Linear theory of the oscillation of a hanging chain

Daniel Bernoulli in 1732 and Euler in 1781.

$m$ is the mass per unit length and $x$ is measured upwards from the bottom. The tension, $T = mgx$, and so the equation of SMALL oscillations is

$$\frac{\partial}{\partial x} \left( T \frac{\partial y}{\partial x} \right) = m \frac{\partial^2 y}{\partial t^2} \quad \text{or} \quad g \frac{\partial}{\partial x} \left( x \frac{\partial y}{\partial x} \right) = \frac{\partial^2 y}{\partial t^2}.$$

**Motion sinusoidal with time**

If we write $y = Y(x) \cos \omega t$, this becomes

$$g \frac{d}{dx} \left( x \frac{dY}{dx} \right) \cos \omega t = -\omega^2 Y \cos \omega t \quad \text{so that} \quad \frac{d}{dx} \left( x \frac{dY}{dx} \right) + \frac{\omega^2}{g} Y = 0.$$

Substitute $u = 2\omega \sqrt{\frac{x}{g}}$ so that $\frac{du}{dx} = \frac{\omega}{\sqrt{gx}}$ and therefore

$$\frac{\omega}{\sqrt{gx}} \frac{d}{du} \left( \frac{\omega x}{\sqrt{gx}} \frac{dY}{du} \right) + \frac{\omega^2}{g} Y = 0 \quad \text{or} \quad \frac{2}{g} \frac{d}{du} \left( \frac{u}{2} \frac{dY}{du} \right) + \frac{\omega^2}{g} Y = 0 \quad \text{and}$$

$$\frac{1}{u} \frac{d}{du} \left( u \frac{dY}{du} \right) + Y = 0.$$

This finally produces

$$\frac{d^2 Y}{du^2} + \frac{1}{u} \frac{dY}{du} + Y = 0.$$

**Bessel functions**

The standard form of Bessel's differential equation is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left( x^2 - \nu^2 \right) y = 0 \quad \text{which has the solution}$$

$$y = AJ_\nu(x) + BY_\nu(x) \quad \text{in which} \quad J_\nu(x) \quad \text{is the Bessel function of the first kind} \quad \text{and} \quad Y_\nu(x) \quad \text{and the Bessel function of the second kind and} \quad A \quad \text{and} \quad B \quad \text{are constants.}$$

In our case we have to replace $y$ by $Y$ and $x$ by $u$ and also set $\nu = 0$.

Bessel's equation is satisfied by Bessel functions of the first kind and Bessel functions of the second kind. However the Bessel functions of the second kind are infinite at the origin and therefore we only want the Bessel function of the first kind. Thus because $\nu = 0$,

$$Y = AJ_0(u) = AJ_0 \left( 2\omega \sqrt{\frac{x}{g}} \right).$$

The series for Bessel function of the first kind is

$$J_\nu(u) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\nu^{2m+\nu} m!(n+m)!} u^{2m} \quad \text{and therefore}$$
\[ Y = A J_0(u) = A \sum_{m=0}^{\infty} \frac{(-1)^m u^{2m}}{2^m (m!)^2} = A \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m (m!)^2} \left( \frac{4\omega^2 x}{g} \right)^m \]

where \( A \) is a constant.

```cpp
#include <iostream>
#include <fstream>
#include <cmath>
#include <cstdlib>
using namespace std;

#define LastPoint 5000

double x[LastPoint + 1], y[LastPoint + 1];
ofstream Chris("Chain.dxf");

int main(void)
{
    double L = 1000.0;
    double A = L / 10.0;
    double gOverOmegaSquared = L / 30.0;
    int LastTerm = 100;
    for(int Point=0; Point <= LastPoint; Point++)
    {
        x[Point] = L * (double)Point / (double)LastPoint;
        y[Point] = 0.0;
    }
    for(int Point = 0; Point <= LastPoint; Point++)
    {
        double mFactorial = 1.0;
        double thingy = 4.0 * x[Point] / gOverOmegaSquared;
        double thingytothem = 1.0;
        double twotothe2m = 1.0;
        double minus1tothem = 1.0;
        for(int m = 0; m <= LastTerm; m++)
        {
            if(m != 0)
            {
                thingytothem *= thingy;
                mFactorial *= (double)m;
                twotothe2m *= 4.0;
                minus1tothem = - minus1tothem;
            }
            y[Point] += A * minus1tothem * thingytothem / (twotothe2m * mFactorial * mFactorial);
        }
    }
    Chris<<"0\nSECTION\n2\nENTITIES\n";
    for(int Point = 0; Point <= LastPoint - 1; Point++)
    {
        Chris<<"0\nLINE\n8\nChain\n";
        Chris<<"10\n" << y[Point] << "\n";
        Chris<<"20\n" << x[Point] << "\n";
        Chris<<"11\n" << y[Point+1] << "\n";
        Chris<<"21\n" << x[Point+1] << "\n";
    }
    Chris<<"0\nLINE\n8\nVertical\n";
    Chris<<"10\n" << 0.0 << "\n";
    Chris<<"20\n" << 0.0 << "\n";
    Chris<<"11\n" << 0.0 << "\n";
    Chris<<"21\n" << L << "\n";
    Chris<<"0\nENDSEC\n0\nEOF\n";
    Chris.close();
    cout<<"DXF file written, end of program\n";
    return 0;
}