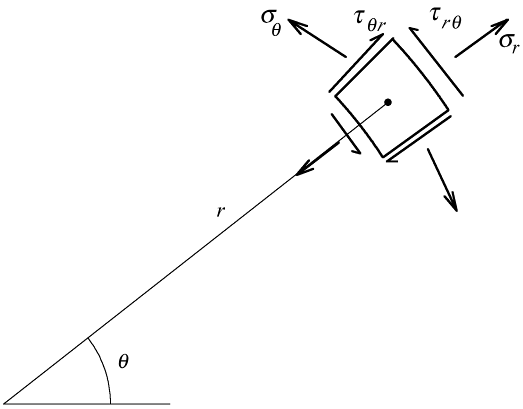
Airy stress function in Polar coordinates



The diagram shows a plan view of an element of a shell in cylindrical polar coordinates, r,θ,z. The horizontal components of membrane stress per unit horizontal length are σ_r, σ_θ, $\tau_{r\theta}$ and $\tau_{\theta r}$. If there is no horizontal load, the equilibrium equations in plan are

$$\frac{\partial}{\partial r}(r\sigma_r)-\sigma_\theta+\frac{\partial\tau_{\theta r}}{\partial\theta}=0, \qquad \frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial}{\partial r}(r\tau_{r\theta})+\tau_{\theta r}=0 \text{ and } \tau_{\theta r}=\tau_{r\theta}.$$

They come from considering element $\delta r\delta\theta$:

Resolving in the radial direction:

$$\frac{\partial}{\partial r}(r\delta\theta\sigma_r)\delta r-2\delta r\sigma_\theta\frac{\delta\theta}{2}+\frac{\partial}{\partial\theta}(\delta r\tau_{\theta r})\delta\theta=0.$$

Resolving in the tangential direction:

$$\frac{\partial}{\partial\theta}(\delta r\sigma_\theta)\delta\theta+\frac{\partial}{\partial r}(r\delta\theta\tau_{r\theta})\delta r+2\delta r\tau_{\theta r}\frac{\delta\theta}{2}=0.$$

Equilibrium of moments:

$$r\delta\theta\tau_{\theta r}\delta r=\delta r\tau_{r\theta}r\delta\theta.$$

The equilibrium equations in plan are automatically satisfied by $\sigma_r=\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$,

$\sigma_\theta=\frac{\partial^2\phi}{\partial r^2}$ and $\tau_{r\theta}=\tau_{\theta r}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$ in which ϕ is an Airy stress function.

Proof:

$$\frac{\partial}{\partial r}(r\sigma_r)-\sigma_\theta+\frac{\partial\tau_{\theta r}}{\partial\theta}$$
$$=\frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial r}+\frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2}\right)-\frac{\partial^2\phi}{\partial r^2}-\frac{\partial^2}{\partial\theta\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$
$$=\frac{\partial^2\phi}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2}\right)-\frac{\partial^2\phi}{\partial r^2}-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2}\right)=0$$

$$\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial}{\partial r}(r\tau_{r\theta})+\tau_{\theta r}$$
$$=\frac{\partial}{\partial\theta}\left(\frac{\partial^2\phi}{\partial r^2}\right)-\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\right)-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$
$$=\frac{\partial^3\phi}{\partial\theta\partial r^2}-\frac{\partial}{\partial r}\left(\frac{\partial^2\phi}{\partial\theta\partial r}-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)=0$$

If z is the height of the shell, the vertical equilibrium equation is

$$w=\left(\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)\frac{\partial^2 z}{\partial r^2}-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial z}{\partial\theta}\right)+\left(\frac{1}{r}\frac{\partial z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2}\right)\frac{\partial^2\phi}{\partial r^2}$$

in which w is downwards the load per unit plan area.

For radial symmetry all the derivatives with respect to θ disappear:

$$w=\frac{1}{r}\frac{d\phi}{dr}\frac{d^2z}{dr^2}+\frac{1}{r}\frac{dz}{dr}\frac{d^2\phi}{dr^2}=\frac{1}{r}\frac{d}{dr}\left(\frac{d\phi}{dr}\frac{dz}{dr}\right).$$