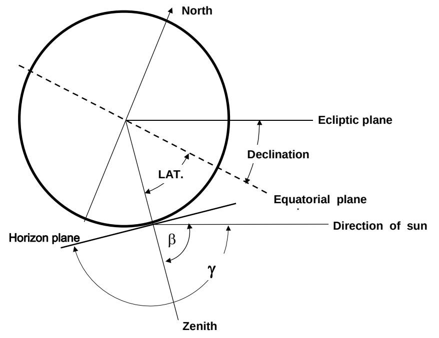

Figure 8.22 – Section of Earth at noon

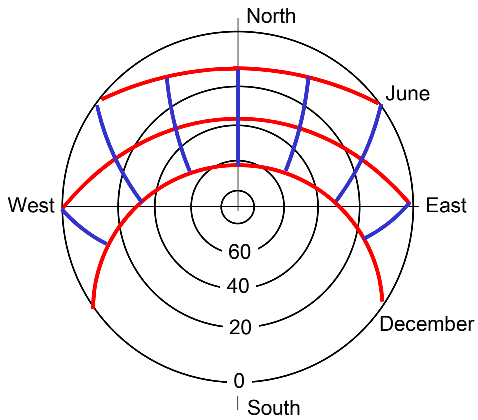

Figure 8.23 – Sunpaths for 51.3° South

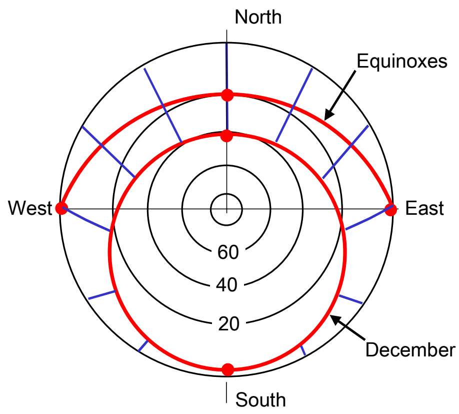

Figure 8.24 – Sunpaths for 70° South

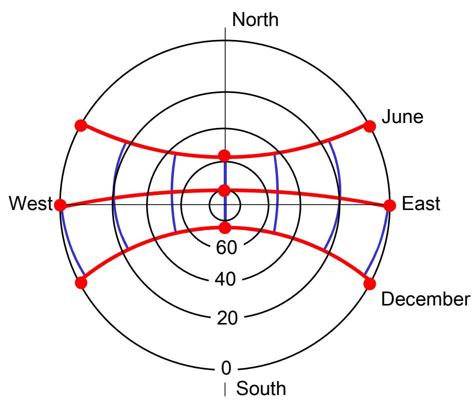

Figure 8.25 – Sunpaths for 10° South