Theory of large rotations

Rotations in three dimensions can be described in a number of different, but equivalent ways, for example the Euler angles, http://en.wikipedia.org/wiki/Euler_angles. Here we will use a vector representing the axis of rotation.

Consider a rotation is of magnitude α radians about the axis defined by the vector $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ in which $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are the Cartesian orthogonal unit base vectors. We will see later that it makes sense to make the magnitude of $\mathbf{a}$ equal to $\tan\frac{\alpha}{2}$. A positive rotation is clockwise looking along $\mathbf{a}$. A negative rotation is anticlockwise, or clockwise looking in the opposite direction.

This definition means that the rotation causes the vector $\mathbf{u}$ to become the vector

$$\mathbf{u}' = \frac{(\mathbf{u} \bullet \mathbf{a})\mathbf{a}}{a^2} + \left(\mathbf{u} - \frac{(\mathbf{u} \bullet \mathbf{a})\mathbf{a}}{a^2}\right)\cos\alpha + \frac{\mathbf{a} \times \mathbf{u}}{a}\sin\alpha$$
$$= \mathbf{u} + \frac{(\mathbf{u} \bullet \mathbf{a})\mathbf{a}}{a^2} + (1 - \cos\alpha) + \frac{\mathbf{a} \times \mathbf{u}}{a}\sin\alpha$$

in which $a = \sqrt{\mathbf{a} \bullet \mathbf{a}}$ and $\frac{\sin\alpha}{a}$ is taken as positive. $\frac{(\mathbf{u} \bullet \mathbf{a})\mathbf{a}}{a^2}$ is the component of $\mathbf{u}$ in the direction of $\mathbf{a}$, $\left(\mathbf{u} - \frac{(\mathbf{u} \bullet \mathbf{a})\mathbf{a}}{a^2}\right)$ is the component of $\mathbf{u}$ perpendicular to $\mathbf{a}$ and $\frac{(\mathbf{a} \times \mathbf{u})\sin\alpha}{a}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{a}$.

If we choose $a = \tan\frac{\alpha}{2}$ then

$$1-\cos\alpha = 1-\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2} = \frac{2\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}} = \frac{2\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{2a^2}{1+a^2}$$

$$\sin\alpha = 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} = \frac{2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}} = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{2a}{1+a^2}$$

so that

$$\mathbf{u}' = \mathbf{u} + \frac{2}{1+a^2}\left((\mathbf{u}\bullet\mathbf{a})\mathbf{a} + \mathbf{a}\times\mathbf{u}\right).$$

In terms of components,

$$\begin{aligned}
\begin{bmatrix} u_1' & u_2' & u_3' \end{bmatrix} &= \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} + \frac{2}{1+a^2}\left[\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right]\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \\
&\quad + \frac{2}{1+a^2}\begin{bmatrix} (a_2u_3 - a_3u_2) & (a_3u_1 - a_1u_3) & (a_1u_2 - a_2u_1) \end{bmatrix} \\
&= \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}
\end{aligned}$$

in which

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{2}{1+a^2}\left[\begin{bmatrix} a_1a_1 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2a_2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3a_3 \end{bmatrix} + \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix}\right].$$

These results can also be written

$$\begin{aligned}
A_{ij} &= \delta_{ij} + \frac{2}{1+a^2}\left(a_ia_j + \varepsilon_{ijk}a_k\right) \\
u_j' &= \sum_{i=1}^{3} u_iA_{ij} = u_iA_{ij}
\end{aligned}$$

where $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = -\varepsilon_{321} = -\varepsilon_{132} = -\varepsilon_{213} = 1$ and all other $\varepsilon_{ijk} = 0$. The $\sum_{i=1}^{3}$ is implied by the Einstein summation convention.

If the rotation is **small**, $A_{ij} = \delta_{ij} + 2\varepsilon_{ijk} a_k$. The factor of 2 arises because the magnitude of the rotation vector is defined in terms of the tangent of one half the angle.

If the rotation $\mathbf{A}$ is followed by the rotation $\mathbf{B}$, the net result is $u'_j = u_i C_{ij}$ in which

$$C_{ij} = A_{ik} B_{kj}$$

$$
\begin{aligned}
&\delta_{ij} + \frac{2}{1+c^2}\left(c_i c_j + \varepsilon_{ijm} c_m\right) \\
&= \left(\delta_{ik} + \frac{2}{1+a^2}\left(a_i a_k + \varepsilon_{ikp} a_p\right)\right)\left(\delta_{kj} + \frac{2}{1+b^2}\left(b_k b_j + \varepsilon_{kjq} b_q\right)\right) \\
&= \delta_{ij} + \frac{2}{1+b^2}\left(b_i b_j + \varepsilon_{ijq} b_q\right) + \frac{2}{1+a^2}\left(a_i a_j + \varepsilon_{ijp} a_p\right) \\
&\quad + \frac{4}{\left(1+a^2\right)\left(1+b^2\right)}\left(a_i a_k b_k b_j + \varepsilon_{ikp} a_p b_k b_j + a_i a_k \varepsilon_{kjq} b_q + \varepsilon_{ikp} a_p \varepsilon_{kjq} b_q\right) \\
&= \delta_{ij} + \frac{2}{1+b^2}\left(b_i b_j + \varepsilon_{ijq} b_q\right) + \frac{2}{1+a^2}\left(a_i a_j + \varepsilon_{ijp} a_p\right) \\
&\quad + \frac{4}{\left(1+a^2\right)\left(1+b^2\right)}\left(a_i a_k b_k b_j - \left(\varepsilon_{ipq} b_j + a_i \varepsilon_{jpq}\right) a_p b_q - \left(\delta_{ij}\delta_{pq} - \delta_{iq}\delta_{pj}\right) a_p b_q\right) \\
&= \delta_{ij} + \frac{2}{1+b^2}\left(b_i b_j + \varepsilon_{ijq} b_q\right) + \frac{2}{1+a^2}\left(a_i a_j + \varepsilon_{ijp} a_p\right) \\
&\quad + \frac{4}{\left(1+a^2\right)\left(1+b^2\right)}\left(\mathbf{a} \bullet \mathbf{b} a_i b_j - \left(\varepsilon_{ipq} b_j + a_i \varepsilon_{jpq}\right) a_p b_q - \delta_{ij} \mathbf{a} \bullet \mathbf{b} + b_i a_j\right)
\end{aligned}
$$

Multiplying both sides by ε_{ijk}

$$\frac{4c_k}{1+c^2} = \frac{4b_k}{1+b^2} + \frac{4a_k}{1+a^2}$$

$$+\frac{4}{\left(1+a^2\right)\left(1+b^2\right)}\begin{pmatrix}\mathbf{a}\bullet\mathbf{b}\varepsilon_{ijk}a_ib_j \\ -\left(\left(\delta_{jp}\delta_{kq}-\delta_{jq}\delta_{kp}\right)b_j - \left(\delta_{ip}\delta_{kq}-\delta_{iq}\delta_{kp}\right)a_i\right)a_pb_q + \varepsilon_{ijk}b_ia_j\end{pmatrix}$$

$$= \frac{4b_k}{1+b^2} + \frac{4a_k}{1+a^2} + \frac{4}{\left(1+a^2\right)\left(1+b^2\right)}\begin{pmatrix}\left(\mathbf{a}\bullet\mathbf{b}-1\right)\varepsilon_{ijk}a_ib_j \\ -\left(a_jb_k - b_ja_k\right)b_j + \left(a_ib_k - b_ia_k\right)a_i\end{pmatrix}$$

$$= \frac{4b_k}{1+b^2} + \frac{4a_k}{1+a^2} + \frac{4}{\left(1+a^2\right)\left(1+b^2\right)}\begin{pmatrix}\left(\mathbf{a}\bullet\mathbf{b}-1\right)\varepsilon_{ijk}a_ib_j \\ -\left(\mathbf{a}\bullet\mathbf{b}b_k - b^2a_k\right) + \left(a^2b_k - \mathbf{a}\bullet\mathbf{b}a_k\right)\end{pmatrix}$$

$$= \frac{4}{1+a^2}\left(1+\frac{b^2-\mathbf{a}\bullet\mathbf{b}}{1+b^2}\right)a_k + \frac{4}{1+b^2}\left(1+\frac{a^2-\mathbf{a}\bullet\mathbf{b}}{1+a^2}\right)b_k - \frac{4\left(1-\mathbf{a}\bullet\mathbf{b}\right)\varepsilon_{ijk}a_ib_j}{\left(1+a^2\right)\left(1+b^2\right)}$$

$$= \frac{4}{1+a^2}\left(1+\frac{b^2-\mathbf{a}\bullet\mathbf{b}}{1+b^2}\right)a_k + \frac{4}{1+b^2}\left(1+\frac{a^2-\mathbf{a}\bullet\mathbf{b}}{1+a^2}\right)b_k - \frac{4\left(1-\mathbf{a}\bullet\mathbf{b}\right)\varepsilon_{ijk}a_ib_j}{\left(1+a^2\right)\left(1+b^2\right)}$$

$$= \frac{4\left(1-\mathbf{a}\bullet\mathbf{b}\right)\left(a_k + b_k - \varepsilon_{ijk}a_ib_j\right)}{\left(1+a^2\right)\left(1+b^2\right)}$$

so that

$$\frac{\mathbf{c}}{1+c^2} = \frac{\left(1-\mathbf{a}\bullet\mathbf{b}\right)\left(\mathbf{a}+\mathbf{b}-\mathbf{a}\times\mathbf{b}\right)}{\left(1+a^2\right)\left(1+b^2\right)}$$

and

$$\frac{\mathbf{c}\bullet\mathbf{c}}{\left(1+c^2\right)^2} = \frac{c^2}{\left(1+c^2\right)^2} = \frac{\left(1-\mathbf{a}\bullet\mathbf{b}\right)^2\left(a^2+2\mathbf{a}\bullet\mathbf{b}+b^2+\left(\mathbf{a}\times\mathbf{b}\right)\bullet\left(\mathbf{a}\times\mathbf{b}\right)\right)}{\left(1+a^2\right)^2\left(1+b^2\right)^2}$$

$$= \frac{\left(1-\mathbf{a}\bullet\mathbf{b}\right)^2\left(a^2+2\mathbf{a}\bullet\mathbf{b}+b^2+\left(a^2b^2-\left(\mathbf{a}\bullet\mathbf{b}\right)^2\right)\right)}{\left(1+a^2\right)^2\left(1+b^2\right)^2}$$

$$= \frac{\left(1-\mathbf{a}\bullet\mathbf{b}\right)^2\left(\left(1+a^2\right)\left(1+b^2\right)-\left(1-\left(\mathbf{a}\bullet\mathbf{b}\right)\right)^2\right)}{\left(1+a^2\right)^2\left(1+b^2\right)^2}$$

$$= \frac{\dfrac{\left(1+a^2\right)\left(1+b^2\right)}{\left(1-\mathbf{a}\bullet\mathbf{b}\right)^2}-1}{\left(\dfrac{\left(1+a^2\right)\left(1+b^2\right)}{\left(1-\mathbf{a}\bullet\mathbf{b}\right)^2}\right)^2}$$

Therefore

$$1 + c^2 = \frac{\left(1 + a^2\right)\left(1 + b^2\right)}{\left(1 - \mathbf{a} \bullet \mathbf{b}\right)^2}$$

and

$$\mathbf{c} = \frac{\mathbf{a} + \mathbf{b} - \mathbf{a} \times \mathbf{b}}{1 - \mathbf{a} \bullet \mathbf{b}}.$$

Thus $\mathbf{c} = \mathbf{a} + \mathbf{b}$ only if both $\mathbf{a}$ and $\mathbf{b}$ are small.